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WEIGHTED LIPSCHITZ ESTIMATES FOR COMMUTATORS OF ONE–SIDED OPERATORS ON ONE–SIDED TRIEBEL–LIZORKIN SPACES

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Abstract. Using the extrapolation of one-sided weights, we establish the boundedness of commutators generated by weighted Lipschitz functions and one-sided singular integral operators from weighted Lebesgue spaces to weighted one-sided Triebel-Lizorkin spaces. The corresponding results for commutators of one-sided discrete square functions are also obtained.

1. Introduction

The study of one-sided operators was motivated not only as the generalization of the theory of both-sided ones but also by the requirement in ergodic theory. In [22], Sawyer studied the weighted theory of one-sided maximal Hardy-Littlewood operators in depth for the first time. Since then, numerous papers have appeared, among which we choose to refer to [2], [3], [5], [14], [13], [16] about one-sided operators, [1], [15], [17], [21] about one-sided spaces and so on. Interestingly, lots of results show that for a class of smaller operators (one-sided operators) and a class of wider weights (one-sided weights), many famous results in harmonic analysis still hold.

Recently, Lorente and Riveros introduced the commutators of one-sided operators. In [10], they investigated the weighted boundedness for commutators generated by several one-sided operators (one-sided discrete square functions, one-sided fractional operators, one-sided maximal operators of a certain type) and BMO functions. Recall that a locally integrable function f is said to belong to $BMO(\mathbb{R})$ if

$$\|f\|_{BMO} = \sup_I \frac{1}{|I|}\int_I |f - f_I| < \infty,$$

where I denotes any bounded interval and $f_I = \frac{1}{|I|}\int_I f(y)dy$. In [11], [12], they obtained the weighted inequalities for commutators of a certain kind of one-sided operators (one-sided singular integrals and other one-sided operators appeared in [10]) and the weighted BMO functions.

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Very recently, Fu and Lu [4] introduced a class of one-sided Triebel-Lizorkin spaces and studied the boundedness for commutators (with symbol $b \in \mathrm{Lip}_\alpha$) of one-sided Calerón-Zygmund singular integral operators and one-sided fractional integral operators. A function $b \in \mathrm{Lip}_\alpha$, $0 < \alpha < 1$, if it satisfies

$$\|b\|_{\mathrm{Lip}_\alpha} = \sup_{x,h \in \mathbb{R}, h \neq 0} \frac{|b(x+h) - b(x)|}{|h|^\alpha} < \infty.$$

And it has the following equivalent form [18]:

$$\|f\|_{\mathrm{Lip}_\alpha} \approx \sup_I \frac{1}{|I|^{1+\alpha}} \int_I |f - f_I| \approx \sup_I \frac{1}{|I|^\alpha} \left(\frac{1}{|I|} \int_I |f - f_I|^q \right)^{\frac{1}{q}},$$

where $1 \leqslant q < \infty$. Obviously, if $\alpha = 0$, then $f \in BMO$. In fact, BMO and Lip_α are the special cases of Campanato spaces (cf. [19], [26]). It should be noted that just like functions in BMO may be unbounded, such as $\log|x|$, the functions in Lip_α are not necessarily bounded either, for example $|x|^\alpha \in \mathrm{Lip}_\alpha$. Therefore, it is also meaningful to investigate the commutators generated by operators and Lipschitz functions (cf. [7], [9], [18]).

Inspired by the above results, we concentrate on the boundedness for commutators (with symbol b belonging to weighted Lipschitz spaces) of one-sided singular operators as well as one-sided discrete square functions from weighted Lebesgue spaces to weighted one-sided Triebel-Lizorkin spaces.

In [18], Paluszyński introduced a kind of Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,\infty}$. Fu and Lu [4] gave their one-sided versions.

DEFINITION 1.1. [4] For $0 < \alpha < 1$ and $1 < p < \infty$, one-sided Triebel-Lizorkin spaces $\dot{F}_{p,+}^{\alpha,\infty}$ and $\dot{F}_{p,-}^{\alpha,\infty}$ are defined by

$$\|f\|_{\dot{F}_{p,+}^{\alpha,\infty}} \approx \left\| \sup_{h>0} \frac{1}{h^{1+\alpha}} \int_x^{x+h} |f - f_{[x,x+h]}| \right\|_{L^p} < \infty,$$

$$\|f\|_{\dot{F}_{p,-}^{\alpha,\infty}} \approx \left\| \sup_{h>0} \frac{1}{h^{1+\alpha}} \int_{x-h}^{x} |f - f_{[x-h,x]}| \right\|_{L^p} < \infty.$$

REMARK 1.1. It is clear that $\dot{F}_p^{\alpha,\infty} \subsetneqq \dot{F}_{p,+}^{\alpha,\infty}$, $\dot{F}_p^{\alpha,\infty} \subsetneqq \dot{F}_{p,-}^{\alpha,\infty}$, $\dot{F}_{p,+}^{\alpha,\infty} \cap \dot{F}_{p,-}^{\alpha,\infty} = \dot{F}_p^{\alpha,\infty}$.

Furthermore, the weighted one-sided Triebel-Lizorkin spaces have been defined in [4].

DEFINITION 1.2. For $0 < \alpha < 1$, $1 < p < \infty$ and an appropriate weight ω, the weighted one-sided Triebel-Lizorkin spaces $\dot{F}_{p,+}^{\alpha,\infty}(\omega)$ and $\dot{F}_{p,-}^{\alpha,\infty}(\omega)$ are defined by

$$\|f\|_{\dot{F}_{p,+}^{\alpha,\infty}(\omega)} \approx \left\| \sup_{h>0} \frac{1}{h^{1+\alpha}} \int_x^{x+h} |f - f_{[x,x+h]}| \right\|_{L^p(\omega)} < \infty,$$

and

$$\|f\|_{\dot{F}^{\alpha,\infty}_{p,-}(\omega)} \approx \left\| \sup_{h>0} \frac{1}{h^{1+\alpha}} \int_{x-h}^{x} |f - f_{[x-h,x]}| \right\|_{L^p(\omega)} < \infty.$$

One of the main objects of our study is the one-sided singular integral operator. Assume that $K \in L^1(\mathbb{R}\setminus\{0\})$, K is said to be a Calderón-Zygmund kernel if the following properties are satisfied:
(a) There exists a positive constant B_1 such that

$$\left| \int_{\varepsilon<|x|<N} K(x)dx \right| \leqslant B_1,$$

for all ε and N with $0<\varepsilon<N$, and the limit $\lim_{\varepsilon\to 0^+} \int_{\varepsilon<|x|<1} K(x)dx$ exists.
(b) There exists a positive constant B_2 such that

$$|K(x)| \leqslant \frac{B_2}{|x|},$$

for all $x \neq 0$.
(c) There exists a positive constant B_3 such that

$$|K(x-y) - K(x)| \leqslant \frac{B_3|y|}{|x|^2},$$

for any x and y with $|x| > 2|y|$.

The singular integral with Calderón-Zygmund kernel K is defined by

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}} K(x-y)f(y)dy.$$

DEFINITION 1.3. [2] A one-sided singular integral T^+ is a singular integral associated to a Calderón-Zygmund kernel K with support in $(-\infty,0)$:

$$T^+f(x) = \lim_{\varepsilon\to 0^+} \int_{x+\varepsilon}^{\infty} K(x-y)f(y)dy.$$

Similarly, when the support of K is in $(0,+\infty)$,

$$T^-f(x) = \lim_{\varepsilon\to 0^+} \int_{-\infty}^{x-\varepsilon} K(x-y)f(y)dy.$$

The other main object in this paper is the one-sided discrete square function. As it is known, the discrete square function is of interest in ergodic theory and has been extensively studied (cf. [8]).

DEFINITION 1.4. The one-sided discrete square function S^+ is defined by

$$S^+f(x) = \left(\sum_{n\in\mathbb{Z}} |A_nf(x) - A_{n-1}f(x)|^2 \right)^{\frac{1}{2}}.$$

for locally integrable f, where $A_nf(x) = \frac{1}{2^n}\int_x^{x+2^n} f(y)dy$.

It is easy to see that $S^+f(x) = \|U^+f(x)\|_{l^2}$, where U^+ is the sequence valued operator

$$U^+f(x) = \int_{\mathbb{R}} H(x-y)f(y)dy, \tag{1.1}$$

where

$$H(x) = \left\{ \frac{1}{2^n}\chi_{(-2^n,0)}(x) - \frac{1}{2^{n-1}}\chi_{(-2^{n-1},0)}(x) \right\}_{n\in\mathbb{Z}}.$$

(See [25]).

DEFINITION 1.5. The one-sided Hardy-Littlewood maximal operators M^+ and M^- are defined by

$$M^+f(x) = \sup_{h>0}\frac{1}{h}\int_x^{x+h}|f(y)|dy,$$

and

$$M^-f(x) = \sup_{h>0}\frac{1}{h}\int_{x-h}^{x}|f(y)|dy,$$

for locally integrable f.

The good weights for these operators are one-sided weights. Sawyer [22] introduced the one-sided A_p classes A_p^+, A_p^-, which are defined by the following conditions:

$$A_p^+: \quad \sup_{a<b<c}\frac{1}{(c-a)^p}\int_a^b w(x)\,dx\left(\int_b^c w(x)^{1-p'}\,dx\right)^{p-1} < \infty,$$

$$A_p^-: \quad \sup_{a<b<c}\frac{1}{(c-a)^p}\int_b^c w(x)\,dx\left(\int_a^b w(x)^{1-p'}\,dx\right)^{p-1} < \infty,$$

when $1 < p < \infty$; also, for $p = 1$,

$$A_1^+: \quad M^-w(x) \leqslant Cw(x), \quad a.e.,$$
$$A_1^-: \quad M^+w(x) \leqslant Cw(x), \quad a.e.$$

and

$$A_\infty^+: \quad A_\infty^+ = \bigcup_{p\geqslant 1} A_p^+.$$

Let's recall the definition of weighted Lipschitz spaces given in [6], also see [7].

DEFINITION 1.6. For $f \in L_{loc}(\mathbb{R})$, $\mu \in A_\infty$, $1 \leqslant p \leqslant \infty$, $0 < \beta < 1$, we say that f belongs to the weighted Lipschitz space $Lip^p_{\beta,\mu}$ if

$$\|f\|_{Lip^p_{\beta,\mu}} = \sup_I \frac{1}{\mu(I)^\beta}\left[\frac{1}{\mu(I)}\int_I |f(x)-f_I|^p\mu(x)^{1-p}dx\right]^{\frac{1}{p}} < \infty,$$

when $1 \leqslant p < \infty$, and

$$\|f\|_{Lip^{\infty}_{\beta,\mu}} = \sup_I \frac{1}{\mu(I)^{\beta}} \sup_{x\in I} |f(x) - f_I|\mu(x)^{-1} < \infty,$$

where I denotes any bounded interval and $f_I = \frac{1}{|I|}\int_I f$.

The weighted Lipschitz space $Lip^p_{\beta,\mu}$ is a Banach space (modulo constants). Set $Lip_{\beta,\mu} = Lip^1_{\beta,\mu}$, By [6], when $\mu \in A_1$, then the spaces $Lip^p_{\beta,\mu}$ coincide, and the norms $\|\cdot\|_{Lip^p_{\beta,\mu}}$ are equivalent for different p with $1 \leqslant p \leqslant \infty$, thus $\|\cdot\|_{Lip^p_{\beta,\mu}} \sim \|\cdot\|_{Lip_{\beta,\mu}}$ for any $1 \leqslant p \leqslant \infty$. It is clear that for $\mu \equiv 1$, the space $Lip_{\beta,\mu}$ is the classical Lipschitz space Lip_{β}. Therefore, weighted Lipschitz spaces are generalizations of the classical Lipschitz spaces.

DEFINITION 1.7. [10] For appropriate b, the commutators of T^+ and S^+ are defined by

$$T_b^+ f(x) = \int_x^{\infty} (b(x) - b(y))K(x-y)f(y)dy,$$

and

$$S_b^+ f(x) = \left\| \int_{\mathbb{R}} (b(x) - b(y))H(x-y)f(y)dy \right\|_{l^2},$$

respectively.

Now, we formulate our main results as follows.

THEOREM 1.1. *Assume that* $1 < p < \infty$, $v \in A_p$ *and* $w \in A_p^+$ *are such that* $\mu^{1+\alpha} = (\frac{v}{w})^{\frac{1}{p}}$ *for some* $0 < \alpha < 1$ *and* $\mu \in A_1$. *Then, for* $b \in Lip_{\alpha,\mu}$, *there exists* $C > 0$ *such that*

$$\|T_b^+ f\|_{\dot{F}^{\alpha,\infty}_{p,+}(w)} \leqslant C\|f\|_{L^p(v)},$$

for all bounded f *with compact support.*

THEOREM 1.2. *Assume that* $1 < p < \infty$, $v \in A_p$ *and* $w \in A_p^+$ *are such that* $\mu^{1+\alpha} = (\frac{v}{w})^{\frac{1}{p}}$ *for some* $0 < \alpha < 1 - \frac{1}{1+\varepsilon(w,v)}$ *and* $\mu \in A_1$. *Then, for* $b \in Lip_{\alpha,\mu}$, *there exists* $C > 0$ *such that*

$$\|S_b^+ f\|_{\dot{F}^{\alpha,\infty}_{p,+}(w)} \leqslant C\|f\|_{L^p(v)},$$

for all bounded f *with compact support.*

REMARK 1.2. In Theorem 1.2, $\varepsilon(w,v)$ is a positive number depending only on w,v. Since the condition that is satisfied by H in (1.1) is weaker than that of Calderón-Zygmund kernel K (see [25]), naturally, the requirement of α in Theorem 1.2 should be stronger.

We remark that like [11], [12], we will continue to use the one-sided maximal functions to control the commutators of the two operators in this paper. The difference is that, by definition of one-sided Triebel-Lizorkin spaces, the proof in this paper goes without using of one-sided sharp maximal operators.

In Section 2, we will give some necessary lemmas. Then we will prove Theorem 1.1 in Section 3. In the last section, we will give the proof of Theorem 1.2. Throughout this paper the letter C will be used to denote various constants, and the various uses of the letter do not, however, denote the same constant.

2. Preliminaries

In order to prove our results, we will first introduce some necessary lemmas.

LEMMA 2.1. [15] *Suppose that* $\omega \in A_1^-$, *then there exists* $\varepsilon_1 > 0$ *such that for all* $1 < r \leqslant 1+\varepsilon_1$, $w^r \in A_1^-$.

The primary tool in our proofs is an extrapolation theorem appeared in [12].

LEMMA 2.2. [12] *Let* ν *be a weight and* T *a sublinear operator defined in* $C_c^\infty(\mathbb{R})$ *and satisfying*

$$\|\tau T f\|_\infty \leqslant C\|\sigma f\|_\infty,$$

for all τ *and* σ *such that* $\sigma = \nu\tau$, $\tau^{-1} \in A_1^-$ *and* $\sigma^{-1} \in A_1$. *Then for* $1 < p < \infty$,

$$\|T f\|_{L^p(w)} \leqslant C\|f\|_{L^p(v)},$$

holds whenever $w \in A_p^+$ *and* $v = \nu^p w \in A_p$.

Based on Lemma 2.3 in [12], we get the following estimate which is essential to the proofs of Theorem 1.1 and 1.2.

LEMMA 2.3. *Let* $0 < \alpha < 1$, $\mu \in A_1$ *and* $b \in Lip_{\alpha,\mu}$. *Assume that* τ *and* $\sigma = \mu^{1+\alpha}\tau$ *are such that* $\tau^{-1} \in A_1^-$ *and* $\sigma^{-1} \in A_1$. *Then there exists* $\varepsilon_2 > 0$ *such that for all* $1 < r < 1+\varepsilon_2$,

$$\frac{1}{|I|^\alpha}\left(\frac{1}{|I|}\int_I |b(y)-b_I|^r \sigma^{-r} dy\right)^{1/r} \leqslant C\|b\|_{Lip_{\alpha,\mu}} \tau^{-1}(x), \quad a.e.\ x \in \mathbb{R}.$$

where $I = [x, x+h]$.

Proof. Since $\tau^{-1} \in A_1^-$, $\sigma^{-1} \in A_1$, by Lemma 2.1, there exists $\varepsilon_1 > 0$ such that for all $1 < r \leqslant 1+\varepsilon_1$, $\tau^{-r} \in A_1^-$, $\sigma^{-r} \in A_1$. By Theorem 4.4 (Page 272) in [24] and the fact that $\mu \in A_1$, we have

$$\frac{1}{|I|^\alpha}\left(\frac{1}{|I|}\int_I |b(y)-b_I|^r \sigma^{-r}(y) dy\right)^{1/r}$$

$$\leqslant C\frac{1}{|I|^\alpha}\left\{\frac{1}{|I|}\int_I \sup_{\substack{J\ni y\\ J\subset I}}\left(\frac{1}{|J|}\int_J |b(t)-b_J| dt\right)^r \sigma^{-r}(y) dy\right\}^{1/r}$$

$$\leqslant C\frac{1}{|I|^{\alpha}}\left\{\frac{1}{|I|}\int_I \sup_{\substack{J\ni y\\ J\subset I}}\left(\frac{\mu(J)}{|J|}\right)^{(1+\alpha)r}|J|^{\alpha r}\left(\frac{1}{\mu(J)^{1+\alpha}}\int_J|b(t)-b_J|dt\right)^r\sigma^{-r}(y)dy\right\}^{1/r}$$
$$\leqslant C\|b\|_{Lip_{\alpha,\mu}}\frac{1}{|I|^{\alpha}}\left\{\frac{1}{|I|}\int_I|I|^{\alpha r}\mu(y)^{(1+\alpha)r}\sigma^{-r}(y)dy\right\}^{1/r}$$
$$\leqslant C\|b\|_{Lip_{\alpha,\mu}}\left\{\frac{1}{|I|}\int_I\tau^{-r}(y)dy\right\}^{1/r}$$
$$\leqslant C\|b\|_{Lip_{\alpha,\mu}}\tau^{-1}(x),$$

for almost all $x\in\mathbb{R}$. □

LEMMA 2.4. *Assume that $b\in Lip_{\alpha,\mu}$, $\mu\in A_1$, $x\in\mathbb{R}$ and $h>0$. For each $j\in\mathbb{Z}^+$, let $I_j=[x,x+2^jh]$, $j\geqslant 3$. Then*

$$\frac{1}{h^{\alpha}}|b_{I_{j+1}}-b_{I_3}|\leqslant C\|b\|_{Lip_{\alpha,\mu}}\frac{2^{4\alpha}(1-2^{(j-2)\alpha})}{1-2^{\alpha}}\mu(x)^{1+\alpha}.$$

Proof. Since $\mu\in A_1$, we have

$$\frac{1}{h^{\alpha}}|b_{I_m}-b_{I_{m+1}}|\leqslant\frac{1}{h^{\alpha}}\frac{1}{|I_m|}\int_{I_m}|b(t)-b_{I_{m+1}}|dt\leqslant C2^{(m+1)\alpha}\left(\frac{\mu(I_{m+1})}{|I_{m+1}|}\right)^{1+\alpha}\|b\|_{Lip_{\alpha,\mu}}$$
$$\leqslant C2^{(m+1)\alpha}\|b\|_{Lip_{\alpha,\mu}}\mu(x)^{1+\alpha}.$$

Therefore,

$$\frac{1}{h^{\alpha}}|b_{j+1}-b_{I_3}|\leqslant\frac{1}{h^{\alpha}}\sum_{m=3}^{j}|b_{I_m}-b_{I_{m+1}}|\leqslant C\|b\|_{Lip_{\alpha,\mu}}\mu(x)^{1+\alpha}\sum_{m=3}^{j}2^{(m+1)\alpha}$$
$$\leqslant C\|b\|_{Lip_{\alpha,\mu}}\frac{2^{4\alpha}(1-2^{(j-2)\alpha})}{1-2^{\alpha}}\mu(x)^{1+\alpha}.$$

The lemma is proved. □

Using some notations of [10], [11] and [12], we will prove Theorem 1.1 and Theorem 1.2, respectively.

3. Weighted estimates for commutators of one-sided singular integrals

Proof of Theorem 1.1. Let λ be an arbitrary constant. Then

$$T_b^+f(x)=T^+((\lambda-b)f)(x)+(b(x)-\lambda)T^+f(x).$$

Let $x\in\mathbb{R}$, $h>0$, $J=[x,x+8h]$. Write $f=f_1+f_2$, where $f_1=f\chi_J$, set $\lambda=b_J$. Then

$$\frac{1}{h^{1+\alpha}}\int_x^{x+2h}|T_b^+f(y)-(T_b^+f)_{[x,x+2h]}|dy$$
$$\leqslant\frac{2}{h^{1+\alpha}}\int_x^{x+2h}|T_b^+f(y)-T^+((b-b_J)f_2)(x+2h)|dy$$

$$\begin{aligned}
&\leqslant \frac{2}{h^{1+\alpha}}\int_x^{x+2h}|T^+((b-b_J)f_1)(y)|dy\\
&\quad+\frac{2}{h^{1+\alpha}}\int_x^{x+2h}|T^+((b-b_J)f_2)(y)-T^+((b-b_J)f_2)(x+2h)|dy\\
&\quad+\frac{2}{h^{1+\alpha}}\int_x^{x+2h}|b(y)-b_J||T^+f(y)|dy\\
&=2(I(x)+II(x)+III(x)).
\end{aligned}$$

By definition of Calderón-Zygmund kernel, we have

$$II(x)\leqslant C\frac{1}{h^{1+\alpha}}\int_x^{x+2h}\int_{x+8h}^{\infty}\frac{x+2h-y}{(t-(x+2h))^2}|b(t)-b_J||f(t)|dtdy.$$

Consider the following three sublinear operators defined on C_c^∞:

$$\begin{aligned}
M_1^+f(x)&=\sup_{h>0}\frac{1}{h^{1+\alpha}}\int_x^{x+2h}|T^+((b-b_J)f\chi_J)(y)|dy,\\
M_2^+f(x)&=\sup_{h>0}\frac{1}{h^{1+\alpha}}\int_x^{x+2h}\int_{x+8h}^{\infty}\frac{x+2h-y}{(t-(x+2h))^2}|b(t)-b_J||f(t)|dtdy, \qquad (3.1)\\
M_3^+g(x)&=\sup_{h>0}\frac{1}{h^{1+\alpha}}\int_x^{x+2h}|b(y)-b_{[x,x+8h]}||g(y)|dy.
\end{aligned}$$

The above inequalities imply that

$$\begin{aligned}
&\frac{1}{h^{1+\alpha}}\int_x^{x+2h}|T_b^+f(y)-(T_b^+f)_{[x,x+2h]}|dy\\
&\quad\leqslant C\left(M_1^+f(x)+M_2^+f(x)+M_3^+(T^+f)(x)\right).
\end{aligned} \qquad (3.2)$$

Now, let's discuss the boundedness of these three operators. For M_1^+. Assume that τ and $\sigma=\mu^{1+\alpha}\tau$ are such that $\tau^{-1}\in A_1^-$ and $\sigma^{-1}\in A_1$. Let $1<r<1+\varepsilon_2$, where ε_2 is as in Lemma 2.3. By Hölder's inequality, Lemma 2.3 and the fact that T^+ is bounded from $L^r(\mathbb{R})$ to $L^r(\mathbb{R})$ [2], we get

$$\begin{aligned}
&\frac{1}{h^{1+\alpha}}\int_x^{x+2h}|T^+((b-b_J)f\chi_J)(y)|dy\\
&\leqslant\frac{C}{h^\alpha}\left(\frac{1}{h}\int_x^{x+2h}|T^+((b-b_J)f\chi_J)(y)|^rdy\right)^{1/r}\\
&\leqslant\frac{C}{h^\alpha}\left(\frac{1}{h}\int_x^{x+8h}|(b(y)-b_J)f(y)|^rdy\right)^{1/r}\\
&\leqslant C\|f\sigma\|_\infty\frac{1}{h^\alpha}\left(\frac{1}{h}\int_x^{x+8h}|b(y)-b_J|^r\sigma^{-r}(y)dy\right)^{1/r}\\
&\leqslant C\|b\|_{Lip_{\alpha,\mu}}\|f\sigma\|_\infty\tau^{-1}(x).
\end{aligned}$$

Therefore,

$$\|\tau M_1^+f\|_\infty\leqslant C\|f\sigma\|_\infty.$$

Then by Lemma 2.2, for $w \in A_p^+$ and $v = \mu^{(1+\alpha)p} w \in A_p$, we have

$$\|M_1^+ f\|_{L^p(w)} \leqslant C\|f\|_{L^p(v)}. \tag{3.3}$$

For M_2^+, let $I_j = [x, x+2^j h]$, $j \in \mathbb{Z}^+$. Then

$$\begin{aligned}
&\frac{1}{h^{1+\alpha}} \int_x^{x+2h} \int_{x+8h}^{\infty} \frac{x+2h-y}{(t-(x+2h))^2} |b(t)-b_J||f(t)|dtdy \\
&\leqslant \frac{C}{h^\alpha} \int_x^{x+2h} \sum_{j=3}^{\infty} \int_{x+2^j h}^{x+2^{j+1}h} \frac{|b(t)-b_J|}{(t-(x+2h))^2} |f(t)|dtdy \\
&\leqslant C \sum_{j=3}^{\infty} \frac{1}{(2^j-2)^2 h^{1+\alpha}} \int_{x+2^j h}^{x+2^{j+1}h} |b(t)-b_J||f(t)|dt \\
&\leqslant C \sum_{j=3}^{\infty} \frac{2^{j+1}}{(2^j-2)^2} \left(\frac{1}{2^{j+1}h^{1+\alpha}} \int_{I_{j+1}} |b(t)-b_{I_{j+1}}||f(t)|dt \right. \\
&\qquad \left. + \frac{1}{2^{j+1}h^{1+\alpha}} \int_{I_{j+1}} |b_{I_{j+1}}-b_J||f(t)|dt \right) \\
&= C \sum_{j=3}^{\infty} \frac{1}{2^j} \big(II_1(x) + II_2(x) \big).
\end{aligned} \tag{3.4}$$

By Hölder's inequality and Lemma 2.3, we have

$$\begin{aligned}
II_1(x) &= \frac{1}{2^{j+1}h^{1+\alpha}} \int_{I_{j+1}} |b(t)-b_{I_{j+1}}||f(t)|dt \\
&\leqslant \frac{1}{h^\alpha} \left(\frac{1}{2^{j+1}h} \int_{I_{j+1}} |b(t)-b_{I_{j+1}}|^r |f|^r dt \right)^{1/r} \\
&\leqslant \|f\sigma\|_\infty \frac{1}{h^\alpha} \left(\frac{1}{2^{j+1}h} \int_{I_{j+1}} |b(t)-b_{I_{j+1}}|^r \sigma^{-r} dt \right)^{1/r} \\
&\leqslant C 2^{(j+1)\alpha} \|b\|_{Lip_{\alpha,\mu}} \|f\sigma\|_\infty \tau^{-1}(x).
\end{aligned} \tag{3.5}$$

Since $\sigma^{-1} \in A_1$, then by Lemma 2.4,

$$\begin{aligned}
II_2(x) &= \frac{1}{2^{j+1}h^{1+\alpha}} \int_{I_{j+1}} |b_{I_{j+1}}-b_J||f(t)|dt \\
&\leqslant \frac{1}{h^\alpha} |b_{I_{j+1}}-b_J| \|f\sigma\|_\infty \frac{1}{|I_{j+1}|} \int_{I_{j+1}} \sigma^{-1} dt \\
&\leqslant C\|b\|_{Lip_{\alpha,\mu}} \frac{2^{4\alpha}(1-2^{(j-2)\alpha})}{1-2^\alpha} \mu(x)^{1+\alpha} \|f\sigma\|_\infty \sigma^{-1}(x) \\
&= C\|b\|_{Lip_{\alpha,\mu}} \frac{2^{4\alpha}(1-2^{(j-2)\alpha})}{1-2^\alpha} \|f\sigma\|_\infty \tau^{-1}(x).
\end{aligned} \tag{3.6}$$

Then (3.4)-(3.6) indicate that

$$\begin{aligned} &\frac{1}{h^{1+\alpha}}\int_x^{x+2h}\int_{x+8h}^{\infty}\frac{x+2h-y}{(t-(x+2h))^2}|b(t)-b_J||f(t)|dtdy \\ &\leqslant C\|b\|_{Lip_{\alpha,\mu}}\|f\sigma\|_\infty\tau^{-1}(x)\sum_{j=3}^{\infty}\frac{1}{2^j}\left(2^{(j+1)\alpha}+\frac{2^{4\alpha}(1-2^{(j-2)\alpha})}{1-2^\alpha}\right) \\ &\leqslant C\|b\|_{Lip_{\alpha,\mu}}\|f\sigma\|_\infty\tau^{-1}(x), \end{aligned} \tag{3.7}$$

where the last inequality is due to the fact that $0<\alpha<1$. Consequently,

$$\|\tau M_2^+ f\|_\infty \leqslant C\|f\sigma\|_\infty.$$

Then by Lemma 2.2, for $w\in A_p^+$ and $v=\mu^{(1+\alpha)p}w\in A_p$, we have

$$\|M_2^+ f\|_{L^p(w)} \leqslant C\|f\|_{L^p(v)}. \tag{3.8}$$

For M_3^+. By Hölder's inequality and Lemma 2.4, we get

$$\begin{aligned} &\frac{1}{h^{1+\alpha}}\int_x^{x+2h}|b(y)-b_J||g(y)|dy \\ &\leqslant \frac{C}{h^\alpha}\left(\frac{1}{h}\int_x^{x+2h}|b(y)-b_J|^r|g(y)|^r dy\right)^{1/r} \\ &\leqslant C\|g\sigma\|_\infty\frac{1}{h^\alpha}\left(\frac{1}{h}\int_x^{x+8h}|b(y)-b_J|^r\sigma^{-r}(y)dy\right)^{1/r} \\ &= C\|b\|_{Lip_{\alpha,\mu}}\|g\sigma\|_\infty\tau^{-1}(x). \end{aligned}$$

Thus,

$$\|\tau M_3^+ g\|_\infty \leqslant C\|g\sigma\|_\infty.$$

From Lemma 2.2, we get

$$\|M_3^+ g\|_{L^p(w)} \leqslant C\|g\|_{L^p(v)}, \tag{3.9}$$

where $w\in A_p^+$ and $v=\mu^{(1+\alpha)p}w\in A_p$. Since T^+ is bounded from $L^p(v)$ to $L^p(v)$ [2], it follows that

$$\|M_3^+(T^+ f)\|_{L^p(w)} \leqslant C\|T^+ f\|_{L^p(v)} \leqslant C\|f\|_{L^p(v)}. \tag{3.10}$$

Consequently, by (3.2), (3.3), (3.8) and (3.10), we obtain

$$\|T_b^+ f\|_{\dot F_{p,+}^{\alpha,\infty}(\omega)} \approx \left\|\sup_{h>0}\frac{1}{h^{1+\alpha}}\int_x^{x+h}|T_b^+ f-(T_b^+ f)_{[x,x+h]}|\right\|_{L^p(\omega)} \leqslant C\|f\|_{L^p(v)}.$$

This completes the proof of Theorem 1.1. □

4. Weighted estimates for commutators of one-sided discrete square functions

Proof of Theorem 1.2. The procedure of this proof is analogous to that of Theorem 1.1. Let λ be an arbitrary constant. Then

$$\begin{aligned} S_b^+ f(x) &= \left\| \int_{\mathbb{R}} (b(x)-b(y)) H(x-y) f(y) dy \right\|_{l^2} \\ &\leqslant \left\| (b(x)-\lambda) \int_{\mathbb{R}} H(x-y) f(y) dy \right\|_{l^2} + \left\| \int_{\mathbb{R}} H(x-y)(b(y)-\lambda) f(y) dy \right\|_{l^2} \\ &= |b(x)-\lambda| S^+ f(x) + S^+((b-\lambda)f)(x). \end{aligned}$$

Let $x \in \mathbb{R}$, $h > 0$ and let $j \in \mathbb{Z}$ be such that $2^j \leqslant h < 2^{j+1}$. Set $J = [x, x+2^{j+3}]$. Write $f = f_1 + f_2$, where $f_1 = f\chi_J$, set $\lambda = b_J$. Then

$$\begin{aligned} &\frac{1}{h^{1+\alpha}} \int_x^{x+2h} |S_b^+ f(y) - (S_b^+ f)_{[x,x+2h]}| dy \\ &\leqslant \frac{2}{h^{1+\alpha}} \int_x^{x+2h} |S_b^+ f(y) - S^+((b-b_J)f_2)(x)| dy \\ &\leqslant \frac{2}{h^{1+\alpha}} \int_x^{x+2h} |S^+((b-b_J)f_1)(y)| dy \\ &\quad + \frac{2}{h^{1+\alpha}} \int_x^{x+2h} |S^+((b-b_J)f_2)(y) - S^+((b-b_J)f_2)(x)| dy \\ &\quad + \frac{2}{h^{1+\alpha}} \int_x^{x+2h} |b(y)-b_J||S^+ f(y)| dy \\ &= 2(L(x) + LL(x) + LLL(x)). \end{aligned}$$

By definition, we have

$$\begin{aligned} LL(x) &\leqslant \frac{1}{h^{1+\alpha}} \int_x^{x+2^{j+2}} \|U^+((b-b_J)f_2)(y) - U^+((b-b_J)f_2)(x)\|_{l^2} \\ &\leqslant \frac{1}{h^{1+\alpha}} \int_x^{x+2^{j+2}} \int_{x+2^{j+3}}^{\infty} |(b(t)-b_J)f(t)| \|H(y-t) - H(x-t)\|_{l^2} dt dy. \end{aligned}$$

Define sublinear operators:

$$\begin{aligned} M_4^+ f(x) &= \sup_{j\in\mathbb{Z}} \frac{1}{2^{j(1+\alpha)}} \int_x^{x+2^{j+2}} |S^+((b-b_J)f\chi_J)(y)| dy, \\ M_5^+ f(x) &= \sup_{j\in\mathbb{Z}} \frac{1}{2^{j(1+\alpha)}} \int_x^{x+2^{j+2}} \int_{x+2^{j+3}}^{\infty} |(b(t)-b_J)f(t)| \|H(y-t) - H(x-t)\|_{l^2} dt dy. \end{aligned}$$

It follows that

$$\begin{aligned} &\frac{1}{h^{1+\alpha}} \int_x^{x+2h} |S_b^+ f(y) - (S_b^+ f)_{[x,x+2h]}| dy \\ &\leqslant C\left(M_4^+ f(x) + M_5^+ f(x) + M_3^+(S^+ f)(x)\right), \end{aligned} \tag{4.1}$$

where M_3^+ is defined in (3.1). It follows from (3.10) that

$$\|M_3^+(S^+f)\|_{L^p(w)} \leqslant C\|S^+f\|_{L^p(v)}.$$

By Theorem A in [25], we have

$$\|S^+f\|_{L^p(v)} \leqslant C\|f\|_{L^p(v)}.$$

Therefore,

$$\|M_3^+(S^+f)\|_{L^p(w)} \leqslant C\|f\|_{L^p(v)} \tag{4.2}$$

holds for $w \in A_p^+$ and $v = \mu^{(1+\alpha)p}w \in A_p$.

Next we shall prove that M_4^+, M_5^+ are all bounded from $L^p(v)$ to $L^p(w)$. For M_4^+. Assume that τ and $\sigma = \mu^{1+\alpha}\tau$ are such that $\tau^{-1} \in A_1^{-1}$ and $\sigma^{-1} \in A_1$. By Hölder's inequality, Lemma 2.3 and the fact that S^+ is bounded from $L^r(\mathbb{R})$ to $L^r(\mathbb{R})$ [25], we get

$$\begin{aligned}
&\frac{1}{2^{j(1+\alpha)}}\int_x^{x+2^{j+2}} |S^+((b-b_J)f\chi_J)(y)|dy \\
&\leqslant \frac{C}{2^{j\alpha}}\left(\frac{1}{2^j}\int_x^{x+2^{j+2}} |S^+((b-b_J)f\chi_J)(y)|^r dy\right)^{1/r} \\
&\leqslant \frac{C}{2^{j\alpha}}\left(\frac{1}{2^j}\int_x^{x+2^{j+3}} |(b(y)-b_J)f(y)|^r dy\right)^{1/r} \\
&\leqslant C\|f\sigma\|_\infty \frac{1}{2^{j\alpha}}\left(\frac{1}{2^j}\int_x^{x+2^{j+3}} |b(y)-b_J|^r \sigma^{-r}(y)dy\right)^{1/r} \\
&= C\|b\|_{Lip_{\alpha,\mu}}\|f\sigma\|_\infty \tau^{-1}(x).
\end{aligned}$$

Therefore,

$$\|\tau M_4^+ f\|_\infty \leqslant C\|f\sigma\|_\infty.$$

Then by Lemma 2.2, the inequality

$$\|M_4^+ f\|_{L^p(w)} \leqslant C\|f\|_{L^p(v)} \tag{4.3}$$

holds for $w \in A_p^+$ and $v = \mu^{(1+\alpha)p}w \in A_p$.

For M_5^+, let $I_j = [x, x+2^j]$, $j \in \mathbb{Z}$. Then

$$\begin{aligned}
&\int_{x+2^{j+3}}^\infty |(b(t)-b_J)f(t)|\|H(y-t)-H(x-t)\|_{l^2}dt \\
&\leqslant \sum_{k=j+3}^\infty \int_{x+2^k}^{x+2^{k+1}} |(b(t)-b_{I_{k+1}})f(t)|\|H(y-t)-H(x-t)\|_{l^2}dt \\
&\quad + \sum_{k=j+3}^\infty |b_{I_{k+1}}-b_J| \int_{x+2^k}^{x+2^{k+1}} |f(t)|\|H(y-t)-H(x-t)\|_{l^2}dt \\
&= LL_1(x) + LL_2(x).
\end{aligned} \tag{4.4}$$

Since τ and $\sigma = \mu^{1+\alpha}\tau = (\frac{v}{w})^{\frac{1}{p}}\tau$ are such that $\tau^{-1} \in A_1^-$ and $\sigma^{-1} \in A_1 \subset A_1^-$, by Lemma 2.1, there exists $\varepsilon > 0$ such that when $1 < r < 1+\varepsilon$, $\tau^{-r} \in A_1^-$ and $\sigma^{-r} \in A_1^-$. Since $\alpha < 1 - \frac{1}{1+\varepsilon}$, we can choose $r > 1$ such that $\alpha < \frac{1}{r'}$, then by Hölder's inequality and Lemma 2.3,

$$LL_1(x) \leqslant C \sum_{k=j+3}^{\infty} \left(\int_{I_{k+1}} |b(t) - b_{I_{k+1}}|^r |f(t)|^r dt \right)^{\frac{1}{r}} \times \left(\int_{x+2^k}^{x+2^{k+1}} \|H(y-t) - H(x-t)\|_{l^2}^{r'} dt \right)^{\frac{1}{r'}}.$$

By Theorem 1.6 in [25], for all $y \in [x, x+2^{j+3}]$, the kernel H satisfies

$$\left(\int_{x+2^k}^{x+2^{k+1}} \|H(y-t) - H(x-t)\|_{l^2}^{r'} dt \right)^{\frac{1}{r'}} \leqslant C \frac{2^{\frac{j}{r'}}}{2^k}. \tag{4.5}$$

Therefore

$$\begin{aligned} LL_1(x) &\leqslant C\|f\sigma\|_\infty \sum_{k=j+3}^{\infty} \frac{2^{\frac{j}{r'}}}{2^k} \left(\int_{I_{k+1}} |b(t) - b_{I_{k+1}}|^r \sigma^{-r}(t) dt \right)^{\frac{1}{r}} \\ &\leqslant C\|f\sigma\|_\infty \|b\|_{Lip_{\alpha,\mu}} \tau^{-1}(x) \sum_{k=j+3}^{\infty} \frac{2^{\frac{j}{r'}}}{2^k} |I_{k+1}|^{\alpha+\frac{1}{r}} \\ &\leqslant C\|f\sigma\|_\infty \|b\|_{Lip_{\alpha,\mu}} \tau^{-1}(x) \sum_{k=j+3}^{\infty} \frac{2^{\frac{j}{r'}}}{2^k} 2^{(k+1)(\alpha+\frac{1}{r})} \\ &\leqslant C2^{j\alpha} \|b\|_{Lip_{\alpha,\mu}} \|f\sigma\|_\infty \tau^{-1}(x). \end{aligned} \tag{4.6}$$

By the same proof as in Lemma 2.4 we can get that

$$|b_{I_{k+1}} - b_J| \leqslant \sum_{m=j+3}^{k} |b_{I_{m+1}} - b_{I_m}| \leqslant C(2^{j\alpha} + 2^{k\alpha}) \|b\|_{Lip_{\alpha,\mu}} \mu^{1+\alpha}(x)$$

Then by (4.5), Hölder's inequality and the fact that $\sigma^{-r} \in A_1^-$, $\alpha < \frac{1}{r'}$, we have

$$\begin{aligned} LL_2(x) &\leqslant C\|b\|_{Lip_{\alpha,\mu}} \mu^{1+\alpha}(x) \sum_{k=j+3}^{\infty} \frac{2^{\frac{j}{r'}}(2^{j\alpha} + 2^{k\alpha})}{2^k} \left(\int_{I_{k+1}} |f(t)|^r dt \right)^{\frac{1}{r}} \\ &\leqslant C\|b\|_{Lip_{\alpha,\mu}} \|f\sigma\|_\infty \mu^{1+\alpha}(x) \sum_{k=j+3}^{\infty} \frac{2^{\frac{j}{r'}}(2^{j\alpha} + 2^{k\alpha})}{2^k} \left(\int_{I_{k+1}} \sigma^{-r}(t) dt \right)^{\frac{1}{r}} \\ &\leqslant C2^{j\alpha} \|b\|_{Lip_{\alpha,\mu}} \|f\sigma\|_\infty \tau^{-1}(x). \end{aligned} \tag{4.7}$$

Following from (4.4), (4.6) and (4.7), we get

$$\int_{x+2^{j+3}}^{\infty} |(b(t) - b_J) f(t)| \|H(y-t) - H(x-t)\|_{l^2} dt \leqslant C2^{j\alpha} \|b\|_{Lip_{\alpha,\mu}} \|f\sigma\|_\infty \tau^{-1}(x).$$

Consequently,

$$\frac{1}{2^{j(1+\alpha)}}\int_x^{x+2^{j+2}}\int_{x+2^{j+3}}^{\infty}|(b(t)-b_J)f(t)|\|H(y-t)-H(x-t)\|_{l^2}dtdy$$
$$\leqslant C\|b\|_{Lip_{\alpha,\mu}}\|f\sigma\|_\infty\tau^{-1}(x).$$

Therefore,

$$\|\tau M_5^+ f\|_\infty \leqslant C\|f\sigma\|_\infty.$$

Then by Lemma 2.2, the inequality

$$\|M_5^+ f\|_{L^p(w)} \leqslant C\|f\|_{L^p(v)} \tag{4.8}$$

holds for $w\in A_p^+$ and $v=\mu^{(1+\alpha)p}w\in A_p$. Then Theorem 1.2 follows form (4.1)-(4.3) and (4.8). □

REMARK 4.1. It should be noted that by the well-known extrapolation theorem appeared in [23] and the similar estimate of Lemma 2.3, we can also obtain the corresponding boundedness for commutators generated by 'both-sided' singular integrals and weighted Lipschitz functions from weighted Lebesgue spaces to weighted 'both-sided' Triebel-Lizorkin spaces. We leave the completion of the proof to the interested readers.

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