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ON SOME NEW WEIGHTED EULER SEQUENCE SPACES AND COMPACT OPERATORS

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Abstract. In this paper, we define some new Euler sequence spaces and construct Schauder basis of these spaces. Moreover, we determine their $\beta-$duals and characterize some related matrix classes. Finally, we give the characterization of some classes of compact operators on these spaces by using the Hausdorff measure of noncompactness.

1. Introduction

Let ω be the real linear space of all real sequences under the natural algebraic operations. Any vector subspace of ω is called a *sequence space*. We write ℓ_∞, c and c_0 for the spaces of all bounded, convergent, null sequences, respectively. Also, by cs, ℓ_1 and ℓ_p, we denote the spaces of all sequences associated with convergent, absolutely and $p-$absolutely convergent series, respectively; where $1 < p < \infty$. Moreover, we shall write ϕ for the set of all finite sequences that terminate in zeros, $e = (1,1,...)$ and $e^{(n)}$ for the sequence whose only non-zero term is 1 in the $n-$th place for each $n \in \mathbb{N} = \{0,1,2,...\}$.

Let $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of real numbers. We write A_n for the sequence in the $n-$th row of A, that is $A_n = (a_{nk})_{k=0}^{\infty}$ for every $n \in \mathbb{N}$. In addition, if $x = (x_k) \in \omega$ then we define the $A-$transform of x as the sequence $Ax = (A_n(x))_{n\in\mathbb{N}}$, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k \quad (n \in \mathbb{N}) \tag{1}$$

provided the series on the right side converges for each $n \in \mathbb{N}$. For any subsets $X, Y \subset \omega$ we denote by (X,Y) the class of all matrices A such that $A : X \to Y$. Thus $A \in (X,Y)$ if and only if $Ax = (A_n(x))_{n\in\mathbb{N}} \in Y$ for all $x \in X$. The matrix domain of an infinite matrix A in X is defined by

$$X_A = \{x = (x_k) \in \omega : Ax \in X\}. \tag{2}$$

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The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has recently been employed by several authors, see for instance [1], [10], [15], [17], [29].

Let $0<r<1$ and let $E^r=\left(e^r_{nk}\right)$ be the Euler matrix of order r, defined by

$$e^r_{nk}=\begin{cases}\binom{n}{k}(1-r)^{n-k}r^k, & 0\leqslant k\leqslant n\\ 0, & k>n\end{cases}.$$

In the literature, the matrix domain X_{E^r} is called the Euler sequence space whenever X is a normed or paranormed sequence space. The Euler sequence spaces have been studied by many authors as can be seen in [2], [3], [9], [11–14], [19], [31], [32].

In the present paper, we introduce some new Euler sequence spaces. The paper is organized as follows:

In Section 2 we give some notations used in this paper. The sequence spaces $e^\theta_{w,0}$, $e^\theta_{w,c}$ and $e^\theta_{w,\infty}$ are defined in Section 3. Also, we prove that these spaces are linearly isomorphic to the spaces c_0, c, ℓ_∞, respectively, and construct the bases of the spaces $e^\theta_{w,0}$ and $e^\theta_{w,c}$. In Section 4 we determine the $\beta-$duals of the sequence spaces $e^\theta_{w,0}$, $e^\theta_{w,c}$, $e^\theta_{w,\infty}$ and characterize some related matrix classes. Finally, we examine some classes of compact operators on the spaces $e^\theta_{w,0}$ and $e^\theta_{w,\infty}$ by applying the Hausdorff measure of noncompactness.

2. Notations and preliminaries

A sequence space X is called a *FK space* if it is a complete linear metric space with continuous coordinates $p_n: X\to\mathbb{R}$ $(n\in\mathbb{N})$, where $\mathbb{R}$ denotes the real field and $p_n(x)=x_n$ for all $x=(x_k)\in X$ and every $n\in\mathbb{N}$. A BK space is a normed FK space, that is, a BK space is a Banach space with continuous coordinates. The sequence spaces ℓ_∞, c, c_0 are BK spaces with $\|x\|_{\ell_\infty}=\sup_k|x_k|$. Further, the space ℓ_p is a BK space with the usual norm defined by $\|x\|_{\ell_p}=\left(\sum_{k=0}^\infty|x_k|^p\right)^{1/p}$, where $1\leqslant p<\infty$.

A sequence $(b^{(n)})_{n=0}^\infty$ in a linear metric space X is called *Schauder basis* if for every $x\in X$ there is a unique sequence (α_n) of scalars such that $x=\sum_{k=0}^\infty \alpha_n b^{(n)}$.

Let S_X denote the unit sphere in a normed linear space X, that is, $S_X=\{x\in X: \|x\|=1\}$. Let X and Y be Banach spaces then $B(X,Y)$ of all bounded linear operators from X to Y is a Banach space with the operator norm defined by $\|L\|=\sup\{\|L(x)\|: \|x\|\leqslant 1\}$ for all $L\in B(X,Y)$.

If $(X,\|.\|)$ is a normed sequence space, then we write

$$\|a\|^*_X=\sup_{x\in S_X}\left|\sum_{k=0}^\infty a_kx_k\right|, \tag{3}$$

for $a=(a_k)\in\omega$ provided the expression on the right side exists and is finite which is the case whenever X is a BK space and $a\in X^\beta$, where

$$X^\beta=\{a=(a_k)\in\omega: ax=(a_kx_k)\in cs \text{ for all } x=(x_k)\in X\}$$

is the β-dual of X.

By M_X we denote the collection of all bounded subsets of a metric space (X,d). If $Q \in M_X$, then the *Hausdorff measure of noncompactness of the set* Q, denoted by $\chi(Q)$, is defined by

$$\chi(Q) := \inf\left\{ \varepsilon > 0 : Q \subset \bigcup_{i=1}^{n} B(x_i, r_i),\ x_i \in X,\ r_i < \varepsilon\ (i = 1,2,\ldots,n),\ n \in \mathbb{N}\backslash\{0\} \right\}.$$

The function $\chi : M_X \to [0,\infty)$ is called the *Hausdorff measure of noncompactness*.

The basic properties of the Hausdorff measure of noncompactness can be found in [16].

Let X and Y be Banach spaces. Then, a linear operator $L : X \to Y$ is said to be *compact* if the domain of L is all of X and $L(Q)$ is a totally bounded subset of Y for every $Q \in M_X$. Equivalently, we say that L is compact if its domain is all of X and for every bounded sequence (x_n) in X, the sequence (Lx_n) has a convergent subsequence in Y.

3. The weighted Euler sequence spaces $e^{\theta}_{w,0}$, $e^{\theta}_{w,c}$ and $e^{\theta}_{w,\infty}$

Recently, Demiriz and Çakan [9] have introduced the Euler sequence space $e^r(u,p)$ as follows:

$$e^r(u,p) = \left\{ x = (x_k) \in \omega : \sum_{n=0}^{\infty} \left| \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k u_k x_k \right|^{p_n} < \infty \right\} \ (0 < r < 1),$$

where $u = (u_k)$ is a arbitrary sequence and $p = (p_k)$ is a bounded sequence of strictly positive real numbers. It is obvious that $e^r(u,p)$ space is a generalization of the Euler spaces $e^r(p)$ and e^r_p defined by Kara et al. [12] and Altay et al. [3], respectively.

Quite recently, Lashkaripour and Talebi [14] have defined the Euler weighted sequence space $e^{\theta}_{w,p}$ as below:

$$e^{\theta}_{w,p} = \left\{ x = (x_k) \in \omega : \sum_{n=0}^{\infty} w_n \left| \sum_{k=0}^{n} \binom{n}{k} (1-\theta)^{n-k} \theta^k x_k \right|^{p} < \infty \right\},$$

where $0 < p < 1$, $0 < \theta \leqslant 1$ and $w = (w_n)$ is a decreasing non-negative sequence of real numbers.

Now, following Demiriz and Çakan [9] and Lashkaripour and Talebi [14], we introduce the Euler weighted sequence spaces $e^{\theta}_{w,0}$, $e^{\theta}_{w,c}$ and $e^{\theta}_{w,\infty}$ as follows:

$$e^{\theta}_{w,0} = \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} \left(w_n \sum_{k=0}^{n} \binom{n}{k} (1-\theta)^{n-k} \theta^k x_k \right) = 0 \right\},$$

$$e^{\theta}_{w,c} = \left\{ x = (x_k) \in \omega : \exists l \in \mathbb{R} \text{ such that } x - le \in e^{\theta}_{w,0} \right\}.$$

and

$$e^{\theta}_{w,\infty}=\left\{x=(x_k)\in\omega:\sup_n\left|w_n\sum_{k=0}^{n}\binom{n}{k}(1-\theta)^{n-k}\theta^k x_k\right|<\infty\right\},$$

where $0<\theta<1$ and $w_n\neq 0$ for all $n\in\mathbb{N}$ and $x-le=(x_k-l)_{k\in\mathbb{N}}$.

With the notation of (2), we can redefine the spaces $e^{\theta}_{w,0}$, $e^{\theta}_{w,c}$ and $e^{\theta}_{w,\infty}$ by

$$e^{\theta}_{w,0}=(c_0)_{E^{\theta,w}},\quad e^{\theta}_{w,c}=(c)_{E^{\theta,w}}\ \text{and}\ e^{\theta}_{w,\infty}=(\ell_\infty)_{E^{\theta,w}}, \tag{4}$$

where $E^{\theta,w}=(e^{\theta,w}_{nk})$ denotes the weighted Euler matrix, i.e,

$$e^{\theta,w}_{nk}=\begin{cases} w_n\binom{n}{k}(1-\theta)^{n-k}\theta^k, & 0\leqslant k\leqslant n\\ 0, & k>n\end{cases}$$

for all $k,n\in\mathbb{N}$.

For any sequence $x=(x_k)$, we define the sequence $y=(y_k)$, which will be frequently used, as the $E^{\theta,w}$-transform of x, i.e., $y=E^{\theta,w}x$ (the associated sequence) and so we have that

$$y_k=w_k\sum_{j=0}^{k}\binom{k}{j}(1-\theta)^{k-j}\theta^j x_j\ (k\in\mathbb{N}). \tag{5}$$

The following theorem is essential in the sequel.

THEOREM 1. *Let λ be any of the sequence spaces $e^{\theta}_{w,0}$, $e^{\theta}_{w,c}$ or $e^{\theta}_{w,\infty}$. Then, λ is a BK-space with the norm $\|x\|_\lambda=\left\|E^{\theta,w}x\right\|_{\ell_\infty}$, that is*

$$\|x\|_\lambda=\sup_n\left|(E^{\theta,w}x)_n\right|. \tag{6}$$

Proof. Since (4) holds and c_0, c and ℓ_∞ are BK-spaces with respect to their natural norms and the matrix $E^{\theta,w}$ is a triangle, Theorem 4.3.2 of Wilansky [34, p.63] gives the fact that any $\lambda\in\{e^{\theta}_{w,0},e^{\theta}_{w,c},e^{\theta}_{w,\infty}\}$ is a BK-space with the given norm (6). This completes the proof. □

REMARK 1. Let λ be any of the sequence spaces $e^{\theta}_{w,0}$, $e^{\theta}_{w,c}$ or $e^{\theta}_{w,\infty}$. Then, one can easily check that λ is not of absolute type, i.e. we have $\|x\|_\lambda\neq\||x|\|_\lambda$ with $|x|=(|x_k|)$ for at least one sequence $x=(x_k)\in\lambda$.

THEOREM 2. *The sequence spaces $e^{\theta}_{w,0}$, $e^{\theta}_{w,c}$ and $e^{\theta}_{w,\infty}$ of non-absolute type are linearly isomorphic to the spaces c_0, c and ℓ_∞, respectively, that is $e^{\theta}_{w,0}\cong c_0$, $e^{\theta}_{w,c}\cong c$ and $e^{\theta}_{w,\infty}\cong\ell_\infty$.*

Proof. We give the proof only for the space $e^{\theta}_{w,0}$ since the proofs for other spaces are similar. It is clear, from the definition of $E^{\theta,w}$, that the map $x\to y=E^{\theta,w}x$ is linear and injective.

Furthermore, let $y=(y_k)\in c_0$ and define the sequence $x=(x_k)$ by

$$x_k=\sum_{j=0}^{k}\binom{k}{j}\frac{1}{w_j}(\theta-1)^{k-j}\theta^{-k}y_j\ \ (k\in\mathbb{N}). \tag{7}$$

Then we have that

$$(E^{\theta,w}x)_n=w_n\sum_{k=0}^{n}\binom{n}{k}(1-\theta)^{n-k}\theta^k\left(\sum_{j=0}^{k}\binom{k}{j}\frac{1}{w_j}(\theta-1)^{k-j}\theta^{-k}y_j\right)=y_n$$

This shows that $E^{\theta,w}x=y\in c_0$. Thus, we deduce that $x\in e^{\theta}_{w,0}$ and that $E^{\theta,w}$ is surjective.

Moreover, we have for every $x\in e^{\theta}_{w,0}$ that

$$\|y\|_{\ell_\infty}=\left\|E^{\theta,w}x\right\|_{\ell_\infty}=\|x\|_{e^{\theta}_{w,0}}$$

which means that $E^{\theta,w}$ is norm preserving. Consequently, $E^{\theta,w}$ is a linear bijection which shows that the spaces $e^{\theta}_{w,0}$ and c_0 are linearly isomorphic and this concludes the proof. □

THEOREM 3. *The inclusions $c_0\subset e^{\theta}_{w,0}$, $c\subset e^{\theta}_{w,c}$ and $\ell_\infty\subset e^{\theta}_{w,\infty}$ strictly hold.*

Proof. This theorem can be proved similarly as the appropriate theorems for spaces $e^{\theta}_0, e^{\theta}_c$ and e^{θ}_∞ in [2] or [3]. □

THEOREM 4. *Define the sequences $b^{(k)}=(b^{(k)}_n)_{n\in\mathbb{N}}$ and $b^{(-1)}=(b^{(-1)}_n)$ by*

$$b^{(k)}_n=\begin{cases}0, & 0\leqslant n<k\\ \binom{n}{k}\frac{1}{w_k}(\theta-1)^{n-k}\theta^{-n}, & n\geqslant k\end{cases}$$

and

$$b^{(-1)}_n=\sum_{j=0}^{n}\binom{n}{j}\frac{1}{w_j}(\theta-1)^{n-j}\theta^{-n}\ (n\in\mathbb{N}).$$

Then:

(a) The sequence $(b^{(k)})_{k=0}^{\infty}$ is a basis for the space $e^{\theta}_{w,0}$ and every $x\in e^{\theta}_{w,0}$ has a unique representation of the form

$$x=\sum_k(E^{\theta,w}x)_k b^{(k)}.$$

(b) $(b^{(k)})_{k=-1}^{\infty}$ is a Schauder basis for $e^{\theta}_{w,c}$ and every $x\in e^{\theta}_{w,c}$ has a unique representation of the form

$$x=lb^{(-1)}+\sum_k[(E^{\theta,w}x_k-l]b^{(k)},$$

where $l=\lim_{k\to\infty}(E^{\theta,w}x)_k$.

Proof. This is an immediate consequence of [10, Lemma 2.3]. □

4. Certain matrix transformations on the spaces $e^{\theta}_{w,0}$, $e^{\theta}_{w,c}$ and $e^{\theta}_{w,\infty}$

In this section we determine the β-dual of the spaces $e^{\theta}_{w,0}$, $e^{\theta}_{w,c}$ and $e^{\theta}_{w,\infty}$ and characterize various matrix mappings on these spaces.

The following lemma is essential for our results.

LEMMA 1. [33] *$A=(a_{nk})\in(c_0,c)$ if and only if there exists*

$$\lim_{n\to\infty} a_{nk}=\alpha_k \ (k\in\mathbb{N}) \tag{8}$$

and

$$\sup_{n\in\mathbb{N}}\sum_{k=0}^{\infty}|a_{nk}|<\infty. \tag{9}$$

THEOREM 5. *Consider the sets d_1, d_2, d_3 and d_4 defined as follows*

$$d_1=\left\{a=(a_k)\in\omega:\sup_{m\in\mathbb{N}}\sum_{k=0}^{m}\left|\frac{1}{w_k}\sum_{j=k}^{m}\binom{j}{k}(\theta-1)^{j-k}\theta^{-j}a_j\right|<\infty\right\},$$

$$d_2=\left\{a=(a_k)\in\omega:\frac{1}{w_k}\sum_{j=k}^{\infty}\binom{j}{k}(\theta-1)^{j-k}\theta^{-j}a_j \text{ exists for each } k\in\mathbb{N}\right\},$$

$$d_3=\left\{a=(a_k)\in\omega:\lim_{m\to\infty}\sum_{k=0}^{m}\left(\frac{1}{w_k}\sum_{j=k}^{m}\binom{j}{k}(\theta-1)^{j-k}\theta^{-j}a_j\right) \text{ exists}\right\}$$

and

$$d_4=\left\{a=(a_k)\in\omega:\lim_{m\to\infty}\sum_{k=0}^{m}\left|\frac{1}{w_k}\sum_{j=k}^{m}\binom{j}{k}(\theta-1)^{j-k}\theta^{-j}a_j\right|=\sum_{k=0}^{\infty}\left|\frac{1}{w_k}\sum_{j=k}^{\infty}\binom{j}{k}(\theta-1)^{j-k}\theta^{-j}a_j\right|\right\}.$$

Then $\left\{e^{\theta}_{w,0}\right\}^{\beta}=d_1\cap d_2$, $\{e^{\theta}_{w,c}\}^{\beta}=d_1\cap d_2\cap d_3$ *and* $\{e^{\theta}_{w,\infty}\}^{\beta}=d_2\cap d_4$.

Proof. We only give a proof for the space $e^{\theta}_{w,0}$. Let $a=(a_k)\in\omega$. Recalling the identity (7) we have

$$\begin{aligned}\sum_{k=0}^{m}a_kx_k&=\sum_{k=0}^{m}a_k\left[\sum_{j=0}^{k}\binom{k}{j}\frac{1}{w_j}(\theta-1)^{k-j}\theta^{-k}y_j\right]\\&=\sum_{k=0}^{m}\left[\frac{1}{w_k}\sum_{j=k}^{m}\binom{j}{k}(\theta-1)^{j-k}\theta^{-j}a_j\right]y_k=\sum_{k=0}^{m}\tilde{a}_k(m)y_k,\end{aligned} \tag{10}$$

where

$$\tilde{a}_k(m) = \frac{1}{w_k}\sum_{j=k}^{m}\binom{j}{k}(\theta-1)^{j-k}\theta^{-j}a_j.$$

Then one can define a new matrix $A = (a_{mk})$ by

$$a_{mk} = \begin{cases} \tilde{a}_k(m), & 0 \leqslant k \leqslant m \\ 0, & k > m \end{cases} \quad (k, m \in \mathbb{N})$$

so that

$$\sum_{k=0}^{m} a_k x_k = \sum_{k=0}^{m} \tilde{a}_k(m) y_k = \sum_{k=0}^{m} a_{mk} y_k = (Ay)_m.$$

From the last identity we see that $(a_k x_k) \in cs$ for $x = (x_k) \in e^{\theta}_{w,o}$ if and only if $Ay \in c$ for $y \in c_0$. Therefore, using the relations (8) and (9) from Lemma 1, we conclude that

$$\frac{1}{w_k}\sum_{j=k}^{\infty}\binom{j}{k}(\theta-1)^{j-k}\theta^{-j}a_j \text{ exists for each } k \in \mathbb{N}$$

and

$$\sup_{m\in\mathbb{N}}\sum_{k=0}^{m}\left|\frac{1}{w_k}\sum_{j=k}^{m}\binom{j}{k}(\theta-1)^{j-k}\theta^{-j}a_j\right| < \infty,$$

which shows that $\left\{e^{\theta}_{w,0}\right\}^{\beta} = d_1 \cap d_2$. $\square$

For an infinite matrix, $A = (a_{nk})$, we shall write for brevity that

$$\tilde{a}_{nk}(m) = \frac{1}{w_k}\sum_{j=k}^{m}\binom{j}{k}(\theta-1)^{j-k}\theta^{-j}a_{nj}$$

and

$$\tilde{a}_{nk} = \frac{1}{w_k}\sum_{j=k}^{\infty}\binom{j}{k}(\theta-1)^{j-k}\theta^{-j}a_{nj} \tag{11}$$

for all n, $k \in \mathbb{N}$ and all $m \geqslant k$ provided that the series on the right hand to be convergent. Then, $A = (\tilde{a}_{nk})$ is the so-called associated matrix. Further, let $x, y \in \omega$ be connected by the relation (5) and (7). Then we have by (10) that

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m} \tilde{a}_{nk}(m) y_k \quad (n,\ m \in \mathbb{N}). \tag{12}$$

In particular, let $x \in e^{\theta}_{w,c}$ and $A_n = (a_{nk})_{k=0}^{\infty} \in \left\{e^{\theta}_{w,c}\right\}^{\beta}$ for all $n \in \mathbb{N}$. Then, we obtain, by passing to limit in (12) as $m \to \infty$ and using Theorem 5, that

$$\sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} \tilde{a}_{nk} y_k \quad (n \in \mathbb{N}),$$

which gives the equality

$$\sum_{k=0}^{\infty} a_{nk}x_k = \sum_{k=0}^{\infty} \tilde{a}_{nk}(y_k - l) + l\sum_{k=0}^{\infty} \tilde{a}_{nk} \quad (n \in \mathbb{N}), \tag{13}$$

where $l = \lim_{k\to\infty} y_k$.

Now let $\mathscr{F}$ be the family of all finite subsets in $\mathbb{N}$ and consider the following conditions;

$$\sup_n \left(\sum_{k=0}^{\infty} |\tilde{a}_{nk}|\right) < \infty, \tag{14}$$

$$\lim_{m\to\infty} \sum_{k=0}^{\infty} |\tilde{a}_{nk}(m)| = \sum_{k=0}^{\infty} |\tilde{a}_{nk}| \quad (n \in \mathbb{N}), \tag{15}$$

$$\tilde{a}_{nk} \text{ exists for all } k,n \in \mathbb{N}, \tag{16}$$

$$\sup_{m\in\mathbb{N}} \sum_{k=0}^{m} |\tilde{a}_{nk}(m)| < \infty \quad (n \in \mathbb{N}), \tag{17}$$

$$\lim_{n\to\infty} \tilde{a}_{nk} = \tilde{a}_k \quad (k \in \mathbb{N}), \tag{18}$$

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} |\tilde{a}_{nk} - \tilde{a}_k| = 0, \tag{19}$$

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \tilde{a}_{nk} = \alpha, \tag{20}$$

$$\sum_{k=0}^{\infty} \tilde{a}_{nk} \text{ converges for all } n \in \mathbb{N}, \tag{21}$$

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} |\tilde{a}_{nk}| = 0, \tag{22}$$

$$\lim_{n\to\infty} \tilde{a}_{nk} = 0 \text{ for all } k \in \mathbb{N}, \tag{23}$$

$$\sup_{K\in\mathscr{F}} \left(\sum_{n=0}^{\infty} \left|\sum_{k\in K} \tilde{a}_{nk}\right|^p\right) < \infty \quad (1 \leqslant p < \infty), \tag{24}$$

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \tilde{a}_{nk} = 0. \tag{25}$$

Then, by using (13) and by combining Theorem 5 with the results of Stieglitz and Tietz [33], we immediately derive the following results:

THEOREM 6. *For an infinite matrix A we have:*

(a) $A \in \left(e^{\theta}_{w,\infty}, \ell_\infty\right)$ if and only if (14), (15) and (16) hold.
(b) $A \in \left(e^{\theta}_{w,c}, \ell_\infty\right)$ if and only if (14), (16) and (17) hold.
(c) $A \in \left(e^{\theta}_{w,0}, \ell_\infty\right)$ if and only if (14), (16) and (17) hold.

THEOREM 7. *For an infinite matrix A we have:*

(a) $A \in \left(e^{\theta}_{w,\infty}, c\right)$ if and only if (15), (16), (18) and (19) hold.
(b) $A \in \left(e^{\theta}_{w,c}, c\right)$ if and only if (14), (16), (17), (18) and (20) hold.
(c) $A \in \left(e^{\theta}_{w,0}, c\right)$ if and only if (14), (16), (17) and (18) hold.

THEOREM 8. *For an infinite matrix A we have:*

(a) $A \in \left(e^{\theta}_{w,\infty}, c_0\right)$ if and only if (15), (16) and (22) hold.
(b) $A \in \left(e^{\theta}_{w,c}, c_0\right)$ if and only if (14), (16), (17), (23) and (25) hold.
(c) $A \in \left(e^{\theta}_{w,0}, c_0\right)$ if and only if (14), (16), (17) and (23) hold.

THEOREM 9. *Let $1 \leqslant p < \infty$. Then, for an infinite matrix A we have:*

(a) $A \in \left(e^{\theta}_{w,\infty}, \ell_p\right)$ if and only if (15), (16) and (24) hold.
(b) $A \in \left(e^{\theta}_{w,c}, \ell_p\right)$ if and only if (16), (17), (21) and (24) hold.
(c) $A \in \left(e^{\theta}_{w,0}, \ell_p\right)$ if and only if (16), (17) and (24) hold.

5. Compact operators on the weighted Euler sequence spaces $e^{\theta}_{w,0}$, $e^{\theta}_{w,\infty}$

In this section, we derive some identities for the Hausdorff measure of noncompactness of certain matrix operators on the spaces of generalized means and apply our results to obtain the necessary and sufficient (or only sufficient) conditions for such operators to be compact.

Recent developments on this particular topic can be found in [5–8], [10], [11], [15], [18], [20–28], [30]. Our consideration will go along the same lines.

We shall need the following known results for our investigation.

LEMMA 2. ([20], Lemma 3.1) *Let X denotes any of the spaces c_0 and ℓ_∞. If $A \in (X, c)$, then we have*

$$\alpha_k = \lim_{n\to\infty} a_{nk} \text{ exists for every } k \in \mathbb{N},$$

$$\alpha = (\alpha_k) \in \ell_1,$$

$$\sup_n \left(\sum_{k=0}^{\infty} |a_{nk} - \alpha_k| \right) < \infty,$$

$$\lim_{n\to\infty} (Ax)_n = \sum_{k=0}^{\infty} \alpha_k x_k \text{ exists for all } x = (x_k) \in X.$$

LEMMA 3. ([20], Lemma 1.1) *Let X denotes any of the spaces c_0, c or ℓ_∞. Then, we have $X^{\beta} = \ell_1$ and $\|a\|^*_X = \|a\|_{\ell_1}$ for all $a \in \ell_1$.*

LEMMA 4. ([34], Theorem 4.2.8) *Let X and Y be BK-spaces. Then we have $(X,Y) \subset B(X,Y)$, that is, every $A \in (X,Y)$ defines a linear operator $L_A \in B(X,Y)$, where $L_A(x) = Ax$ for all $x \in X$.*

LEMMA 5. ([10], Lemma 5.2) *Let $X \supset \phi$ be BK-space and Y be any of the spaces c_0, c or ℓ_∞. If $A \in (X,Y)$, then*

$$\|L_A\| = \|A\|_{(X,\ell_\infty)} = \sup_n \|A_n\|_X^* < \infty.$$

LEMMA 6. ([20], Lemma 1.5) *Let $Q \in M_{c_0}$ and $P_r : c_0 \to c_0$ $(r \in \mathbb{N})$ be the operator defined by $P_r(x) = (x_0, x_1, \ldots, x_r, 0, 0, \ldots)$ for all $x = (x_k) \in c_0$. Then, we have*

$$\chi(Q) = \lim_{r\to\infty}\left(\sup_{x\in Q}\|(I-P_r)(x)\|_{\ell_\infty}\right),$$

where I is the identity operator on c_0.

Further, we know by [16, Theorem 1.10] that every $z = (z_n) \in c$ has a unique representation $z = \bar{z}e + \sum_{n=0}^{\infty}(z_n - \bar{z})e^{(n)}$, where $\bar{z} = \lim_{n\to\infty} z_n$. Thus, we define the projectors $P_r : c \to c$ $(r \in \mathbb{N})$ by

$$P_r(z) = \bar{z}e + \sum_{n=0}^{r}(z_n - \bar{z})e^{(n)}; \quad (r \in \mathbb{N}) \tag{26}$$

for all $z = (z_n) \in c$ with $\bar{z} = \lim_{n\to\infty} z_n$. In this situation, the following result gives an estimate for the Hausdorff measure of noncompactness in the BK space c.

LEMMA 7. ([20], Lemma 1.6) *Let $Q \in M_c$ and $P_r : c \to c$ $(r \in \mathbb{N})$ be the projector onto the linear span of $\left\{e, e^{(0)}, e^{(1)}, \ldots, e^{(r)}\right\}$. Then, we have*

$$\frac{1}{2}.\lim_{r\to\infty}\left(\sup_{x\in Q}\|(I-P_r)(x)\|_{\ell_\infty}\right) \leqslant \chi(Q) \leqslant \lim_{r\to\infty}\left(\sup_{x\in Q}\|(I-P_r)(x)\|_{\ell_\infty}\right),$$

where I is the identity operator on c.

The next lemma is related to the Hausdorff measure of noncompactness of a bounded linear operator.

LEMMA 8. ([16], Theorem 2.25, Corollary 2.26) *Let X and Y be Banach spaces and $L \in B(X,Y)$. Then we have*

$$\|L\|_\chi = \chi(L(S_X)) \tag{27}$$

and

$$L \in K(X,Y) \text{ if and only if } \|L\|_\chi = 0. \tag{28}$$

The following results will be needed in establishing our results.

LEMMA 9. *Let X denotes any of the spaces $e^{\theta}_{w,0}$ or $e^{\theta}_{w,\infty}$. If $a=(a_k)\in X^{\beta}$ then $\tilde{a}=(\tilde{a}_k)\in \ell_1$ and the equality*

$$\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} \tilde{a}_k y_k \tag{29}$$

holds for every $x=(x_k)\in X$, where $y=(y_k)$ is the associated sequence defined by (5) and

$$\tilde{a}_k = \frac{1}{w_k}\sum_{j=k}^{\infty}\binom{j}{k}(\theta-1)^{j-k}\theta^{-j}a_j \quad (k\in\mathbb{N}).$$

Proof. This follows directly from the definition of the associate sequence (10) and [33]. □

LEMMA 10. *Let X denotes any of the spaces $e^{\theta}_{w,0}$ or $e^{\theta}_{w,\infty}$. Then, we have*

$$\|a\|^*_X = \|\tilde{a}\|_{\ell_1} = \sum_{k=0}^{\infty}|\tilde{a}_k| < \infty$$

for all $a=(a_k)\in X^{\beta}$, where $\tilde{a}=(\tilde{a}_k)$ is as in Lemma 9.

Proof. Let Y be the respective one of the spaces c_0 or ℓ_∞ and take any $a=(a_k)\in X^{\beta}$. Then, we have by Lemma 9 that $\tilde{a}=(\tilde{a}_k)\in\ell_1$ and the equality (29) holds for all sequences $x=(x_k)\in X$ and $y=(y_k)\in Y$ which are connected by the relation (5). Further, it follows by (6) that $x\in S_X$ if and only if $y\in S_Y$. Therefore, we derive from (3) and (29) that

$$\|a\|^*_X = \sup_{x\in S_X}\left|\sum_{k=0}^{\infty} a_k x_k\right| = \sup_{y\in S_Y}\left|\sum_{k=0}^{\infty}\tilde{a}_k y_k\right| = \|\tilde{a}\|^*_Y$$

and since $\tilde{a}\in\ell_1$, we obtain from Lemma 3 that

$$\|a\|^*_X = \|\tilde{a}\|^*_Y = \|\tilde{a}\|_{\ell_1} < \infty$$

which concludes the proof. □

LEMMA 11. *Let X be any of the spaces $e^{\theta}_{w,0}$ or $e^{\theta}_{w,\infty}$, Y the respective one of the spaces c_0 or ℓ_∞, Z a sequence space and $A=(a_{nk})$ an infinite matrix. If $A\in(X,Z)$, then $\tilde{A}\in(Y,Z)$ such that $Ax=\tilde{A}y$ for all sequences $x\in X$ and $y\in Y$ which are connected by the relation (5), where $\tilde{A}=(\tilde{a}_{nk})$ is the associated matrix defined as in (11).*

Proof. This is immediate by [20, Lemma 2.3]. □

Now, let $A=(a_{nk})$ be an infinite matrix and $\tilde{A}=(\tilde{a}_{nk})$ the associated matrix defined by (11). Then, we have the following result.

THEOREM 10. *Let X denotes any of the spaces $e^{\theta}_{w,0}$ or $e^{\theta}_{w,\infty}$. Then, we have*
(a) If $A\in(X,c_0)$, then

$$\|L_A\|_{\chi}=\limsup_{n\to\infty}\sum_{k=0}^{\infty}|\tilde{a}_{nk}|. \tag{30}$$

(b) If $A\in(X,c)$, then

$$\frac{1}{2}.\limsup_{n\to\infty}\sum_{k=0}^{\infty}|\tilde{a}_{nk}-\tilde{a}_k|\leqslant\|L_A\|_{\chi}\leqslant\limsup_{n\to\infty}\sum_{k=0}^{\infty}|\tilde{a}_{nk}-\tilde{a}_k|, \tag{31}$$

where $\tilde{a}_k$ is defined as in (18) for all $k\in\mathbb{N}$.
(c) If $A\in(X,\ell_\infty)$, then

$$0\leqslant\|L_A\|_{\chi}\leqslant\limsup_{n\to\infty}\sum_{k=0}^{\infty}|\tilde{a}_{nk}|. \tag{32}$$

Proof. Let us remark that the limes superiors in (30) and (32) are finite by Theorems 8 and 6. Also, by combining Lemmas 2 and 11, we deduce the same for limes superiors in (31).

We write $S=S_X$ for short. Then, we obtain by (27) and Lemma 4 that

$$\|L_A\|_{\chi}=\chi(AS). \tag{33}$$

For (a) we have $AS\in M_{c_0}$. Thus, it follows by applying Lemma 6 that

$$\chi(AS)=\lim_{r\to\infty}\left(\sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_\infty}\right), \tag{34}$$

where $P_r:c_0\to c_0$ $(r\in\mathbb{N})$ is the operator defined by $P_r(x)=(x_0,x_1,\ldots,x_r,0,0,\ldots)$ for all $x=(x_k)\in c_0$. This yields that $\|(I-P_r)(Ax)\|_{\ell_\infty}=\sup_{n>r}|(Ax)_n|$ for all $x\in X$ and every $r\in\mathbb{N}$. Therefore, by using (14) and Lemma 10, we have for every $r\in\mathbb{N}$ that

$$\sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_\infty}=\sup_{n>r}\|A_n\|^*_X=\sup_{n>r}\left\|\tilde{A}_n\right\|_{\ell_1}.$$

This and (34) imply that

$$\chi(AS)=\lim_{r\to\infty}\left(\sup_{n>r}\left\|\tilde{A}_n\right\|_{\ell_1}\right)=\limsup_{n\to\infty}\left\|\tilde{A}_n\right\|_{\ell_1}.$$

Hence, we obtain (30).

To prove (b), we have $AS \in M_c$. Thus, we are going to apply Lemma 7 to get an estimate for the value of $\chi(AS)$ in (33). For this, let $P_r : c \to c \ (r \in \mathbb{N})$ be the projectors defined by (26). Then, we have for every $r \in \mathbb{N}$ that $(I-P_r)(z) = \sum_{n=r+1}^{\infty}(z_n - \bar{z})e^{(n)}$ and hence,

$$\|(I-P_r)(z)\|_{\ell_\infty} = \sup_{n>r}|z_n - \bar{z}| \tag{35}$$

for all $z = (z_n) \in c$ and every $r \in \mathbb{N}$, where $\bar{z} = \lim_{n\to\infty} z_n$ and I is identity operator on c.

Now, by using (33) and applying Lemma 7 we obtain

$$\frac{1}{2}\cdot \lim_{r\to\infty}\left(\sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_\infty}\right) \leqslant \|L_A\|_\chi \leqslant \lim_{r\to\infty}\left(\sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_\infty}\right). \tag{36}$$

On the other hand, for X equal to $e^{\theta}_{w,0}$ or $e^{\theta}_{w,\infty}$ let Y be c_0 or ℓ_∞, respectively. Also, for every given $x \in X$, let $y \in Y$ be the associated sequence defined by (5). Since, $A \in (X,c)$, we have by Lemma 11 that $\tilde{A} \in (Y,c)$ and $Ax = \tilde{A}y$. Further, it follows from Lemma 2 that the limits $\tilde{\alpha}_k = \lim_{n\to\infty}\tilde{a}_{nk}$ exists for all k, $(\tilde{a}_k) \in \ell_1 = Y^\beta$ and $\lim_{n\to\infty}(\tilde{A}y)_n = \sum_{k=0}^{\infty}\tilde{\alpha}_k y_k$. Consequently, we derive from (35) that

$$\begin{aligned}\|(I-P_r)(Ax)\|_{\ell_\infty} &= \left\|(I-P_r)(\tilde{A}y)\right\|_{\ell_\infty}\\ &= \sup_{n>r}\left|(\tilde{A}y)_n - \sum_{k=0}^{\infty}\tilde{\alpha}_k y_k\right|\\ &= \sup_{n>r}\left|\sum_{k=0}^{\infty}(\tilde{a}_{nk} - \tilde{\alpha}_k)y_k\right|\end{aligned}$$

for all $r \in \mathbb{N}$. Moreover, since $x \in S = S_X$ if and only if $y \in S_Y$ we obtain by (3) and Lemma 3 that

$$\begin{aligned}\|(I-P_r)(Ax)\|_{\ell_\infty} &= \sup_{n>r}\left(\sup_{y\in S_Y}\left|\sum_{k=0}^{\infty}(\tilde{a}_{nk} - \tilde{\alpha}_k)y_k\right|\right)\\ &= \sup_{n>r}\left\|\tilde{A}_n - \tilde{\alpha}\right\|_Y^*\\ &= \sup_{n>r}\left\|\tilde{A}_n - \tilde{\alpha}\right\|_{\ell_1}\end{aligned}$$

for all $r \in \mathbb{N}$. Thus, we get (31) from (36).

Finally, to prove (c) we define the projectors $P_r : \ell_\infty \to \ell_\infty \ (r \in \mathbb{N})$ as in the proof of part (a) for all $x = (x_k) \in \ell_\infty$. Then, we have

$$AS \subset P_r(AS) + (I-P_r)(AS); \quad (r \in \mathbb{N}).$$

Thus, it follows by the elementary properties of the function χ that

$$\begin{aligned}0 &\leqslant \chi(AS) \leqslant \chi(P_r(AS)) + \chi((I-P_r)(AS))\\ &= \chi((I-P_r)(AS)) \leqslant \sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_\infty}\\ &= \sup_{n>r}\left\|\tilde{A}_n\right\|_{\ell_1}\end{aligned}$$

for all $r \in \mathbb{N}$ and hence,

$$0 \leqslant \chi(AS) \leqslant \lim_{r\to\infty}\left(\sup_{n>r}\left\|\tilde{A}_n\right\|_{\ell_1}\right) = \limsup_{r\to\infty}\left\|\tilde{A}_n\right\|_{\ell_1}.$$

This and (33) together imply (32) and complete the proof. □

COROLLARY 1. *Let X denotes any of the spaces $e^{\theta}_{w,0}$ or $e^{\theta}_{w,\infty}$. Then, we have*
(a) If $A \in (X, c_0)$, then

$$L_A \text{ is compact if and only if } \lim_{n\to\infty}\sum_{k=0}^{\infty}|\tilde{a}_{nk}| = 0.$$

(b) If $A \in (X, c)$, then

$$L_A \text{ is compact if and only if } \lim_{n\to\infty}\sum_{k=0}^{\infty}|\tilde{a}_{nk} - \tilde{\alpha}_k| = 0.$$

(c) If $A \in (X, \ell_\infty)$, then

$$L_A \text{ is compact if } \lim_{n\to\infty}\sum_{k=0}^{\infty}|\tilde{a}_{nk}| = 0.$$

Proof. This result follows from Theorem 10 by using (28). □

Finally, we have the following observation.

COROLLARY 2. *For every matrix $A \in (e^{\theta}_{w,\infty}, c_0)$ or $A \in (e^{\theta}_{w,\infty}, c)$, the operator L_A is compact.*

Proof. Let $A \in (e^{\theta}_{w,\infty}, c_0)$. Then we have by Theorem 8 (a) that $\lim_{n\to\infty}(\sum_{k=0}^{\infty}|\tilde{a}_{nk}|) = 0$. This leads us with Corollary 1(a) to the consequence that L_A is compact. Similarly, if $A \in (e^{\theta}_{w,\infty}, c)$ then, from Theorem 7(a), we have that $\lim_{n\to\infty}(\sum_{k=0}^{\infty}|\tilde{a}_{nk} - \tilde{\alpha}_k|) = 0$, where $\tilde{\alpha}_k = \lim_{n\to\infty}\tilde{a}_{nk}$ for all k. Hence, we deduce from Corollary 1(b) that L_A is compact. □

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REFERENCES

[1] B. ALTAY AND F. BAŞAR, *Some paranormed sequence spaces of non-absolute type derived by weighted mean*, J. Math. Anal. Appl. **319**, 2 (2006), 494–508.

[2] B. ALTAY AND F. BAŞAR, *Some Euler sequence spaces of non-absolute type*, Ukrainian Math. J. **57**, 1 (2005), 1–17.

[3] B. ALTAY, F. BAŞAR AND M. MURSALEEN, *On the Euler sequence spaces which include the spaces, ℓ_p and ℓ_∞, I*, Inform Sci. **176**, 10 (2006), 1450–1462.

[4] F. BAŞAR AND M. KIRIŞÇI, *Almost convergence and generalized difference matrix*, Comput. Math. Appl. **61**, 3 (2010), 602–611.

[5] M. BAŞARIR AND E. E. KARA, *On some difference sequence spaces of weighted means and compact operators*, Ann. Funct. Anal. **2**, 2 (2011), 116–131.

[6] M. BAŞARIR AND E. E. KARA, *On compact operators on the Riesz $B^{(m)}$ difference sequence space*, Iranian J. Sci. Technol. **35**, A4 (2011), 279–285.

[7] M. BAŞARIR AND E. E. KARA, *On the B difference sequence space derived by generalized weighted mean and compact operators*, J. Math. Anal. Appl. **391**, 1 (2012), 67–81.

[8] M. BAŞARIR AND E. E. KARA, *On compact operators on the Riesz $B^{(m)}$ difference sequence spaces II*, Iranian J. Sci. Technol. **36**, A3 (2012), 371–376.

[9] S. DEMIRIZ, C. ÇAKAN, *On some new paranormed Euler sequence spaces and Euler core*, Acta Math. Sin. (Engl. Ser.) **26**, 7 (2010), 1207–1222.

[10] I. DJOLOVIĆ AND E. MALKOWSKY, *Matrix transformations and compact operators on some new m^{th} order difference sequence spaces*, Appl. Math. Comput. **198**, 2 (2008), 700–714.

[11] E. E. KARA AND M. BAŞARIR, *On compact operators and some Euler $B^{(m)}$ difference sequence spaces*, J. Math. Anal. Appl. **379**, 2 (2011), 499–511.

[12] E. E. KARA, M. ÖZTÜRK AND M. BAŞARIR, *Some topological and geometrical properties of generalized Euler sequence spaces*, Math. Slovaca **60**, 3 (2010), 385–398.

[13] V. KARAKAYA AND H. POLAT, *Some new paranormed sequence spaces defined by Euler and difference operators*, Acta Sci. Math. (Szeged) **76**, 1–2 (2010), 87–100.

[14] R. LASHKARIPOUR AND G. TALEBI, *Lower bound for matrix operators on the Euler weighted sequence space $e^{\theta}_{w,p}$ $(0 < p < 1)$*, Acta Sci. Math. (Szeged) **61**, 1 (2012), 1–12.

[15] B. DE MALAFOSSE AND E. MALKOWSKY, *On the measure of noncompactness of linear operators in spaces of strongly α-summable and bounded sequences*, Period. Math. Hungar. **55**, 2 (2007), 129–148.

[16] E. MALKOWSKY AND V. RAKOČEVIĆ, *An introduction into the theory of sequence spaces and measures of noncompactness*, Zb. Rad. (Beogr.) Math. Institut. SANU **9**, 17 (2000), 143–234.

[17] E. MALKOWSKY AND E. SAVAŞ, *Matrix transformations between sequence spaces of generalized weighted mean*, Appl. Math. Comput. **147**, 2 (2004), 333–345.

[18] M. MURSALEEN, *Application of measure of noncompactness to infinite system of differential equations*, Canadian Math. Bull. 2011, doi:10.4153/CMB-2011-170-7.

[19] M. MURSALEEN, F. BAŞAR AND B. ALTAY, *On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞, II*, Nonlinear Anal. **65**, 3 (2006), 707–717.

[20] M. MURSALEEN, V. KARAKAYA, H. POLAT AND N. ŞIMŞEK, *Measure of noncompactness of matrix operators on some difference sequence spaces of weighted means*, Comput. Math. Appl. **62**, 2 (2011), 814–820.

[21] M. MURSALEEN AND A. K. NOMAN, *Applications of the Hausdorff measure of noncompactness in some sequence spaces of weighted means*, Comput. Math. Appl. **60**, 5 (2010), 245–258.

[22] M. MURSALEEN AND A. K. NOMAN, *Applications of Hausdorff measure of noncompactness in the spaces of generalized means*, Math. Inequal. Appl., Preprint.

[23] M. MURSALEEN AND A. K. NOMAN, *Compactness by the Hausdorff measure of noncompactness*, Nonlinear Anal. **73**, 8 (2010), 2541–2557.

[24] M. MURSALEEN AND A. K. NOMAN, *Compactness of matrix operators on some new difference sequence spaces*, Linear Algebra Appl. doi:10.1016/j.laa.2011.06.014.

[25] M. MURSALEEN AND A. K. NOMAN, *Compactness by the Hausdorff measure of noncompactness*, Nonlinear Anal. **436**, 1 (2012), 41–52.

[26] M. MURSALEEN AND A. K. NOMAN, *The Hausdorff measure of noncompactness of matrix operators on some BK spaces*, Operators and Matrices **5**, 3 (2011), 473–486.

[27] M. MURSALEEN AND A. K. NOMAN, *On σ-conservative matrices and compact operators on the space V_σ*, Appl. Math. Lett. **24**, 9 (2011), 1554–1560.
[28] M. MURSALEEN AND A. K. NOMAN, *On generalized means and some related sequence spaces*, Comput. Math. Appl. **61**, 4 (2011), 988–999.
[29] M. MURSALEEN AND A. K. NOMAN, *On some new difference sequence spaces of non-absolute type*, Math. Comput. Modelling **52**, 3–4 (2010), 603–617.
[30] M. MURSALEEN AND S. A. MOHIUDDINE, *Applications of measures of noncompactness to the infinite system of differential equations in ℓ_p spaces*, Nonlinear Anal. **75**, 4 (2012), 2111–2115.
[31] H. POLAT AND F. BAŞAR, *Some Euler spaces of difference sequences of order m*, Acta Math. Sci. **27B**, 2 (2007), 254–266.
[32] H. POLAT AND B. ALTAY, *On some new Euler difference sequence spaces*, Southeast Asian Bull. Math. **30**, 2 (2006), 209–220.
[33] M. STIEGLITZ AND H. TIETZ, *Matrix transformationen von folgenräumen eine ergebnisübersicht*, Math. Z. **154**, 2 (1977), 1–16.
[34] A. WILANSKY, *Summability Through Functional Analysis*, North-Holland Math. Studies **85**, Amsterdam, 1984.

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