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UPPER BOUND FOR SPECTRA OF JENSEN OPERATOR AND ITS APPLICATION TO REVERSE ARITHMETIC–GEOMETRIC MEANS

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Abstract. In this paper we consider Jensen's operator, which includes bounded self-adjoint operator on Hilbert space, and establish the optimal upper bound for Jensen's operator by means of discrete Jensen's functional. The obtained results are applied to operator means, then we get refinements of numerous reverse arithmetic-geometric operators mean inequalities on Hilbert space.

2. Introduction

Let H be a Hilbert space and let $\mathscr{B}_h(H)$ be the semi-space of all bounded self-adjoint operators on H. Besides, let $\mathscr{B}^+(H)$ denote the set of all positive operators in $\mathscr{B}_h(H)$. Throughout this paper, for $0 \leqslant \mu \leqslant 1$, $A,B \in \mathscr{B}^+(H)$, the following notations are defined:

$$A\nabla_\mu B = (1-\mu)A + \mu B, \qquad A\sharp_\mu B = A^{1/2}(A^{-1/2}BA^{-1/2})^\mu A^{1/2},$$

see F. Kubo and T. Ando [8]. When $\mu = 1/2$ we write $A\nabla B$ and $A\sharp B$, respectively. The Specht ratio [11] denoted by

$$S(t) = \frac{t^{\frac{1}{t-1}}}{e\log t^{\frac{1}{t-1}}} \quad \text{for} \quad t>0, t\neq 1; \quad \text{and} \quad S(1) = \lim_{t\to 1} S(t) = 1$$

has the following properties.

(i) $S(h) = S(1/h) \geqslant 1$ for $h > 0$.

(ii) $S(h)$ is a monotone increasing function on $(1,+\infty)$.

(iii) $S(h)$ is a monotone decreasing function on $(0,1)$.

We start from the famous arithmetic-geometric operator mean inequality:

$$A\nabla_\mu B \geqslant A\sharp_\mu B, \quad \mu \in [0,1].$$

And its reverse inequality was given in [12] with the Specht ratio as follows:

$$A\nabla_\mu B \leqslant S(h) A\sharp_\mu B, \quad \mu \in [0,1],$$

where $0 < aI_H \leqslant A,B \leqslant bI_H$ and $h = \frac{b}{a}$.

Afterward, an improvement of the reverse inequality was given in [3] as follows:

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Paper MIA-17-47

THEOREM F. *If* $0 < aI_H \leqslant A, B \leqslant bI_H$, $\mu \in [0,1]$, *then*

$$A\nabla_\mu B - 2r(A\nabla B - A\sharp B) \leqslant S(\sqrt{h})A\sharp_\mu B;$$

$$A\nabla_\mu B - A\sharp_\mu B - 2r(A\nabla B - A\sharp B) \leqslant \omega L(\sqrt{a},\sqrt{b})\log S(\sqrt{h})A.$$

where $r = \min\{\mu, 1-\mu\}$, $\omega = \max\{\sqrt{a},\sqrt{b}\}$, $L(a,b) = \frac{a-b}{\log a - \log b}$, $h = \frac{b}{a}$.

See [1, 2, 5, 6] for more related developments of the arithmetic-geometric means inequality.

Recently, M. Krnić et al. [7] introduced Jensen's operator, which includes bounded self-adjoint operator on Hilbert space, and established some bounds for spectra of Jensen's operator. The obtained results are then applied to operator means. In such a way, they get refinements and conversions of numerous mean inequalities for Hilbert space operators. But the reverse weighted arithmetic-geometric operator mean inequalities were not obtained.

In this paper we also consider Jensen's operator, and establish optimal upper bound for Jensen's operator by means of discrete Jensen's functional. Based on this, we obtain the reverse weighted arithmetic-geometric operator mean inequalities.

3. Upper bound for spectra of Jensen operator

In the last few years converses Jensen inequality have been largely investigated and many published literatures are related to this topic. Now the given lemma is a special case of the Pečarić et al.'s result and it is stated as follows.

LEMMA 1. [6,10] *If* f *is convex function on* $[a,b]$, $x_i \in [a,b]$, $i = 1,2,\cdots,n$, $\sum p_i = 1$, *then*

$$\sum p_i f(x_i) - f(\sum p_i x_i) \leqslant T_f(a,b), \tag{1}$$

where $T_f(a,b) = \max\{(1-\mu)f(a) + \mu f(b) - f((1-\mu)a + \mu b); \mu \in [0,1]\}$.

LEMMA 2. [6,12] *For* $0 < a < b$,

$$T_{-\log}(a,b) = \max\left\{\log\frac{(1-\mu)a+\mu b}{a^{1-\mu}b^{\mu}}; \mu \in [0,1]\right\} = \log S(b/a). \tag{2}$$

In particular,

$$T_{-\log}(a,1) = S(1/a) = S(a), \qquad T_{-\log}(1,b) = S(b),$$

and

$$\begin{aligned} T_{\exp}(\log a, \log b) &= \max\{(1-\mu)a + \mu b - a^{1-\mu}b^{\mu}; \mu \in [0,1]\} \\ &= L(a,b)\log S(b/a). \end{aligned} \tag{3}$$

Let $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be continuous convex function and $\mathscr{F}([a,b],\mathbb{R})$ denote the set of all continuous convex functions on $[a,b]$, I_H denote identity operator on Hilbert space. Now Jensen's operator [2]: $\mathscr{J} : \mathscr{F}([a,b],\mathbb{R}) \times \mathscr{B}_h(H) \times [a,b] \times \mathbb{R}_+^2 \to \mathscr{B}^+(H)$ is defined as

$$\mathscr{J}(f,D,\delta,\mathbf{p}) = p_1 f(D) + p_2 f(\delta) I_H - f(p_1 D + p_2 \delta I_H), \tag{4}$$

where $\mathbf{p} = (p_1,p_2) \in \mathbb{R}_+^2$ with $p_1 + p_2 = 1$, $aI_H \leqslant D \leqslant bI_H$ and $a < b$.

THEOREM 3. *If $\mathscr{J}$ is an operator defined by (4), then*

$$\mathscr{J}(f,D,\delta,\mathbf{p}) \leqslant T_f(a,b) I_H \leqslant \left[f(a) + f(b) - 2f\left(\frac{a+b}{2}\right)\right] I_H. \tag{5}$$

Proof. We consider Jensen's functional

$$\jmath(f,x,\delta,\mathbf{p}) = p_1 f(x) + p_2 f(\delta) - f(p_1 x + p_2 \delta)$$

as a function in variable x. Lemma 1 yields the following inequality:

$$\jmath(f,x,\delta,\mathbf{p}) \leqslant T_f(a,b). \tag{6}$$

Now if $D \in \mathscr{B}_h(H)$ satisfies $aI_H \leqslant D \leqslant bI_H$, then, according to monotonicity property for operator function, we also insert D in the above inequality. As a result we get relation

$$\mathscr{J}(f,D,\delta,\mathbf{p}) \leqslant T_f(a,b) I_H.$$

Since $f : [a,b] \to \mathbb{R}$ is a convex function, then

$$\begin{aligned}
&\mu f(a) + (1-\mu) f(b) - f(\mu a + (1-\mu) b) \\
&\leqslant \max\{\mu, 1-\mu\} \left[f(a) + f(b) - 2f\left(\frac{a+b}{2}\right)\right] \\
&\leqslant f(a) + f(b) - 2f\left(\frac{a+b}{2}\right).
\end{aligned}$$

The proof is completed. □

4. Application to reverse A-G mean inequality

COROLLARY 4. *If $0 < aA \leqslant B \leqslant bA$ for some $a < 1 < b$, then*

$$A \nabla_\mu B \leqslant \max\{S(a), S(b)\} A \sharp_\mu B \quad \text{for} \quad 0 \leqslant \mu \leqslant 1. \tag{7}$$

Proof. First of all, we note that

$$T_{-\log}(a,1) = \log S(a) \text{ and } T_{-\log}(1,b) = \log S(b)$$

by Lemma 2. Let $D = A^{-1/2}BA^{-1/2}$ and $F = E([a,1))$ where $E(\cdot)$ is the family of spectral projections of D. Put $H_1 = FH$, $H_2 = (I_H - F)H = E([1,b])H$ and

$$D = D_1 \oplus D_2 \quad \text{on} \quad H_1 \oplus H_2.$$

By the assumption, we have $aI_H \leqslant D \leqslant bI_H$ and moreover

$$\text{(i) } aI_{H_1} \leqslant D_1 \leqslant I_{H_1} \quad \text{and} \quad \text{(ii) } I_{H_2} \leqslant D_2 \leqslant bI_{H_2}.$$

We now apply Theorem 3 for $f(x) = -\log x$, $\delta = 1$ and $p_1 = \mu$ for (i) and (ii) respectively.

The case (i): It follows from Lemma 2 that

$$\log((\mu D_1 + (1-\mu)I_{H_1})D_1^{-\mu}) \leqslant T_{-\log}(a,1)I_{H_1} = \log S(a)I_{H_1},$$

so that

$$(\mu D_1 + (1-\mu)I_{H_1})D_1^{-\mu} \leqslant S(a)I_{H_1}.$$

The case (ii): It follows from Lemma 2 that

$$\log((\mu D_2 + (1-\mu)I_{H_2})D_2^{-\mu}) \leqslant \log S(b)I_{H_2},$$

so that

$$(\mu D_2 + (1-\mu)I_{H_2})D_2^{-\mu} \leqslant S(b)I_{H_2}.$$

Combining with (i) and (ii) in above, we have

$$(\mu D + (1-\mu)I_H)D^{-\mu} \leqslant S(a)I_{H_1} \oplus S(b)I_{H_2} \leqslant \max\{S(a), S(b)\}I_H.$$

Hence it implies that

$$(\mu D + (1-\mu)I_H) \leqslant \max\{S(a), S(b)\}D^{\mu}.$$

Finally, multiplying $A^{1/2}$ on both sides, we have the desired inequality

$$A\nabla_\mu B \leqslant \max\{S(a), S(b)\}A\sharp_\mu B. \quad \square$$

COROLLARY 5. [12] *If* $0 < aI_H \leqslant A, B \leqslant bI_H$ *for some* $a < b$, *then*

$$A\nabla_\mu B \leqslant S(b/a)A\sharp_\mu B \quad \textit{for} \quad 0 \leqslant \mu \leqslant 1. \tag{8}$$

Proof. Let $D = A^{-1/2}BA^{-1/2}$, then $0 \leqslant \frac{a}{b} \leqslant D \leqslant \frac{b}{a}$. By the same way as in the proof of Corollary 4 and $S(b/a) = S(a/b)$, we get the desired inequality. $\square$

COROLLARY 6. *The assumption as in Corollary 4. Then*

$$A\nabla_\mu B - A\sharp_\mu B \leqslant \max\{L(a,1)\log S(a), L(1,b)\log S(b)\}A, \tag{9}$$

where $L(a,b)$ *is defined as in Theorem F.*

Proof. Let $D = D_1 \oplus D_2$ on $H_1 \oplus H_2$ be as in the proof of Corollary 4. Then it is easily seen that $aI_H \leqslant D \leqslant bI_H$, $aI_{H_1} \leqslant D_1 \leqslant 0$, $0 \leqslant D_2 \leqslant bI_{H_2}$ and

$$T(\exp, \log D_i, 0, (\mu, 1-\mu)) = \mu D_i + (1-\mu)I_{H_i} - D_i^{\mu} \ (i = 1, 2).$$

Applying Theorem 3, we have

$$\mu D_1 + (1-\mu)I_{H_1} - D_1^{\mu} \leqslant L(a,1)\log S(a)I_{H_1}$$

and

$$\mu D_2 + (1-\mu)I_{H_2} - D_2^{\mu} \leqslant L(1,b)\log S(b)I_{H_2}.$$

Combining with them and multiplying $A^{1/2}$ on its both sides, we have the required inequality. □

COROLLARY 7. [12] *If* $0 < aI_H \leqslant A, B \leqslant bI_H$ *for some* $a < b$, $0 \leqslant \mu \leqslant 1$, *then*

$$A\nabla_{\mu}B - A\sharp_{\mu}B \leqslant L(1,h)\log S(h)A, \tag{10}$$

where L(1,h) is defined as in Theorem F, $h = b/a$.

Proof. Let $D = \log(A^{-1/2}BA^{-1/2})$, then $-\log h \leqslant D \leqslant \log h$. By the same way as in the proof of Corollary 6 and using $L(1,1/h) \leqslant L(1,h)$, $S(1/h) = S(h)$, we get the desired operator inequality. □

THEOREM 8. *If* $0 < aA \leqslant B \leqslant bA$ *for some* $a < 1 < b$, $0 \leqslant \mu \leqslant 1$, *then*

$$A\nabla_{\mu}B - 2r(A\nabla B - A\sharp B) \leqslant \max\{S(\sqrt{a}), S(\sqrt{b})\}A\sharp_{\mu}B, \tag{11}$$

where $r = \min\{\mu, 1-\mu\}$.

Proof. If $0 \leqslant \mu \leqslant \frac{1}{2}$, then $0 \leqslant 2\mu \leqslant 1$. Since $0 < aA \leqslant B \leqslant bA$ admits that $\sqrt{a}A \leqslant A\sharp B \leqslant \sqrt{b}A$, we substitute B by $A\sharp B$ and μ by 2μ in (7) of Corollary 4, then

$$A\nabla_{2\mu}(A\sharp B) \leqslant \max\{S(\sqrt{a}), S(\sqrt{b})\}A\sharp_{2\mu}(A\sharp B),$$

equivalently,

$$A\nabla_{\mu}B - 2\mu(A\nabla B - A\sharp B) \leqslant \max\{S(\sqrt{a}), S(\sqrt{b})\}A\sharp_{\mu}B. \tag{12}$$

If $\frac{1}{2} \leqslant \mu \leqslant 1$, then $0 \leqslant 1-\mu \leqslant \frac{1}{2}$. $0 < aA \leqslant B \leqslant bA$ admits $0 < \frac{1}{b}B \leqslant A \leqslant \frac{1}{a}B$, then, by the inequality (12), we have

$$B\nabla_{1-\mu}A - 2(1-\mu)(B\nabla A - B\sharp A) \leqslant \max\{S(\sqrt{1/b}), S(\sqrt{1/a})\}B\sharp_{1-\mu}A,$$

then, by $S(\sqrt{1/a}) = S(\sqrt{a})$ and $S(\sqrt{1/b}) = S(\sqrt{b})$, we have

$$A\nabla_{\mu}B - 2(1-\mu)(A\nabla B - A\sharp B) \leqslant \max\{S(\sqrt{a}), S(\sqrt{b})\}A\sharp_{\mu}B. \tag{13}$$

Therefore, for $0 \leqslant \mu \leqslant 1$, we have

$$A\nabla_{\mu}B - 2r(A\nabla B - A\sharp B) \leqslant \max\{S(\sqrt{a}), S(\sqrt{b})\}A\sharp_{\mu}B,$$

where $r = \min\{\mu, 1-\mu\}$. □

THEOREM 9. [3] *If* $0 < aI_H \leqslant A, B \leqslant bI_H$ *for some* $a < b$, $0 \leqslant \mu \leqslant 1$, *then*

$$A\nabla_\mu B - 2r(A\nabla B - A\sharp B) \leqslant S(\sqrt{b/a})A\sharp_\mu B, \tag{14}$$

where $r = \min\{\mu, 1-\mu\}$.

Proof. From the assumption $0 < aI_H \leqslant A, B \leqslant bI_H$, we have

$$\sqrt{a/b}A \leqslant A\sharp B \leqslant \sqrt{b/a}A, \quad \sqrt{a/b}B \leqslant B\sharp A \leqslant \sqrt{b/a}B,$$

If $0 \leqslant \mu \leqslant \frac{1}{2}$, by $\sqrt{a/b}A \leqslant A\sharp B \leqslant \sqrt{b/a}A$, we substitute B by $A\sharp B$ and μ by 2μ in the inequality (7) of Corollary 4, then

$$A\nabla_{2\mu}(A\sharp B) \leqslant \max\{S(\sqrt{b/a}), S(\sqrt{a/b})\}A\sharp_{2\mu}(A\sharp B),$$

using $S(\sqrt{b/a}) = S(\sqrt{a/b})$, we have

$$A\nabla_\mu B - 2\mu(A\nabla B - A\sharp B) \leqslant S(\sqrt{b/a})A\sharp_\mu B. \tag{15}$$

If $\frac{1}{2} \leqslant \mu \leqslant 1$, then $0 \leqslant 1-\mu \leqslant \frac{1}{2}$. By the inequality (15) and $\sqrt{a/b}B \leqslant B\sharp A \leqslant \sqrt{b/a}B$, we have

$$B\nabla_{1-\mu}A - 2(1-\mu)(B\nabla A - B\sharp A) \leqslant S(\sqrt{b/a})B\sharp_{1-\mu}A.$$

Or, equivalently,

$$A\nabla_\mu B - 2(1-\mu)(A\nabla B - A\sharp B) \leqslant S(\sqrt{b/a})A\sharp_\mu B. \tag{16}$$

Therefore,

$$A\nabla_\mu B - 2r(A\nabla B - A\sharp B) \leqslant S(\sqrt{b/a})A\sharp_\mu B, \tag{17}$$

where $r = \min\{\mu, 1-\mu\}$. □

THEOREM 10. *If* $0 < aA \leqslant B \leqslant bA$ *for some* $a < 1 < b$, $0 \leqslant \mu \leqslant 1$, *then*

$$\begin{aligned} &A\nabla_\mu B - A\sharp_\mu B - 2r(A\nabla B - A\sharp B) \\ &\leqslant \max\{L(1,\sqrt{1/a})\log S(\sqrt{a}), L(1,\sqrt{1/b})\log S(\sqrt{b})\}bA, \end{aligned}$$

where $r = \min\{\mu, 1-\mu\}$.

Proof. If $0 \leqslant \mu \leqslant \frac{1}{2}$, since $\sqrt{a}A \leqslant A\sharp B \leqslant \sqrt{b}A$, then, substitute B by $A\sharp B$ and μ by 2μ in (9) of Corollary 6,

$$\begin{aligned} A\nabla_{2\mu}(A\sharp B) - A\sharp_{2\mu}(A\sharp B) &= A\nabla_\mu B - A\sharp_\mu B - 2\mu(A\nabla B - A\sharp B) \\ &\leqslant \max\{L(1,\sqrt{a})\log S(\sqrt{a}), L(1,\sqrt{b})\log S(\sqrt{b})\}A. \end{aligned} \tag{18}$$

If $\frac{1}{2} \leqslant \mu \leqslant 1$, then $0 \leqslant 1-\mu \leqslant \frac{1}{2}$. By $\frac{1}{b}B \leqslant A \leqslant \frac{1}{a}B$ and the inequality (18), then

$$\begin{aligned}&B\nabla_{1-\mu}A - B\sharp_{1-\mu}A - 2(1-\mu)(B\nabla A - B\sharp A)\\ &\leqslant \max\{L(1,\sqrt{1/a})\log S(\sqrt{a}), L(1,\sqrt{1/b})\log S(\sqrt{b})\}B,\end{aligned}$$

by $B \leqslant bA$, we have

$$\begin{aligned}&A\nabla_{\mu}B - A\sharp_{\mu}B - 2(1-\mu)(A\nabla B - A\sharp B)\\ &\leqslant \max\{L(1,\sqrt{1/a})\log S(\sqrt{a}), L(1,\sqrt{1/b})\log S(\sqrt{b})\}bA. \qquad (19)\end{aligned}$$

Therefore, for $0 \leqslant \mu \leqslant 1$, we have

$$\begin{aligned}&A\nabla_{\mu}B - A\sharp_{\mu}B - 2r(A\nabla B - A\sharp B)\\ &\leqslant \max\{L(1,\sqrt{1/a})\log S(\sqrt{a}), L(1,\sqrt{1/b})\log S(\sqrt{b})\}bA,\end{aligned}$$

where $r = \min\{\mu, 1-\mu\}$. □

THEOREM 11. [3] *If* $0 < aI_H \leqslant A, B \leqslant bI_H$ *for some* $a < b$, $0 \leqslant \mu \leqslant 1$, *then*

$$A\nabla_{\mu}B - A\sharp_{\mu}B - 2r(A\nabla B - A\sharp B) \leqslant bL(1,\sqrt{h})\log S(\sqrt{h}), \qquad (20)$$

where $r = \min\{\mu, 1-\mu\}, h = b/a$.

Proof. From the assumption $0 < aI_H \leqslant A, B \leqslant bI_H$, we have

$$1/\sqrt{h}A \leqslant A\sharp B \leqslant \sqrt{h}A, \quad 1/\sqrt{h}B \leqslant B\sharp A \leqslant \sqrt{h}B.$$

If $0 \leqslant \mu \leqslant \frac{1}{2}$, then, substitute B by $A\sharp B$ and μ by 2μ in (9) of Corollary 6, and using $L(1/\sqrt{h},1) \leqslant L(1,\sqrt{h})$ and $S(1/\sqrt{h}) = S(\sqrt{h})$, we have

$$\begin{aligned}&A\nabla_{2\mu}(A\sharp B) - A\sharp_{2\mu}(A\sharp B) = A\nabla_{\mu}B - A\sharp_{\mu}B - 2\mu(A\nabla B - A\sharp B)\\ &\leqslant L(1,\sqrt{h})\log S(\sqrt{h})A \leqslant bL(1,\sqrt{h})\log S(\sqrt{h}). \qquad (21)\end{aligned}$$

If $\frac{1}{2} \leqslant \mu \leqslant 1$, then $0 \leqslant 1-\mu \leqslant \frac{1}{2}$. By the inequality (21), we have

$$B\nabla_{1-\mu}A - B\sharp_{1-\mu}A - 2(1-\mu)(B\nabla A - B\sharp A) \leqslant L(1,\sqrt{h})\log S(\sqrt{h})B,$$

then

$$A\nabla_{\mu}B - A\sharp_{\mu}B - 2(1-\mu)(A\nabla B - A\sharp B) \leqslant bL(1,\sqrt{h})\log S(\sqrt{h}). \qquad (22)$$

Therefore, we have

$$A\nabla_{\mu}B - A\sharp_{\mu}B - 2r(A\nabla B - A\sharp B) \leqslant bL(1,\sqrt{h})\log S(\sqrt{h})$$

for $0 \leqslant \mu \leqslant 1$, where $r = \min\{\mu, 1-\mu\}$. □

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