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# MONOTONE TRANSFORMATIONS ON $B(H)$ WITH RESPECT TO THE LEFT–STAR AND THE RIGHT–STAR PARTIAL ORDER

GREGOR DOLINAR, ALEXANDER GUTERMAN AND JANKO MAROVT

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*Abstract.* Let $H$ be an infinite dimensional complex Hilbert space, and let $B(H)$ be the set of all bounded linear operators on $H$. In the paper equivalent definitions for the left-star and the right-star partial orders on $B(H)$ are given and bijective additive maps on $B(H)$ which preserve the left-star or the right-star partial order in both directions are characterized.

## 1. Introduction

Let $M_n$ be the algebra of all $n \times n$ complex matrices and let $\operatorname{Im} A$ denote the image of $A \in M_n$. Drazin defined in [2] a partial order on $M_n$, named the star partial order, in the following way:

$$A \underset{*}{\leqslant} B \quad \text{if and only if} \quad A^*A = A^*B \text{ and } AA^* = BA^*. \tag{1}$$

Here $A^*$ stands for the conjugate transpose of $A$. Baksalary and Mitra introduced in [1] notions of order which are related.

DEFINITION 1. The left-star partial order on $M_n$ is a relation defined by

$$A {*\leqslant} B \quad \text{if and only if} \quad A^*A = A^*B \text{ and } \operatorname{Im} A \subseteq \operatorname{Im} B.$$

DEFINITION 2. The right-star partial order on $M_n$ is a relation defined by

$$A {\leqslant *} B \quad \text{if and only if} \quad AA^* = BA^* \text{ and } \operatorname{Im} A^* \subseteq \operatorname{Im} B^*.$$

It was shown in [1, Section 2] that both relations are partial orders and they are related to the star order in the following way:

$$A {*\leqslant} B \quad \text{and} \quad A {\leqslant *} B \quad \text{if and only if} \quad A \underset{*}{\leqslant} B \quad \text{for every} \quad A, B \in M_n.$$

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Many other partial orders can be also defined on $M_n$ (see [9]). For example, Hartwig [7] introduced the rank substractivity order as follows

$$A \overset{-}{\leqslant} B \quad \text{if and only if} \quad \text{rank }(B-A) = \text{rank } B - \text{rank } A.$$

He also observed that there exists another equivalent definition of the rank substractivity order, namely

$$A \overset{-}{\leqslant} B \quad \text{if and only if} \quad A^-A = A^-B \text{ and } AA^- = BA^-$$

where $A^-$ is a generalized inner inverse of $A$. The partial order $\overset{-}{\leqslant}$ is thus usually called the minus partial order. It was observed in [1, Theorem 2.1] that for every $A, B \in M_n$ each of the relations $A{*}{\leqslant} B$ or $A{\leqslant}{*}B$ implies $A \overset{-}{\leqslant} B$.

Let $H$ be an infinite-dimensional complex Hilbert space, and let $B(H)$ be the algebra of all bounded linear operators on $H$. In [14] Šemrl extended the minus partial order from $M_n$ to $B(H)$. Since $A \in B(H)$ has a generalized inner inverse if and only if its image is closed (see for example [11]), Šemrl found an appropriate equivalent definition of the minus partial order on $M_n$ without using inner inverses, and then extended this definition to $B(H)$. More precisely, he proved that for $A, B \in M_n$ we have $A \overset{-}{\leqslant} B$ if and only if there exist idempotent matrices $P, Q \in M_n$ such that $\operatorname{Im} P = \operatorname{Im} A$, $\operatorname{Ker} A = \operatorname{Ker} Q$, $PA = PB$ and $AQ = BQ$. Using the same equations, only adding the closure on $\operatorname{Im} A$ since the image of a bounded idempotent operator is closed, he then extended the concept of the minus partial order from $M_n$ to $B(H)$. In the same paper [14] Šemrl also described the structure of corresponding automorphisms for the minus partial order on $B(H)$.

Motivated by Šemrl's results Dolinar and Marovt introduced in [4] a similar equivalent definition of the star partial order $\underset{*}{\leqslant}$ on $B(H)$. Namely, they showed that the usual definition of the star partial order (1) for $B(H)$ is equivalent to the following definition: for $A, B \in B(H)$, $A \underset{*}{\leqslant} B$ if and only if there exist self-adjoint idempotent operators $P, Q \in B(H)$ such that $\operatorname{Im} P = \overline{\operatorname{Im} A}$, $\operatorname{Ker} A = \operatorname{Ker} Q$, $PA = PB$ and $AQ = BQ$. In addition, bijective additive continuous maps which preserve the star partial order in both directions on a set of all compact operators from $B(H)$, where $H$ is a separable Hilbert space, were characterized by Dolinar, Guterman, and Marovt in [3]. The authors restricted themselves in [3] to the set of all compact operators in $B(H)$ in order to use rank one operators in the proof and it is known that there exists a Hilbert space $H$ and an operator $A \in B(H)$ such that there is no rank one operator $C \in B(H)$ with $C \underset{*}{\leqslant} A$. We will show (see Lemma 10) that this problem does not occur in the case of the left-star or the right-star partial order on $B(H)$ and hence the characterization of the corresponding automorphisms will not be restricted to the set of all compact operators from $B(H)$.

It is the aim of this paper to continue with the study of partial orders on $B(H)$. First we extend the definition of left-star and right-star partial orders from $M_n$ to $B(H)$. Then we characterize automorphisms on $B(H)$ for the case of left-star and right-star partial orders and explain the structure of bijective additive converters between left-star and right-star orders. To indicate some possibilities for a further research we conclude the paper with an observation about left-star and right-star partial orders on the set of all Moore-Penrose invertible elements of a unital $C^*$-algebra.

## 2. The definition of left-star and right-star partial orders on $B(H)$

Let us introduce two relations on $B(H)$ and then prove that these relations are partial order relations which generalize left and right-star orders.

DEFINITION 3. For $A,B \in B(H)$ we write $A{*}\!\leqslant B$ if and only if there exist a self-adjoint idempotent $P \in B(H)$ and an idempotent $Q \in B(H)$, such that

$$\operatorname{Im} P = \overline{\operatorname{Im} A}, \tag{2}$$

$$\operatorname{Ker} A = \operatorname{Ker} Q, \tag{3}$$

$$PA = PB, \tag{4}$$

$$AQ = BQ. \tag{5}$$

We call this order *the left-star partial order* on $B(H)$.

DEFINITION 4. For $A,B \in B(H)$ we write $A \leqslant\!{*} B$ if and only if there exist an idempotent $P \in B(H)$ and a self-adjoint idempotent $Q \in B(H)$, such that

$$\operatorname{Im} P = \overline{\operatorname{Im} A}, \quad \operatorname{Ker} A = \operatorname{Ker} Q, \quad PA = PB, \quad AQ = BQ.$$

We call this order *the right-star partial order* on $B(H)$.

It will be proved that relations ${*}\!\leqslant$ from Definition 3 and $\leqslant\!{*}$ from Definition 4 are partial orders on $B(H)$. We will formulate and prove our results only for the left-star partial order, the right-star partial order can be treated in the same way by the symmetry.

PROPOSITION 1. *Let $A,B \in B(H)$. If $A{*}\!\leqslant B$ then $A \overset{-}{\leqslant} B$. If $A \underset{*}{\leqslant} B$ then $A{*}\!\leqslant B$.*

*Proof.* Both implications follow directly from the definitions of $\overset{-}{\leqslant}$ and $\underset{*}{\leqslant}$. □

Next lemma is formulated and proved in a similar way as [14, Theorem 2] and [4, Lemma 3].

LEMMA 2. *Let $A,B \in B(H)$. The following statements are equivalent:*

*(i)* $A{*}\!\leqslant B$.

*(ii) There exist closed subspaces $H_1, H_2, H_3 \subseteq H$, such that*

$$A,B : H_1 \oplus H_2 \to H_3 \oplus H_3^{\perp}$$

*have matrix representations*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix},$$

*where $A_1 : H_1 \to H_3$ and $B_1 : H_2 \to H_3^{\perp}$ are bounded linear operators and $A_1$ is injective with $\overline{\operatorname{Im} A_1} = H_3$.*

*(iii)* $\overline{\mathrm{Im}A} \perp \overline{\mathrm{Im}(B-A)}$ *and* $\overline{\mathrm{Im}B^*} = \overline{\mathrm{Im}A^*} \oplus \overline{\mathrm{Im}(B^*-A^*)}$.

*Proof.* First we prove that (i) implies (ii). Let $P$ and $Q \in B(H)$ be idempotent operators satisfying

$$\mathrm{Im}\,P = \overline{\mathrm{Im}A}, \quad P^* = P, \quad \mathrm{Ker}\,A = \mathrm{Ker}\,Q, \quad PA = PB, \quad AQ = BQ.$$

Let $H_1 = \mathrm{Im}\,Q$, $H_2 = \mathrm{Ker}\,Q$, $H_3 = \mathrm{Im}\,P$. Then $H_3^{\perp} = \mathrm{Ker}\,P$ since $P = P^*$ and $H = H_1 \oplus H_2 = H_3 \oplus H_3^{\perp}$. Representing $A$ according to this direct decomposition, we obtain $A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$, where $A_1 : H_1 \to H_3$ is an injective operator by (3) whose image is dense in $H_3$ by (2). From (5) we conclude that $B = \begin{bmatrix} A_1 & B_2 \\ 0 & B_1 \end{bmatrix}$. It follows by (4) that $\mathrm{Im}(B-A) \subseteq \mathrm{Ker}\,P$, hence $B_2 = 0$ and $B$ has the desired form as well.

Second, (ii) implies (iii). There exist idempotents $P, Q \in B(H)$ such that $\mathrm{Im}\,Q = H_1$, $\mathrm{Ker}\,Q = H_2$, $\mathrm{Im}\,P = H_3$, $\mathrm{Ker}\,P = H_3^{\perp}$, and $P = P^*$. Now it is easy to check that $A {*\leqslant} B$. Since $\mathrm{Im}A \subseteq H_3$ and $\mathrm{Im}(B-A) \subseteq H_3^{\perp}$, it follows that $\mathrm{Im}\,A \perp \mathrm{Im}(B-A)$, hence $\overline{\mathrm{Im}A} \perp \overline{\mathrm{Im}(B-A)}$. By Proposition 1, $A \bar{\leqslant} B$ and therefore by [14, Theorem 2] the other equality in (iii) is also true.

Third, (iii) implies (i). There exists an idempotent $P \in B(H)$ such that $\mathrm{Im}\,P = \overline{\mathrm{Im}A}$, $\mathrm{Ker}\,P = \overline{\mathrm{Im}A}^{\perp}$, and $P = P^*$. Then $\mathrm{Im}(B-A) \subseteq \overline{\mathrm{Im}(B-A)} \subseteq \overline{\mathrm{Im}A}^{\perp} = \mathrm{Ker}\,P$. It follows that $P(B-A) = 0$ which implies (4). Since $\overline{\mathrm{Im}\,B^*} = \overline{\mathrm{Im}A^*} \oplus \overline{\mathrm{Im}(B^*-A^*)}$ we can define an idempotent $R \in B(H)$ such that $\mathrm{Im}\,R = \overline{\mathrm{Im}A^*}$ and $\mathrm{Im}(B^*-A^*) \subseteq \mathrm{Ker}\,R$. It follows that $R(B^*-A^*) = 0$. Therefore, for $Q = R^*$ equalities (3) and (5) hold. Hence, $A {*\leqslant} B$. □

The following lemma follows directly from Definition 3 and Definition 4.

LEMMA 3. *Let* $A, B \in B(H)$. *Then* $A {*\leqslant} B$ *if and only if* $A^* {\leqslant *} B^*$.

THEOREM 4. *The relation* ${*\leqslant}$, *defined in Definition 3, is a partial order on* $B(H)$.

*Proof.* 1. $A {*\leqslant} A$ for all $A \in B(H)$ by item (iii) of Lemma 2.

2. If $A {*\leqslant} B$ and $B {*\leqslant} A$ for some $A, B \in B(H)$ then $A \bar{\leqslant} B$ and $B \bar{\leqslant} A$ by Proposition 1. Hence, $A = B$ by [14, Corollary 3].

3. It remains to check transitivity. So, let $A {*\leqslant} B$ and $B {*\leqslant} C$ for some $A, B, C \in B(H)$. We are going to show that the condition (iii) from Lemma 2 is satisfied. First, note that by Proposition 1, $A \bar{\leqslant} B$ and $B \bar{\leqslant} C$, which implies by [14, Corollary 3] that $A \bar{\leqslant} C$, so the second part of (iii) from Lemma 2 is satisfied. To show that $A {*\leqslant} C$ it remains to prove that $\overline{\mathrm{Im}A} \perp \overline{\mathrm{Im}(C-A)}$. By item (ii) of Lemma 2 we have $\overline{\mathrm{Im}A} \subseteq \overline{\mathrm{Im}B} \subseteq \overline{\mathrm{Im}C}$. Since $\overline{\mathrm{Im}B} \perp \overline{\mathrm{Im}(C-B)}$, we may conclude that $\overline{\mathrm{Im}A} \perp \overline{\mathrm{Im}(C-B)}$. Also

$$\mathrm{Im}(C-A) \subseteq \mathrm{Im}(C-B) + \mathrm{Im}(B-A) \subseteq \overline{\mathrm{Im}(C-B)} + \overline{\mathrm{Im}(B-A)}.$$

Since $\overline{\mathrm{Im}A} \perp \overline{\mathrm{Im}(B-A)}$ and $\overline{\mathrm{Im}A} \perp \overline{\mathrm{Im}(C-B)}$, it follows that $\overline{\mathrm{Im}A} \perp \mathrm{Im}(C-A)$. Using the fact that the inner product is continuous we may conclude that $\overline{\mathrm{Im}A} \perp \overline{\mathrm{Im}(C-A)}$. ☐

Let us prove the equivalence of two definitions of left-star partial orders.

THEOREM 5. *The left-star partial order, given by the usual Definition 1 is equivalent to the left-star partial order, given by Definition 3 on $M_n$ and $B(H)$.*

*Proof.* First, let us prove that Definition 3 implies Definition 1. Let $*\leqslant$ be the order defined with Definition 3 and suppose $A*\leqslant B$, $A,B \in B(H)$. Observe that then $\mathrm{Im}A \subseteq \mathrm{Im}B$. From item (iii) of Lemma 2 we have $\overline{\mathrm{Im}A} \perp \overline{\mathrm{Im}(B-A)}$. So, $\langle (B-A)x, Ax\rangle = 0$ and therefore $\langle A^*(B-A)x, x\rangle = 0$, $x \in H$. It follows that $A^*A = A^*B$.

To prove the converse implication, let us assume that for $A,B \in B(H)$ we have $A^*A = A^*B$ and $\mathrm{Im}A \subseteq \mathrm{Im}B$. There exists a unique partial isometry $W$ such that $A = \sqrt{AA^*}W$ is the polar decomposition of $A$ with $\mathrm{Im}W = \overline{\mathrm{Im}\sqrt{AA^*}}$. Let $\widetilde{P} \in B(H)$ be the self-adjoint idempotent such that $\mathrm{Im}\widetilde{P} = \overline{\mathrm{Im}\sqrt{AA^*}}$. Identically as in the proof of [4, Theorem 5] we may show that $\widetilde{P}A = \widetilde{P}B$. Since $A = \sqrt{AA^*}W$, it follows that $\mathrm{Im}A \subseteq \mathrm{Im}\sqrt{AA^*}$ and hence $\overline{\mathrm{Im}A} \subseteq \overline{\mathrm{Im}\sqrt{AA^*}}$. There exists a self-adjoint idempotent $P \in B(H)$ such that $\mathrm{Im}P = \overline{\mathrm{Im}A}$. So, $\mathrm{Im}P \subseteq \mathrm{Im}\widetilde{P}$ which yields $P\widetilde{P} = \widetilde{P}P = P$ and hence $PA = PB$. It follows that $\mathrm{Im}(B-A) \subseteq \mathrm{Ker}P = (\mathrm{Im}P)^{\perp} = \overline{\mathrm{Im}A}^{\perp}$ and therefore $\overline{\mathrm{Im}A} \perp \overline{\mathrm{Im}(B-A)}$. Observe that the left-star partial order implies minus partial order and then by [14, Theorem 2], $\overline{\mathrm{Im}B^*} = \overline{\mathrm{Im}A^*} \oplus \overline{\mathrm{Im}(B^*-A^*)}$. By Lemma 2 the assertion follows. ☐

## 3. Automorphisms of $B(H)$

Let us start with some auxiliary results.

LEMMA 6. *If $P \in B(H)$ is a self-adjoint idempotent and $A*\leqslant P$, then $A$ is a self-adjoint idempotent and $AP = PA = A$.*

*Proof.* Let $P \in B(H)$ be a self-adjoint idempotent and $A*\leqslant P$. By Proposition 1, $A*\leqslant P$ implies $A \overset{-}{\leqslant} P$. By [14] we may conclude that $A$ is an idempotent and that $AP = PA = A$. It remains to show that $A$ is a self-adjoint operator. By Lemma 2 it follows that $\overline{\mathrm{Im}A} \perp \overline{\mathrm{Im}(P-A)}$. So, $\langle (P-A)x, Ax\rangle = 0$ for every $x \in H$ and therefore, since $H$ is a complex Hilbert space, $A^*A = A^*P$. Note that $A^*A$ is a self-adjoint operator. It follows that

$$A^*A = (A^*P)^* = PA = A.$$

This yields that $A$ is a self-adjoint idempotent. ☐

LEMMA 7. *Let $U \in B(H)$ be a unitary operator and $S \in B(H)$ an invertible operator. Let $A,B \in B(H)$. Then*

- *$A *\leqslant B$ if and only if $UAS *\leqslant UBS$,*
- *$A \leqslant* B$ if and only if $SAU \leqslant* SBU$.*

*Proof.* Let $A *\leqslant B$. Then there exist idempotent operators $P, Q \in B(H)$ such that $\operatorname{Im} P = \overline{\operatorname{Im} A}$, $\operatorname{Ker} A = \operatorname{Ker} Q$, $P^* = P$, $PA = PB$ and $AQ = BQ$. Suppose $U \in B(H)$ is a unitary operator and $S \in B(H)$ is an invertible operator. Then $UAS *\leqslant UBS$ for a self-adjoint idempotent $P_1 = UPU^*$ and an idempotent $Q_1 = S^{-1}QS$. Conversely, let $U \in B(H)$ be a unitary operator and $S \in B(H)$ an invertible operator such that $UAS *\leqslant UBS$. Then for a unitary operator $U^*$ and invertible operator $S^{-1}$ it follows by the first part that $A *\leqslant B$.

The second claim can be proved in a similar way. □

We will denote by $I(H)$ the set of all idempotent operators in $B(H)$. Let $x, y \in H$ be nonzero vectors. We denote by $x \otimes y^*$ a rank one operator in $B(H)$ defined in the following way: $(x \otimes y^*)z = \langle z, y \rangle x$, for every $z \in H$. Note that every rank one operator in $B(H)$ can be written in this form.

LEMMA 8. *Let $x, y \in H$ be nonzero vectors and $A \in B(H)$. The following two statements are equivalent:*

*(i)* $x \otimes y^* \leqslant* A$.

*(ii)* $Ay = \langle y, y \rangle x$ *and* $y \in \operatorname{Im} A^*$.

*Proof.* Suppose first that $x \otimes y^* \leqslant* A$ for some nonzero vectors $x, y \in H$ and $A \in B(H)$. So, there exist $P, Q \in I(H)$, $Q = Q^*$ such that $\operatorname{Im} P = \operatorname{Im} x \otimes y^*, \operatorname{Ker} Q = \operatorname{Ker} x \otimes y^*$, $P(x \otimes y^*) = PA$ and $(x \otimes y^*)Q = AQ$. From $\operatorname{Ker} Q = \operatorname{Ker} x \otimes y^*$ we have $\operatorname{Im} Q = \operatorname{Im} y \otimes x^*$, and hence $\operatorname{Im} Q = \operatorname{Lin}\{y\}$. Since $Q$ is a rank one self-adjoint idempotent, it has the following form: $Q = \frac{y}{\|y\|} \otimes \left(\frac{y}{\|y\|}\right)^* = \frac{y \otimes y^*}{\langle y, y \rangle}$. From $(x \otimes y^*)Q = AQ$ it follows that $(x \otimes y^*)Q = \frac{1}{\langle y, y \rangle}(Ay \otimes y^*)$. Note that $x \otimes y^* = (x \otimes y^*)Q$. So, $\langle z, y \rangle x = \frac{1}{\langle y, y \rangle} \langle z, y \rangle Ay$ for every $z \in H$. We may conclude that $Ay = \langle y, y \rangle x$.

Since $\operatorname{Im} P = \operatorname{Im} x \otimes y^*$, we have $P(x \otimes y^*) = x \otimes y^*$ and hence $x \otimes y^* = PA$. So, $y \otimes x^* = A^* P^*$ and hence $y \in \operatorname{Im} A^*$.

Conversely, let $x, y \in H$ be nonzero vectors and $A \in B(H)$ such that $y \in \operatorname{Im} A^*$ and $Ay = \langle y, y \rangle x$. Let $Q = \frac{y \otimes y^*}{\langle y, y \rangle}$. So, $Q$ is a self-adjoint idempotent operator in $B(H)$ and $\operatorname{Ker} Q = \operatorname{Ker} x \otimes y^*$. Also, $(x \otimes y^*)Q = x \otimes y^*$. Let now $z \in H$. Then

$$AQz = \frac{1}{\langle y, y \rangle} A(\langle z, y \rangle y) = \frac{1}{\langle y, y \rangle} \langle z, y \rangle \langle y, y \rangle x = (x \otimes y^*)z$$

and therefore $(x \otimes y^*)Q = AQ$.

Since $y \in \operatorname{Im} A^*$, there exists $u \in H$ such that $A^* u = y$. Let $x \otimes u^* = P$. Then

$$\langle x, u \rangle = \left\langle A\left(\frac{y}{\langle y, y \rangle}\right), u \right\rangle = \left\langle \frac{y}{\langle y, y \rangle}, A^* u \right\rangle = 1.$$

It follows that $P = x \otimes u^*$ is an idempotent. Clearly, $\mathrm{Im}\, P = \mathrm{Im}\, x \otimes y^*$. Let $z \in H$. Then

$$P(x \otimes y^*)z = (x \otimes u^*)(x \otimes y^*)z = \langle z, y \rangle \langle x, u \rangle x = \langle z, y \rangle x = (x \otimes y^*)z$$

and

$$PAz = (x \otimes u^*)Az = \langle Az, u \rangle x = \langle z, A^* u \rangle x = \langle z, y \rangle x = (x \otimes y^*)z.$$

So, $P(x \otimes y^*) = PA$ and hence $x \otimes y^* \leqslant\!* A$. □

The following lemma, which can be proved directly, follows easily from Lemma 3 and Lemma 8.

LEMMA 9. *Let $x, y \in H$ be nonzero vectors and $A \in B(H)$. The following two statements are equivalent:*

*(i)* $x \otimes y^* {*\!\leqslant}\, A$.

*(ii)* $A^* x = \langle x, x \rangle y$ *and* $x \in \mathrm{Im} A$.

We will now show that for every nonzero operator $A \in B(H)$ there exists a rank one operator $C$ such that $C \; {*\!\leqslant}\, A$.

LEMMA 10. *Let $A \in B(H)$ be nonzero. For every nonzero $x \in \mathrm{Im} A$ there exists nonzero $y \in H$ such that $x \otimes y^* {*\!\leqslant}\, A$.*

*Proof.* By Lemma 9 it remains to prove that there exists a nonzero $y \in H$ such that $A^* x = \langle x, x \rangle y$. Suppose that there is no such $y$. So, $A^* \left( \frac{x}{\|x\|^2} \right) = 0$ and hence $\langle A^* x, z \rangle = 0$ for every $z \in H$. Therefore $\langle x, Az \rangle = 0$ for every $z \in H$ which yields that $x$ is orthogonal to $\mathrm{Im} A$. But $x \in \mathrm{Im} A$, hence $x = 0$, a contradiction. □

With the next lemma we will present a similar observation for the right-star partial order.

LEMMA 11. *Let $A \in B(H)$ be nonzero and suppose the image of $A$ is closed. Let $y \in \mathrm{Im} A^*, y \neq 0$. Then there exists a nonzero $l \in H$ such that $y \otimes l^* \leqslant\!* A^*$.*

*Proof.* Since $y \in \mathrm{Im} A^*$, $y \neq 0$, there exists $u \in H$, $u \neq 0$, such that $A^* u = y$. Let $z = \frac{u}{\|u\|^2}$. Then $\|z\| = \frac{1}{\|u\|}$, hence $A^* z = \langle z, z \rangle y$. Suppose $z \notin \mathrm{Im} A$. Since $\mathrm{Im} A = \overline{\mathrm{Im} A}$, it follows that $H = \mathrm{Ker} A^* \oplus \mathrm{Im} A$. Write $z = k + l_1$ where $k \in \mathrm{Ker} A^*$ and $l_1 \in \mathrm{Im} A$. If $l_1 = 0$, then $A^* z = 0$, a contradiction since $z \neq 0$, $y \neq 0$ and $A^* z = \langle z, z \rangle y$. It follows that $l_1 \neq 0$ and

$$A^* z = A^* l_1 = \langle k + l_1, k + l_1 \rangle y = (\langle k, k \rangle + \langle l_1, l_1 \rangle) y.$$

Let $a = \frac{\langle k,k \rangle + \langle l_1, l_1 \rangle}{\langle l_1, l_1 \rangle} y$. So, $a \in \mathrm{Lin}\{y\}$ and $y = \frac{\langle l_1, l_1 \rangle}{\langle k,k \rangle + \langle l_1, l_1 \rangle} a$. It follows that

$$A^* l_1 = \langle l_1, l_1 \rangle a.$$

Recall that $l_1 \in \mathrm{Im}A$. Lemma 8 yields $a \otimes l_1^* \leqslant\!* A^*$. Since $a \in \mathrm{Lin}\{y\}$, there exists $\alpha \in \mathbb{C}\setminus\{0\}$ such that $a = \alpha y$. So, $a \otimes l_1^* = y \otimes (\overline{\alpha} l_1)^*$. Let $l = \overline{\alpha} l_1$ hence $y \otimes l^* \leqslant\!* A^*$. $\square$

Let now $x \otimes y^*$ and $u \otimes v^*$ be two rank one operators in $B(H)$. Let us define the following relation between rank one operators in $B(H)$: we write $x \otimes y^* \sim u \otimes v^*$ if $x$ and $u$ are linearly dependent or $y$ and $v$ are linearly dependent. So, for two rank one operators $A, B \in B(H)$ we write $A \sim B$ if $A$ and $B$ have the same image or the same kernel.

LEMMA 12. *Let $A, B \in B(H)$, $A \neq B$, be rank one operators in $B(H)$. The following two statements are equivalent:*

*(i)* $A \sim B$;

*(ii) There does not exist a rank two operator $C \in B(H)$ such that $A *\!\leqslant C$ and $B *\!\leqslant C$.*

*Proof.* (i) $\Rightarrow$ (ii): Let $A, B \in B(H)$, $A \neq B$, be rank one operators in $B(H)$ and $A \sim B$. Denote $A = x \otimes y^*, B = u \otimes v^*$. Let $C$ be any rank two operator in $B(H)$ such that $A *\!\leqslant C$. Suppose first that $x$ and $u$ are linearly dependent and $y$ and $v$ are also linearly dependent. So, $A = \lambda B$ for some nonzero $\lambda \in \mathbb{C}$. Note that then $B = x \otimes \left(\frac{1}{\lambda} y\right)^*$. Since $A *\!\leqslant C$, by Lemma 9 we have $C^* x = \langle x, x\rangle y$. Suppose that $B *\!\leqslant C$. It follows that $C^* x = \frac{1}{\lambda}\langle x, x\rangle y$. So, $\langle x, x\rangle y = \frac{1}{\lambda}\langle x, x\rangle y$ hence $\lambda = 1$. This yields $A = B$, a contradiction.

Suppose now that $x$ and $u$ are linearly dependent and that $y$ and $v$ are linearly independent. Without loss of generality we may write $A = x \otimes y^*$ and $B = x \otimes v^*$. Suppose that there exists a rank two operator $C \in B(H)$ such that $A, B *\!\leqslant C$. So, $C^* x = \langle x, x\rangle y$ and $C^* x = \langle x, x\rangle v$ hence $y = v$, a contradiction.

Next, suppose that $x$ and $u$ are linearly independent and that $y$ and $v$ are linearly dependent. We may write $A = x \otimes y^*$ and $B = u \otimes y^*$. Again, suppose that there exists a rank two operator $C \in B(H)$ such that $A, B *\!\leqslant C$. So, $C^* x = \langle x, x\rangle y$, $C^* u = \langle u, u\rangle y$, and $x, u \in \mathrm{Im}C$. It follows that

$$C^*\left(\frac{x}{\|x\|^2}\right) = C^*\left(\frac{u}{\|u\|^2}\right)$$

hence $C^*\left(\frac{x}{\|x\|^2} - \frac{u}{\|u\|^2}\right) = 0$. So, $\frac{x}{\|x\|^2} - \frac{u}{\|u\|^2} \in \mathrm{Ker}C^* = (\mathrm{Im}C)^\perp$. Since $x, u \in \mathrm{Im}C$, it follows that $\frac{x}{\|x\|^2} - \frac{u}{\|u\|^2} \in \mathrm{Im}C$. We may conclude that $\frac{x}{\|x\|^2} - \frac{u}{\|u\|^2} = 0$ which yields that $x$ and $u$ are linearly dependent, a contradiction.

(ii) $\Rightarrow$ (i): To prove the converse implication suppose $A \nsim B$. Again, let $A = x \otimes y^*, B = u \otimes v^*$. So, $x$ and $u$ are linearly independent and $y$ and $v$ are linearly independent. Without loss of generality we may assume that $\|x\| = \|u\| = 1$. There exists $z \in \mathrm{Lin}\{x, u\}$ with $\langle x, z\rangle = 0$ and $\langle u, z\rangle \neq 0$. Let $C \in B(H)$ be defined in the following way:

$$C = x \otimes y^* + z \otimes s^* \quad \text{where } s = \frac{v - \langle u, x\rangle y}{\langle u, z\rangle}.$$

Since $v$ and $y$ are linearly independent, it follows that $s$ and $y$ are also linearly independent. Recall that $z$ and $x$ are linearly independent. We may conclude that $C$ is a rank two operator and since $z \in \operatorname{Lin}\{x,u\}$ it follows that $\operatorname{Im} C = \operatorname{Lin}\{x,u\}$. So, $u,x \in \operatorname{Im} C$. Also,

$$C^*x = (y \otimes x^* + s \otimes z^*)x = \langle x,x\rangle y + \langle x,z\rangle s = \langle x,x\rangle y$$

and

$$C^*u = \langle u,x\rangle y + \langle u,z\rangle s = \langle u,x\rangle y + \langle u,z\rangle \frac{v - \langle u,x\rangle y}{\langle u,z\rangle} = v = \langle u,u\rangle v.$$

Hence $x \otimes y^* *\!\leqslant C$ and $u \otimes v^* *\!\leqslant C$. We found a rank two operator $C$ such that $A,B *\!\leqslant C$. $\square$

LEMMA 13. *Let $A,B \in B(H)$. Suppose that for every rank one operator $C \in B(H)$ we have*

$$C *\!\leqslant A \quad \textit{if and only if} \quad C *\!\leqslant B.$$

*Then $A = B$.*

*Proof.* Let $A,B \in B(H)$. Assume that for every rank one operator $C \in B(H)$ we have $C *\!\leqslant A$ if and only if $C *\!\leqslant B$. Let $x \in \operatorname{Im} A$. By Lemma 10 there exists a nonzero $y \in H$ such that $x \otimes y^* *\!\leqslant A$. It follows that $x \otimes y^* *\!\leqslant B$ therefore by Lemma 9, $x \in \operatorname{Im} B$. We proved that $\operatorname{Im} A \subseteq \operatorname{Im} B$. In the same way we prove that $\operatorname{Im} B \subseteq \operatorname{Im} A$, hence $\operatorname{Im} A = \operatorname{Im} B$.

Note again that for every $x \in \operatorname{Im} A = \operatorname{Im} B$ there exists nonzero $y_1 \in H$ such that $x \otimes y_1^* *\!\leqslant A$. But since for every rank one operator $C \in B(H)$, $C *\!\leqslant A$ if and only if $C *\!\leqslant B$, we have that $x \otimes y_1^* *\!\leqslant B$. It follows by Lemma 9 that

$$A^*\left(\frac{x}{\|x\|^2}\right) = B^*\left(\frac{x}{\|x\|^2}\right),$$

hence $A^*x = B^*x$ for every $x \in \operatorname{Im} A = \operatorname{Im} B$. By the continuity of $A^*$ and $B^*$ we may conclude that $A^*x = B^*x$ for every $x \in \overline{\operatorname{Im} A} = \overline{\operatorname{Im} B}$.

Note that $\operatorname{Ker} A^* = \operatorname{Ker} B^*$ and $H = \operatorname{Ker} A^* \oplus \overline{\operatorname{Im} A}$. Take any $x \in H$. Then $x = u + v$ for some $u \in \operatorname{Ker} A^*$ and $v \in \overline{\operatorname{Im} A}$. It follows that

$$A^*x = A^*v = B^*v = B^*x.$$

So, $A^* = B^*$ and therefore $A = B$. $\square$

Let $x,y \in H$ be nonzero. Let us define the following sets of operators:

$$L_x = \{x \otimes v^* : v \in H \backslash \{0\}\} \quad \text{and} \quad R_y = \{z \otimes y^* : z \in H \backslash \{0\}\}.$$

Note that every operator in $L_x$ and every operator in $R_y$ is of rank one.

LEMMA 14. *An operator $A$ is invertible if and only if for every nonzero $x \in H$ and for every nonzero $y \in H$ there exist $B \in L_x$ and $C \in R_y$ such that $B,C *\!\leqslant A$.*

*Proof.* Suppose first that for every nonzero $x \in H$ and for every nonzero $y \in H$ there exist $B \in L_x$ and $C \in R_y$ such that $B, C * \leqslant A$. Let $x \in H$, $x \neq 0$. Then there exist $v \in H \backslash \{0\}$ such that $B = x \otimes v^* * \leqslant A$. By Lemma 9, $x \in \operatorname{Im} A$ hence $A$ is surjective. Similarly, for $y \in H$, $y \neq 0$, there exists $z \in H \backslash \{0\}$ such that $C = z \otimes y^* * \leqslant A$. It follows that $A^* z = \langle z, z \rangle y$ hence $y \in \operatorname{Im} A^*$. This yields $\operatorname{Im} A^* = H$ and hence $\operatorname{Ker} A = \{0\}$. So, $A$ is also injective and hence bijective.

Conversely, suppose $A \in B(H)$ is invertible. Let $x \in H = \operatorname{Im} A$, $x \neq 0$. By Lemma 10 there exists $y \in H \backslash \{0\}$ such that $x \otimes y^* * \leqslant A$. If we denote $B = x \otimes y^*$, then $B \in L_x$ and $B * \leqslant A$. Since $A$ is invertible, $A^*$ is also invertible. Let $y \in H = \operatorname{Im} A^*$, $y \neq 0$. By Lemma 11 there exists a nonzero $u \in H$ such that $y \otimes u^* \leqslant * A^*$, hence by Lemma 3, $u \otimes y^* * \leqslant A$. □

Let us now state the main result of this section.

THEOREM 15. *Let $H$ be an infinite-dimensional complex Hilbert space. Assume that $\phi : B(H) \to B(H)$ is a bijective, additive map such that for every pair $A, B \in B(H)$ we have*

$$A * \leqslant B \quad \text{if and only if} \quad \phi(A) * \leqslant \phi(B).$$

*Then there exist a unitary operator $U \in B(H)$ and an invertible operator $S \in B(H)$, or there exist an antiunitary operator $U : H \to H$ and a bounded, bijective, conjugate-linear map $S : H \to H$ such that*

$$\phi(A) = UAS$$

*for every $A \in B(H)$.*

*Proof.* We split it into the different steps.

1. $\phi(0) = 0$. Since $0 * \leqslant \phi^{-1}(0)$, we have $\phi(0) * \leqslant 0$, thus $\phi(0) = 0$.

2. *$\phi$ preserves rank one.* Denote by $B_1(H)$ the set of all rank one operators in $B(H)$. Let $A \in B_1(H)$, $B \in B(H)$ and suppose $B * \leqslant A$. By Lemma 2 it follows that then either $B = 0$ or $B = A$. Of course, if $B = 0$ or $B = A$, then $B * \leqslant A$. We may conclude that $\phi(B_1(H)) = B_1(H)$.

3. *$\phi$ preserves rank two.* Similarly, $A \in B(H)$ is of rank two if and only if for every $B \in B(H)$ where $B * \leqslant A$ and $B \neq A$ it follows that $B \in \{0\} \cup B_1(H)$. So, $\phi$ maps the set of all rank two operators onto itself.

4. *$\phi$ preserves the relation $\sim$ in both directions.* Indeed, it follows by Lemma 12 that for every pair $A, B \in B_1(H)$ we have:

$$A \sim B \quad \text{if and only if} \quad \phi(A) \sim \phi(B),$$

which is the required condition.

5. *Action of $\phi$ on the sets $L_x$, $R_y$.* Let $x, y \in H$, $x \neq 0$, $y \neq 0$. Note that for every pair of operators $A, B \in L_x$ we have $A \sim B$, and for every pair of operators $C, D \in R_y$ we also have $C \sim D$. Let $\mathscr{T}$ be a subset of $B_1(H)$ such that for every $A, B \in \mathscr{T}$ we have $A \sim B$. Then there exists a nonzero $x \in H$ such that $\mathscr{T} \subseteq L_x$ or there exists a nonzero $y \in H$ such that $\mathscr{T} \subseteq R_y$. Since $\phi$ is bijective and preserves the relation $\sim$ in

both directions, it follows that for every nonzero $x \in H$ there exists a nonzero $u \in H$ such that $\phi(L_x) = L_u$, or there exists a nonzero $y \in H$ such that $\phi(L_x) = R_y$. Similarly, for every nonzero $y \in H$ there exists a nonzero $x \in H$ such that $\phi(R_y) = L_x$, or there exists a nonzero $v \in H$ such that $\phi(R_y) = R_v$. Since $\phi^{-1}$ has the same properties as $\phi$ we may conclude that for every nonzero $u \in H$ there exists a nonzero $x \in H$ such that $\phi(L_x) = L_u$ or there exists a nonzero $y \in H$ such that $\phi(R_y) = L_u$. Similarly, for every nonzero $v \in H$ there exists a nonzero $x \in H$ such that $\phi(L_x) = R_v$, or there exists a nonzero $y \in H$ such that $\phi(R_y) = R_v$.

6. *$\phi$ preserves the invertibility*. Let now $A \in B(H)$ be an invertible operator and suppose $u \in H$ is nonzero. There exists a nonzero $x \in H$ such that $\phi(L_x) = L_u$, or there exists a nonzero $y \in H$ such that $\phi(R_y) = L_u$. Suppose $\phi(L_x) = L_u$. Since $A$ is invertible, it follows by Lemma 14 that there exists $B \in L_x$ such that $B {*\leqslant} A$. So, $\phi(B) {*\leqslant} \phi(A)$. Note that $\phi(B) \in L_u$. Similarly, if $\phi(R_y) = L_u$ there exists $C \in R_y$ such that $\phi(C) {*\leqslant} \phi(A)$ and $\phi(C) \in L_u$. So, since $\phi$ is surjective, we may find for every nonzero $u \in H$ an operator $D \in L_u$ such that $D {*\leqslant} \phi(A)$. In the same way we prove that there exists an operator $E \in R_u$ such that $E {*\leqslant} \phi(A)$. By Lemma 14 we may conclude that $\phi(A)$ is an invertible operator. Since $\phi^{-1}$ has the same properties as $\phi$ it follows that $A \in B(H)$ is invertible if and only if $\phi(A)$ is invertible.

7. *Without loss of generality we may assume that $\phi(I) = I$*. Indeed, $\phi(I)$, where $I$ is the identity operator, is also invertible. By Lemma 7 we may replace the map $\phi$ with the map $\psi : B(H) \to B(H)$ which is defined in the following way: $\psi(A) = \phi(A)\phi^{-1}(I)$. From now on we may and will assume that

$$\phi(I) = I.$$

8. *$\phi$ leaves invariant the set $\mathscr{P}(H)$ of all self-adjoint idempotent operators in $B(H)$*. By Definition 3 it is clear that for every $P \in \mathscr{P}(H)$ we have $P {*\leqslant} I$. So, $\phi(P) {*\leqslant} I$ hence by Lemma 6 $\phi(P)$ is also a self-adjoint idempotent. Since $\phi$ preserves the left-star partial order in both directions, we may conclude that $\phi(\mathscr{P}(H)) = \mathscr{P}(H)$.

9. *Restriction of $\phi$ on $\phi(\mathscr{P}(H))$* . Let $P, Q \in \mathscr{P}(H)$. Lemma 6 yields that if $P {*\leqslant} Q$, then $PQ = QP = P$ and hence $P \leqslant Q$ where $\leqslant$ denotes the usual order (i.e. $P \leqslant Q$ if and only if $PQ = QP = P$) on $\mathscr{P}(H)$. Also, directly from Definition 3 it follows that if $PQ = QP = P$ for $P, Q \in \mathscr{P}(H)$, then $P {*\leqslant} Q$. So, on the set $\mathscr{P}(H)$ the minus order $\overset{-}{\leqslant}$ coincides with the left-star partial order ${*\leqslant}$ . The restriction of $\phi$ to $\mathscr{P}(H)$ is a bijective and additive map from $\mathscr{P}(H)$ to $\mathscr{P}(H)$ which preserves the minus order in both directions.

10. *Action of $\phi$ on the set of closed subspaces in $H$*. We may identify closed subspaces in $H$ with self-adjoint idempotents in $B(H)$. So, the map $\phi$ induces a lattice automorphism, i.e. a bijective map $\omega$ which is defined on the set of all closed subspaces in $H$ and where $M \subseteq N$ if and only if $\omega(M) \subseteq \omega(N)$ for every pair of closed subspaces $M, N$ in $H$. Recall that $H$ is an infinite dimensional complex Hilbert space. By a result of Fillmore and Longstaff [5, Theorem 1] there exists a bicontinuous linear or conjugate linear bijection $S : H \to H$ such that $\omega(M) = SM$ for every closed subspace $M$ in $H$. This means that for our map $\phi$ we have

$$\phi(P_M) = P_{S(M)}$$

for every $P_M \in \mathscr{P}(H)$. Here $P_M$ denotes the self-adjoint idempotent such that $\operatorname{Im} P_M = M$.

11. *We will now show that without loss of generality we may assume that $S: H \to H$ is a unitary or antiunitary operator.* Let $x \in H$ such that $\|x\| = 1$. It follows that $\phi(x \otimes x^*)$ is a rank one self-adjoint idempotent. Denote $\phi(x \otimes x^*) = a \otimes a^*$. So, $\|a\| = 1$. Let $y \in H$, $\|y\| = 1$, such that $\langle x, y\rangle = 0$. Then there exists $b \in H$, $\|b\| = 1$, such that $\phi(y \otimes y^*) = b \otimes b^*$. Suppose that $\langle a, b\rangle \neq 0$. The operator $\phi(x \otimes x^* + y \otimes y^*)$ is a self-adjoint idempotent of rank two. Since $\phi$ is additive, we have $\phi(x \otimes x^* + y \otimes y^*) = a \otimes a^* + b \otimes b^*$, and since this is a rank two operator, we may conclude that $a$ and $b$ are linearly independent. From

$$(a \otimes a^* + b \otimes b^*)(a \otimes a^* + b \otimes b^*) = a \otimes a^* + b \otimes b^*$$

it follows that $\langle b, a\rangle (a \otimes b^*)z = -\langle a, b\rangle (b \otimes a^*)z$ for every $z \in H$. Let $z = b$. So, $\langle b, a\rangle \langle b, b\rangle a = -\langle a, b\rangle \langle b, a\rangle b$. Since $\langle b, a\rangle \neq 0$, we have $a = -\frac{\langle a,b\rangle}{\langle b,b\rangle} b$ hence $a$ and $b$ are linearly dependent, a contradiction. So, $\langle a, b\rangle = 0$.

On one hand $\operatorname{Im} \phi(x \otimes x^*) = \operatorname{Lin}\{a\}$ and on the other hand, since $\phi(P_M) = P_{S(M)}$ for every $P_M \in \mathscr{P}(H)$, we have $\operatorname{Im} \phi(x \otimes x^*) = S(\operatorname{Lin}\{x\}) = \operatorname{Lin}\{Sx\}$. So, $a$ and $Sx$ are linearly dependent hence $a = \alpha Sx$ for some $\alpha \in \mathbb{C} \backslash \{0\}$. Similarly, there exists $\beta \in \mathbb{C} \backslash \{0\}$ such that $b = \beta Sy$. It follows that

$$0 = \langle a, b\rangle = \alpha \overline{\beta} \langle Sx, Sy\rangle = \alpha \overline{\beta} \langle S^* Sx, y\rangle .$$

We proved that $\langle S^* Sx, y\rangle = 0$ for every $y \in H$, $\|y\| = 1$, which is orthogonal to $x$. Note that then for any (fixed) $x \in H$ we have $\langle S^* Sx, y\rangle = 0$ for every $y \in H$ where $\langle x, y\rangle = 0$. So, $S^* Sx$ is a scalar multiple of $x$ hence linear operators $S^* S$ and $I$ are locally linearly dependent. It is known (see for example [10]) that for linear operators which are of rank at least two, local linear dependence implies linear dependence. Hence $S^* S = \mu I$ for some $\mu > 0$. Let $U = \frac{1}{\sqrt{\mu}} S$. Then $U^* U = I$, and since $SS^* = \mu I$, we get $UU^* = I$. We may conclude that $U: H \to H$ is a unitary or antiunitary operator.

12. *Without loss of generality we may assume that $\phi(P) = P$ for every self-adjoint idempotent $P \in B(H)$.* By the previous two steps,

$$\phi(P_M) = P_{U(M)}$$

for every $P_M \in \mathscr{P}(H)$. This implies that

$$\phi(P) = UPU^*$$

for every $P \in \mathscr{P}(H)$.

Let us now define $\psi : B(H) \to B(H)$ in the following way: $\psi(A) = U^* \phi(A) U$. Then $\psi(P) = P$. Without loss of generality we may and will assume that

$$\phi(P) = P \quad \text{for every self-adjoint idempotent } P \in B(H).$$

13. *We are going to show that $\phi(PB(H)P) = PB(H)P$, where $PB(H)P = \{PAP : A \in B(H)\}$, $P \in B(H)$ is a finite rank self-adjoint idempotent of rank $n \geqslant 3$.* Since $\phi^{-1}$

has the same properties as $\phi$, it is enough to show that $\phi(PB(H)P) \subseteq PB(H)P$. It is easy to see that $A \in PB(H)P$ if and only if $\operatorname{Im} A \subseteq \operatorname{Im} P$ and $\operatorname{Ker} P \subseteq \operatorname{Ker} A$.

a). Let us first show that for every rank one operator $A \in PB(H)P$ it follows that $\phi(A) \in PB(H)P$. Suppose $A = \alpha x \otimes y^*$ where $\|x\| = \|y\| = 1$ and $A \in PB(H)P$. Since $A \sim x \otimes x^*$ and $A \sim y \otimes y^*$ and since $x \otimes x^*$ and $y \otimes y^*$ are self-adjoint idempotents, it follows that $\phi(A) \sim \phi(x \otimes x^*) = x \otimes x^*$ and $\phi(A) \sim y \otimes y^*$.

Suppose first that $x$ and $y$ are linearly independent. Denote $\phi(A) = z \otimes w^*$, $z \neq 0$, $w \neq 0$. It follows that $z$ and $x$ are linearly dependent or $w$ and $x$ are linearly dependent. Also, $z$ and $y$ are linearly dependent or $w$ and $y$ are linearly dependent. Since $x$ and $y$ are linearly independent, we may conclude that there exist $\alpha, \beta \in \mathbb{C} \backslash \{0\}$ such that $z = \alpha x$ and $w = \beta y$, or there exist $\gamma, \delta \in \mathbb{C} \backslash \{0\}$ such that $z = \gamma y$ and $w = \delta x$. It follows that $\phi(A) = \lambda x \otimes y^*$ or $\phi(A) = \mu y \otimes x^*, \lambda, \mu \in \mathbb{C} \backslash \{0\}$. In both cases $\phi(A) \in PB(H)P$.

Assume now that $x$ and $y$ are linearly dependent, i.e. $A = \varepsilon x \otimes x^*$ for some $\varepsilon \in \mathbb{C} \backslash \{0,1\}$. Since $\phi^{-1}$ has the same properties as $\phi$, every operator of the form $\lambda u \otimes v^*$ where $u$ and $v$ are linearly independent is the image of the operator $\alpha u \otimes v^*$ or of the operator $\beta v \otimes u^*$ for some nonzero $\alpha$ and $\beta$. Recall that $\phi(x \otimes x^*) = x \otimes x^*$ for every rank one self-adjoint idempotent $x \otimes x^*$. Therefore, there exist $a \in H \backslash \{0\}$ and $\gamma \in \mathbb{C} \backslash \{0,1\}$ such that $\phi(A) = \phi(\varepsilon x \otimes x^*) = \gamma a \otimes a^*$. Again, since $A \sim x \otimes x^*$, we obtain $\gamma a \otimes a^* \sim x \otimes x^*$. It follows that $\phi(A) = \delta x \otimes x^*$ for some $\delta \in \mathbb{C} \backslash \{0,1\}$ and therefore $\phi(A) \in PB(H)P$.

We have just proved that $A \in PB(H)P$ implies $\phi(A) \in PB(H)P$ for every rank one operator $A$.

b). Let now $D \in PB(H)P$ be any nonzero operator. By Lemma 9 and Lemma 10, $\operatorname{Im} D = \cup \{\operatorname{Im} C : C \in B_1(H) \text{ and } C {*\leqslant} D\}$. For every $C = x \otimes y^*$ where $C {*\leqslant} D$ we have $D^* x = \langle x, x \rangle y$ and $x \in \operatorname{Im} D$. So, $\operatorname{Im} C \subseteq \operatorname{Im} D \subseteq \operatorname{Im} P$. Also, $\operatorname{Im} C^* \subseteq \operatorname{Im} D^*$ hence $\operatorname{Ker} P \subseteq \operatorname{Ker} D \subseteq \operatorname{Ker} C$. It follows that $C \in PB(H)P$ and hence $\phi(C) \in PB(H)P$ by the part a).

The map $\phi$ is bijective, preserves the order $*\leqslant$ in both directions and maps the set of all rank one operators onto itself therefore

$$\operatorname{Im} \phi(D) = \cup \{\operatorname{Im} \phi(C) : C \in B_1(H) \text{ and } C {*\leqslant} D\}.$$

Since $\operatorname{Im} \phi(C) \subseteq \operatorname{Im} P$ for every $C \in B_1(H)$ where $C {*\leqslant} D$, it follows that $\operatorname{Im} \phi(D) \subseteq \operatorname{Im} P$.

In order to prove that $\phi(D) \in PB(H)P$ we will show that $\operatorname{Im} \phi(D)^* \subseteq \operatorname{Im} P$. Let us first show that

$$\operatorname{Im} \phi(D)^* = \cup \{\operatorname{Im} \phi(C)^* : C \in B_1(H) \text{ and } \phi(C)^* {\leqslant*} \phi(D)^*\}.$$

For $y \in \operatorname{Im} \phi(D)^*$ there exists by Lemma 11 a nonzero $a \in \operatorname{Lin}\{y\}$ and $l \in H$, $l \neq 0$, such that $a \otimes l^* {\leqslant*} \phi(D)^*$. Also, there exists $C \in B_1(H)$ such that $\phi(C)^* = a \otimes l^*$. Note that $\operatorname{Lin}\{a\} = \operatorname{Lin}\{y\}$ and $\operatorname{Lin}\{a\} = \operatorname{Im} \phi(C)^*$. We proved that for every $y \in \operatorname{Im} \phi(D)^*$ there exists $C \in B_1(H)$ such that $y \in \operatorname{Im} \phi(C)^*$. It follows that

$$\operatorname{Im} \phi(D)^* \subseteq \cup \{\operatorname{Im} \phi(C)^* : \phi(C)^* \in B_1(H) \text{ and } \phi(C)^* {\leqslant*} \phi(D)^*\}.$$

The converse inclusion follows from the fact that $\phi(C)^* \leqslant * \phi(D)^*$ implies $\operatorname{Im}\phi(C)^* \subseteq \operatorname{Im}\phi(D)^*$.

By Lemma 3 and since $\phi$ is bijective and maps the set of all rank one operators onto itself we conclude that

$$\operatorname{Im}\ \phi(D)^* = \cup\{\operatorname{Im}\ \phi(C)^* : C \in B_1(H) \text{ and } C{*}{\leqslant}\, D\}.$$

Clearly, if $A \in PB(H)P$ then $A^* \in PB(H)P$. So, since $D \in PB(H)P$ and hence for every $C \in B_1(H)$ where $C{*}{\leqslant}\, D$ we have $\phi(C) \in PB(H)P$, it follows that $\operatorname{Im}\phi(C)^* \subseteq \operatorname{Im}P$. This implies that $\operatorname{Im}\phi(D)^* \subseteq \operatorname{Im}P$ and hence $\phi(D) \in PB(H)P$.

We proved that if $P$ is any finite rank self-adjoint idempotent of rank greater than $2$, then $\phi(PB(H)P) = PB(H)P$.

14. *Reduction to monotone maps on $M_n$.* Suppose $P \in B(H)$ is a finite rank self-adjoint idempotent of rank $n \geqslant 3$. The set $PB(H)P$ can be then identified with the set $M_n$. The restriction of $\phi$ to $PB(H)P$ may be considered as a bijective, additive map on $M_n$ which preserves the left-star partial order in both directions. As a corollary of a result by Guterman [6, Theorem 3.5] we get that if a bijective additive $T : M_n \to M_n$ preserves the left-star partial order in one direction (i.e., $A{*}{\leqslant}\, B$ implies $\phi(A){*}{\leqslant}\, \phi(B)$), then $T$ belongs to one of the following types: $X \mapsto U_nXS_n$, $X \mapsto U_n\overline{X}S_n$, $X \mapsto U_nX^tS_n$, $X \mapsto U_nX^*S_n$, where $U_n \in M_n$ is unitary and $S_n \in M_n$ is invertible. Here $X^t$ denotes the transpose of $X$, and $\overline{X}$ is the matrix obtained from $X$ by taking complex conjugate values of its entries. It is straightforward to check that only the transformation of type $X \mapsto U_nXS_n$ preserves the left-star order in both directions. Thus we may conclude that $\phi(X) = U_nXS_n$ for any $X \in M_n$. Since $\phi(P) = P$ for every self-adjoint idempotent matrix $P$ in $M_n$, we may conclude that $U_n = S_n = I_n$ and $\phi(A) = A$ for every $A \in M_n$.

Since $P$ is any finite rank self-adjoint idempotent of rank at least three, we may conclude that $\phi(C) = C$ for every rank one $C \in B(H)$. It follows by Lemma 13 that then $\phi(A) = A$ for every $A \in B(H)$.

15. Taking into account the assumptions that were made in Step 7 and in Step 12 about the map $\phi$ we may conclude that there exist a unitary operator $U \in B(H)$ and an invertible operator $S \in B(H)$, or there exist an antiunitary operator $U : H \to H$ and a bounded, bijective, conjugate-linear map $S : H \to H$ such that

$$\phi(A) = UAS$$

for every $A \in B(H)$. $\square$

Symmetrically we can prove a similar result for the right-star partial order on $B(H)$.

THEOREM 16. *Let $H$ be an infinite-dimensional complex Hilbert space. Assume that $\phi : B(H) \to B(H)$ is a bijective, additive map such that for every pair $A, B \in B(H)$ we have*

$$A \leqslant * B \quad \text{if and only if} \quad \phi(A) \leqslant * \phi(B).$$

*Then there exist a unitary operator $U \in B(H)$ and an invertible operator $S \in B(H)$, or there exist an antiunitary operator $U : H \to H$ and a bounded, bijective, conjugate-linear map $S : H \to H$ such that*

$$\phi(A) = SAU$$

*for every $A \in B(H)$.*

As a corollary of Theorem 15 we can also characterize the converters between right-star and left-star partial orders.

COROLLARY 17. *Let $H$ be an infinite-dimensional complex Hilbert space. Assume that $\phi : B(H) \to B(H)$ is a bijective, additive map such that for every pair $A, B \in B(H)$ we have*

$$A \leqslant * B \quad \text{if and only if} \quad \phi(A) * \leqslant \phi(B).$$

*Then there exist a unitary operator $U \in B(H)$ and an invertible operator $S \in B(H)$, or there exist an antiunitary operator $U : H \to H$ and a bounded, bijective, conjugate-linear map $S : H \to H$ such that*

$$\phi(A) = UA^*S$$

*for every $A \in B(H)$.*

*Proof.* By Lemma 3, $A * \leqslant B$ if and only if $A^* \leqslant * B^*$. Thus the transformation $\phi_1 : B(H) \to B(H)$ defined by $\phi_1(X) = \phi(X^*)$ for all $X \in B(H)$ satisfies the conditions

$$A * \leqslant B \quad \text{iff} \quad A^* \leqslant * B^* \quad \text{iff} \quad \phi(A^*) * \leqslant \phi(B^*) \quad \text{iff} \quad \phi_1(A) * \leqslant \phi_1(B)$$

for all $A, B \in B(H)$. Since $\phi$ is bijective and additive, $\phi_1$ is also bijective and additive. So $\phi_1$ satisfies the assumptions of Theorem 15 and hence there exist a unitary operator $U \in B(H)$ and an invertible operator $S \in B(H)$, or there exist an antiunitary operator $U : H \to H$ and a bounded, bijective, conjugate-linear map $S : H \to H$ such that

$$\phi_1(A) = UAS$$

for every $A \in B(H)$. Since $\phi_1(X) = \phi(X^*)$ for every $X \in B(H)$, $\phi$ has the required form. □

To show similar result for a bijective additive map $\phi$ converting $* \leqslant$ to $\leqslant *$ we apply the above corollary to the transformation $\phi^{-1}$.

## 4. Left-star and right-star partial orders on $A^\dagger$

Since the general $C^*$-algebra framework may be a good ground for a further research, let us conclude our paper with an observation about left-star and right-star partial orders on a certain subset of a unital $C^*$-algebra. So, let $\mathscr{A}$ be a $C^*$-algebra with unit $1$. For $a \in \mathscr{A}$ let us consider the following equations

$$(1)\ aba = a, \quad (2)\ bab = b, \quad (3)\ (ab)^* = ab, \quad (4)\ (ba)^* = ba.$$

It is known (see, for example, [12]) that the set of all $b \in \mathscr{A}$ that satisfy equations $(1)-(4)$ is empty or a singleton and when it is a singleton its unique element is called the Moore-Penrose inverse of $a$. We will denote the Moore-Penrose inverse of $a$ by $a^{\dagger}$. The subset of $\mathscr{A}$ consisting of all Moore-Penrose invertible elements of $\mathscr{A}$ will be denoted by $\mathscr{A}^{\dagger}$. Inspired by a paper of Baksalary and Mitra [1], Liu, Benítez and Zhong generalized in [8] the left-star and the right-star partial order to $\mathscr{A}^{\dagger}$ in the following way.

For $a \in \mathscr{A}^{\dagger}$ let $a_l^{\pi} = 1 - a^{\dagger}a$ and $a_r^{\pi} = 1 - aa^{\dagger}$. Observe that $a_l^{\pi}$ and $a_r^{\pi}$ are self-adjoined idempotents. In [8] authors defined for $a, b \in \mathscr{A}^{\dagger}$ the following relations:

$$a * \leqslant b \quad \text{if and only if} \quad a^*a = a^*b \text{ and } b_r^{\pi}a = 0,$$

and

$$a \leqslant * b \quad \text{if and only if} \quad aa^* = ba^* \text{ and } ab_l^{\pi} = 0.$$

They stated that if $A$ and $B$ are complex $n \times n$ matrices, then $B_r^{\pi}A = 0$ if and only if $\operatorname{Im} A \subseteq \operatorname{Im} B$, and $AB_l^{\pi} = 0$ if and only if $\operatorname{Im} A^* \subseteq \operatorname{Im} B^*$. It can be easily proved that the same is also true for Moore-Penrose invertible elements $A, B \in B(H)$. Recall that an operator $A \in B(H)$ has a Moore-Penrose inverse if and only if its image is closed (see, for example, [13]), so on the set of operators from $B(H)$ with a closed image the above orders are equivalent respectively to the left-star and the right-star order presented in this paper.

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*Gregor Dolinar*
*Faculty of Electrical Engineering*
*University of Ljubljana*
*Tržaška 25*
*SI-1000 Ljubljana, Slovenia*
*e-mail:* gregor.dolinar@fe.uni-lj.si

*Alexander Guterman*
*Department of Higher Algebra*
*Faculty of Mathematics and Mechanics*
*Moscow State University*
*Moscow, 119991, GSP-1, Russia*
*e-mail:* guterman@list.ru

*Janko Marovt*
*Faculty of Economics and Business*
*University of Maribor*
*Razlagova 14*
*SI-2000 Maribor, Slovenia*
*e-mail:* janko.marovt@uni-mb.si