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ON $(k,h;m)$–CONVEX MAPPINGS AND APPLICATIONS

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Abstract. In this paper, for given positive integer m and real functions k and h, we prove $(k,h;m)$-convexity of the mapping $\mathbf{p} \to \phi(\mathbf{p}) f\left(\frac{\Phi(\mathbf{p})}{\phi(\mathbf{p})}\right)$ with a convex (increasing) function f and a $(k,h;m)$-convex mapping Φ and a positive $(k,h;m)$-concave mapping ϕ. As application, we establish a subadditivity result for completely monotone and Bernstein functions. We also show monotonity of the mappings $\mathbf{p} \to \frac{\Phi(\mathbf{p})}{\phi(\mathbf{p})}$ and $\mathbf{p} \to f\left(\frac{\Phi(\mathbf{p})}{\phi(\mathbf{p})}\right)$ with respect to a group majorization combined with other preorders.

1. Introduction

A nonempty subset $\mathscr{A}$ of a real linear space $\mathscr{V}$ is called *additive* if $\mathbf{x},\mathbf{y} \in \mathscr{A}$ implies $\mathbf{x}+\mathbf{y} \in \mathscr{A}$.

A mapping $F : \mathscr{A} \to \mathbb{R}$ defined on an additive set $\mathscr{A} \subset \mathscr{V}$ is said to be

(i) *additive* if

$$F(\mathbf{p}+\mathbf{q}) = F(\mathbf{p}) + F(\mathbf{q}) \quad \text{for } \mathbf{p},\mathbf{q} \in \mathscr{A},$$

(ii) *subadditive* if

$$F(\mathbf{p}+\mathbf{q}) \leqslant F(\mathbf{p}) + F(\mathbf{q}) \quad \text{for } \mathbf{p},\mathbf{q} \in \mathscr{A},$$

(iii) *superadditive* if

$$F(\mathbf{p}+\mathbf{q}) \geqslant F(\mathbf{p}) + F(\mathbf{q}) \quad \text{for } \mathbf{p},\mathbf{q} \in \mathscr{A}.$$

For a function $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, $\mathbf{x} = (x_1,x_2,\ldots,x_n) \in I^n$ and $\mathbf{p} \in \mathscr{P}_n^0$, where

$$\mathscr{P}_n^0 = \{\mathbf{p} = (p_1,p_2,\ldots,p_n) : p_i \geqslant 0,\ P_n > 0\} \quad \text{with} \quad P_n = \sum_{i=1}^{n} p_i,$$

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the *Jensen functional* is defined by

$$J(f,\mathbf{x},\mathbf{p}) = \sum_{i=1}^{n} p_i f(x_i) - P_n f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \tag{1}$$

(see [6]).

With the help of the standard inner product $\langle\cdot,\cdot\rangle$ of $\mathbb{R}^n$, equation (1) can be rewritten in the form

$$J(f,\mathbf{x},\mathbf{p}) = \langle \mathbf{p}, f(\mathbf{x})\rangle - \langle \mathbf{p},\mathbf{e}\rangle f\left(\frac{\langle \mathbf{p},\mathbf{x}\rangle}{\langle \mathbf{p},\mathbf{e}\rangle}\right), \tag{2}$$

where $f(\mathbf{x}) = (f(x_1), f(x_2),\ldots, f(x_n))$ and $\mathbf{e} = (1,1,\ldots,1) \in \mathbb{R}^n$.

By Jensen's inequality,

$$J(f,\mathbf{x},\mathbf{p}) \geqslant 0 \quad \text{for a convex function } f.$$

THEOREM A. [6] If $f : I \to \mathbb{R}$ is a convex function then the mapping $\mathbf{p} \to J(f,\mathbf{x},\mathbf{p})$ is superadditive for any $\mathbf{x} \in I^n$, i.e.,

$$J(f,\mathbf{x},\mathbf{p}+\mathbf{q}) \geqslant J(f,\mathbf{x},\mathbf{p}) + J(f,\mathbf{x},\mathbf{q}) \quad \text{for } \mathbf{p},\mathbf{q} \in \mathscr{P}_n^0. \tag{3}$$

In consequence, the mapping $\mathbf{p} \to J(f,\mathbf{x},\mathbf{p})$ is monotone for any $\mathbf{x} \in I^n$, i.e.,

$$J(f,\mathbf{x},\mathbf{p}+\mathbf{q}) \geqslant J(f,\mathbf{x},\mathbf{p}) \quad \text{for } \mathbf{p},\mathbf{q} \in \mathscr{P}_n^0.$$

A result related to Theorem **A** is proved in [9, Theorem 1] which shows superadditivity and monotonicity properties of the so-called *Jessen functional* regarding positive isotonic functionals.

It is not hard to check that property (3) is equivalent to the subadditivity of the mapping

$$\mathbf{p} \to \langle \mathbf{p},\mathbf{e}\rangle f\left(\frac{\langle \mathbf{p},\mathbf{x}\rangle}{\langle \mathbf{p},\mathbf{e}\rangle}\right), \quad \mathbf{p} \in \mathscr{P}_n^0.$$

That is, the convexity of f implies that

$$\langle \mathbf{p}+\mathbf{q},\mathbf{e}\rangle f\left(\frac{\langle \mathbf{p}+\mathbf{q},\mathbf{x}\rangle}{\langle \mathbf{p}+\mathbf{q},\mathbf{e}\rangle}\right) \leqslant \langle \mathbf{p},\mathbf{e}\rangle f\left(\frac{\langle \mathbf{p},\mathbf{x}\rangle}{\langle \mathbf{p},\mathbf{e}\rangle}\right) + \langle \mathbf{q},\mathbf{e}\rangle f\left(\frac{\langle \mathbf{q},\mathbf{x}\rangle}{\langle \mathbf{q},\mathbf{e}\rangle}\right) \quad \text{for } \mathbf{p},\mathbf{q} \in \mathscr{P}_n^0. \tag{4}$$

Let $f : [a,b] \to \mathbb{R}$ be a function and $\mathbf{x} = (x_1,x_2,\ldots,x_n) \in [a,b]^n$, where $[a,b] \subset \mathbb{R}$ is an interval. Assume $\mathbf{p} = (p_1, p_2,\ldots, p_n) \in \mathscr{P}_n^0$. The *Jensen-Mercer functional* is defined by

$$M(f,\mathbf{x},\mathbf{p}) = P_n[f(a)+f(b)] - \sum_{i=1}^{n} p_i f(x_i) - P_n f\left(a+b-\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)$$

(see [8]). Equivalently,

$$M(f,\mathbf{x},\mathbf{p}) = [f(a)+f(b)]\langle \mathbf{p},\mathbf{e}\rangle - \langle \mathbf{p}, f(\mathbf{x})\rangle - \langle \mathbf{p},\mathbf{e}\rangle f\left(\frac{(a+b)\langle \mathbf{p},\mathbf{e}\rangle - \langle \mathbf{p},\mathbf{x}\rangle}{\langle \mathbf{p},\mathbf{e}\rangle}\right), \tag{5}$$

where $f(\mathbf{x}) = (f(x_1), f(x_2), \ldots, f(x_n))$ and $\mathbf{e} = (1,1,\ldots,1) \in \mathbb{R}^n$.

By Mercer's inequality,

$$M(f,\mathbf{x},\mathbf{p}) \geqslant 0 \quad \text{for a convex function } f.$$

THEOREM B. [8] If $f : I \to \mathbb{R}$ is a convex function then the mapping $\mathbf{p} \to M(f,\mathbf{x},\mathbf{p})$ is superadditive for any $\mathbf{x} \in I^n$, i.e.,

$$M(f,\mathbf{x},\mathbf{p}+\mathbf{q}) \geqslant M(f,\mathbf{x},\mathbf{p}) + M(f,\mathbf{x},\mathbf{q}) \quad \text{for } \mathbf{p},\mathbf{q} \in \mathscr{P}_n^0. \tag{6}$$

In consequence, the mapping $\mathbf{p} \to M(f,\mathbf{x},\mathbf{p})$ is monotone for any $\mathbf{x} \in I^n$, i.e.,

$$M(f,\mathbf{x},\mathbf{p}+\mathbf{q}) \geqslant M(f,\mathbf{x},\mathbf{p}) \quad \text{for } \mathbf{p},\mathbf{q} \in \mathscr{P}_n^0.$$

Notice that property (6) is equivalent to the subadditivity of the mapping

$$\mathbf{p} \to \langle \mathbf{p},\mathbf{e}\rangle f\left(\frac{\langle \mathbf{p},\mathbf{y}\rangle}{\langle \mathbf{p},\mathbf{e}\rangle}\right), \quad \mathbf{p} \in \mathscr{P}_n^0,$$

where $\mathbf{y} = (a+b)\mathbf{e} - \mathbf{x}$.

That is, the convexity of f implies that

$$\langle \mathbf{p}+\mathbf{q},\mathbf{e}\rangle f\left(\frac{\langle \mathbf{p}+\mathbf{q},\mathbf{y}\rangle}{\langle \mathbf{p}+\mathbf{q},\mathbf{e}\rangle}\right) \leqslant \langle \mathbf{p},\mathbf{e}\rangle f\left(\frac{\langle \mathbf{p},\mathbf{y}\rangle}{\langle \mathbf{p},\mathbf{e}\rangle}\right) + \langle \mathbf{q},\mathbf{e}\rangle f\left(\frac{\langle \mathbf{q},\mathbf{y}\rangle}{\langle \mathbf{q},\mathbf{e}\rangle}\right) \quad \text{for } \mathbf{p},\mathbf{q} \in \mathscr{P}_n^0. \tag{7}$$

Remind that a nonempty set $\mathscr{D}$ in a real linear space $\mathscr{V}$ is said to be a *convex cone* if (i) $\mathbf{x},\mathbf{y} \in \mathscr{D}$ implies $\mathbf{x}+\mathbf{y} \in \mathscr{D}$, and (ii) $\mathbf{x} \in \mathscr{D}$ and $t \geqslant 0$ imply $t\mathbf{x} \in \mathscr{D}$.

A relation $\prec$ on a real linear space $\mathscr{V}$ is said to be a *cone preorder* if there exists a convex cone $\mathscr{D} \subset \mathscr{V}$ such that for $\mathbf{x},\mathbf{y} \in \mathscr{V}$, $\mathbf{x} \prec \mathbf{y}$ if and only if $\mathbf{y}-\mathbf{x} \in \mathscr{D}$.

In what follows, for a function $\phi : \mathscr{D} \to [0,\infty)$ we denote

$$\mathscr{D}_\phi^0 = \{\mathbf{d} \in \mathscr{D} : \phi(\mathbf{d}) > 0\}.$$

A general result related to (4) and (7) is as follows.

THEOREM C. [3, Theorem 5] Let $\mathscr{D}$ be a convex cone in a linear space $\mathscr{V}$ and $\phi : \mathscr{D} \to [0,\infty)$ be an additive functional on $\mathscr{D}$ and $\Phi : \mathscr{D} \to [0,\infty)$ be a subadditive functional on $\mathscr{D}$.

If $f : [0,\infty) \to \mathbb{R}$ is an increasing convex function, then the mapping

$$\mathbf{p} \to \phi(\mathbf{p}) f\left(\frac{\Phi(\mathbf{p})}{\phi(\mathbf{p})}\right)$$

is subadditive on $\mathscr{D}_\phi^0$, i.e.,

$$\phi(\mathbf{p}+\mathbf{q}) f\left(\frac{\Phi(\mathbf{p}+\mathbf{q})}{\phi(\mathbf{p}+\mathbf{q})}\right) \leqslant \phi(\mathbf{p}) f\left(\frac{\Phi(\mathbf{p})}{\phi(\mathbf{p})}\right) + \phi(\mathbf{q}) f\left(\frac{\Phi(\mathbf{q})}{\phi(\mathbf{q})}\right) \quad \text{for } \mathbf{p},\mathbf{q} \in \mathscr{D}_\phi^0.$$

Recently, sub- and superadditive functions are a research field of growing interest [3, 4, 5, 10, 11]. In [14], the author have extended Theorem C to sub-/superadditive vector-valued mappings $\mathbf{\Phi}$ and ϕ, respectively. The aim of the present paper is give a further generalization of Theorem C for $(k,h;m)$-convex mapping $\mathbf{\Phi}$ and positive $(k,h;m)$-concave mapping ϕ. As applications, we interpret the obtained results for completely monotone and Bernstein functions and utilize them in majorization theory.

2. $(k,h;m)$-convex/concave mappings

We begin with some relevant notation and terminology.

Throughout the paper, $\mathscr{V}$ and $\mathscr{W}$ are real linear spaces.

A relation $\prec$ on a set $\mathscr{S}$ is called a *preorder* if (i) $\mathbf{x} \prec \mathbf{x}$ for $\mathbf{x} \in \mathscr{S}$, and (ii) $\mathbf{x} \prec \mathbf{y}$ and $\mathbf{y} \prec \mathbf{z}$ imply $\mathbf{x} \prec \mathbf{z}$ for $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathscr{S}$.

Given two preorders $\prec_1$ and $\prec_2$ on $\mathscr{A}$, and vectors $\mathbf{x},\mathbf{y} \in \mathscr{A}$, we write

$$\mathbf{x} \prec_{1,2} \mathbf{y} \quad \text{if and only if} \quad \mathbf{x} \prec_1 \mathbf{z} \prec_2 \mathbf{y} \quad \text{for some } \mathbf{z} \in \mathscr{A}. \tag{8}$$

If $(\mathscr{A},\prec_1)$ and $(\mathscr{B},\prec)$ are preordered sets, then a mapping $F : \mathscr{A} \to \mathscr{B}$ is said to be $(\prec_1,\prec)$-*increasing* on $\mathscr{A}$, if for $\mathbf{x},\mathbf{y} \in \mathscr{A}$,

$$\mathbf{x} \prec_1 \mathbf{y} \quad \text{implies} \quad F(\mathbf{x}) \prec F(\mathbf{y}).$$

When $(\mathscr{B},\prec) = (\mathbb{R},\leqslant)$ we write $\prec_1$-*increasing* instead of $(\prec_1,\leqslant)$-*increasing* for brevity.

It is clear that if $(\mathscr{A},\prec_1)$, $(\mathscr{A},\prec_2)$ and $(\mathscr{B},\prec)$ are preordered sets, and a mapping $F : \mathscr{A} \to \mathscr{B}$ is both $(\prec_1,\prec)$-increasing and $(\prec_2,\prec)$-increasing, then for $\mathbf{x},\mathbf{y} \in \mathscr{A}$,

$$\mathbf{x} \prec_{1,2} \mathbf{y} \quad \text{implies} \quad F(\mathbf{x}) \prec F(\mathbf{y}).$$

For $\mathbf{x},\mathbf{y} \in \mathscr{W}$, we write

$$\mathbf{x} \prec_K \mathbf{y} \quad \text{if and only if} \quad \alpha\mathbf{x} = \mathbf{y} \quad \text{for some } \alpha \in [1,\infty). \tag{9}$$

(see [7, p. 121]). So, for $\mathbf{x},\mathbf{y} \in \mathscr{W}$,

$$\mathbf{x} \prec_{K,1} \mathbf{y} \quad \text{if and only if} \quad \mathbf{x} \prec_K \mathbf{z} \prec_1 \mathbf{y} \quad \text{for some } \mathbf{z} \in \mathscr{W} \tag{10}$$

(see e.g. (15)).

A preorder $\prec_1$ on $\mathscr{W}$ is said to be *preserved under multiplying by positive scalars* if for $\mathbf{x},\mathbf{y} \in \mathscr{W}$,

$$\mathbf{x} \prec_1 \mathbf{y} \quad \text{implies} \quad \lambda\mathbf{x} \prec_1 \lambda\mathbf{y} \quad \text{for } \lambda \in (0,\infty).$$

DEFINITION 1. (Cf. [13, Def. 2.1]) Let $k : (0,1) \to \mathbb{R}$ be a given function and $m \geqslant 2$ be a given positive integer. Then a set $\mathscr{D} \subset \mathscr{V}$ is said to be $(k;m)$-*convex* if $k(t_1)\mathbf{p}_1 + \ldots + k(t_m)\mathbf{p}_m \in \mathscr{D}$ for all $\mathbf{p}_1,\ldots,\mathbf{p}_m \in \mathscr{D}$ and $t_1,\ldots,t_m \in (0,1)$ with $t_1 + \ldots + t_m = 1$.

DEFINITION 2. (Cf. [13, Def. 2.4], [2, p. 254]) Let $k,h : (0,1) \to \mathbb{R}$ be two given functions and $m \geqslant 2$ be a given positive integer. A function $F : \mathscr{D} \to \mathbb{R}$ defined on a $(k;m)$-convex set $\mathscr{D} \subset \mathscr{V}$ is said to be $(k,h;m)$-*convex* (resp. $(k,h;m)$-*concave*) if for all $\mathbf{p}_1,\ldots,\mathbf{p}_m \in \mathscr{D}$ and $t_1,\ldots,t_m \in (0,1)$ with $t_1 + \ldots + t_m = 1$,

$$F(k(t_1)\mathbf{p}_1 + \ldots + k(t_m)\mathbf{p}_m) \leqslant (\geqslant)\ h(t_1)F(\mathbf{p}_1) + \ldots + h(t_m)F(\mathbf{p}_m). \tag{11}$$

If equality holds in (11) then F is called $(k,h;m)$-*affine*.

In particular, in (11), if $k(t) = t$ and $h(t) = t^s$, $t \in (0,1)$ and $s \in \mathbb{R}$, then F is said to be s-*convex* (resp. s-*concave*).

See [13] for an interesting discussion on some important classes of functions included in the class of $(k,h;m)$-convex functions for $m=2$.

It is not difficult to prove by induction on m that if k and h are positive and *multiplicative* than $(k,h;m)$-convexity with $m=2$ is equivalent to $(k,h;m)$-convexity for all positive integers $m\geqslant 2$.

DEFINITION 3. (Cf. [13, Def. 2.4]) Let $k,h:(0,1)\to\mathbb{R}$ be two given functions and $m\geqslant 2$ be a given positive integer. Let the linear space $\mathscr{W}$ be equipp with a preorder $\prec_1$. A function $F:\mathscr{D}\to\mathscr{W}$ defined on a $(k;m)$-convex set $\mathscr{D}\subset\mathscr{V}$ is said to be $(k,h;m,\prec_1)$*-convex* (*resp.* $(k,h;m,\prec_1)$-concave) if for all $\mathbf{p}_1,\ldots,\mathbf{p}_m\in\mathscr{D}$ and $t_1,\ldots,t_m\in(0,1)$ with $t_1+\ldots+t_m=1$,

$$F(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)\prec_1(\succ_1)\,h(t_1)F(\mathbf{p}_1)+\ldots+h(t_m)F(\mathbf{p}_m). \tag{12}$$

If equality holds in (12) then F is called $(k,h;m)$*-affine*.

In particular, in (12), if $k(t)=t$ and $h(t)=t^s$, $t\in(0,1)$ and $s\in\mathbb{R}$, then F is said to be $(s,\prec_1)$*-convex* (resp. $(s,\prec_1)$*-concave*).

When $m=2$, we adopt simpler prefix symbols by dropping m off, e.g., $(k,h;\prec_1)$ instead of $(k,h;m,\prec_1)$.

Note that if $\mathscr{D}\subset\mathscr{V}$ is a $(k;m)$-convex set and a functional $\phi:\mathscr{D}\to[0,\infty)$ is $(k,h;m)$-concave on $\mathscr{D}$ with $h(\cdot)>0$ on $(0,1)$, then the set

$$\mathscr{D}_\phi^0=\{\mathbf{d}\in\mathscr{D}:\phi(\mathbf{d})>0\}$$

is $(k;m)$-convex.

Proofs of the next results will be simplified if we first prove a technical lemma.

LEMMA 2.1. *Let $k:(0,1)\to\mathbb{R}$ and $h:(0,1)\to(0,\infty)$ be two given functions and $m\geqslant 2$ be a given positive integer. Let $\Phi:\mathscr{D}\to\mathscr{W}$ and $\phi:\mathscr{D}\to[0,\infty)$ with a $(k;m)$-convex set $\mathscr{D}\subset\mathscr{V}$. Assume that $\prec_1$ is a preorder on $\mathscr{W}$ preserved under multiplying by positive scalars.*

Let $\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_m\in\mathscr{D}_\phi^0$ and $t_1,t_2,\ldots,t_m>0$ with $t_1+t_2+\ldots+t_m=1$. Denote

$$\alpha_i=\frac{h(t_i)\phi(\mathbf{p}_i)}{h(t_1)\phi(\mathbf{p}_1)+\ldots+h(t_m)\phi(\mathbf{p}_m)}\quad\text{for } i=1,2,\ldots,m.$$

Then the following three statements hold.

(i) If Φ is $(k,h;m)$-affine and ϕ is $(k,h;m)$-affine then

$$\frac{\Phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}{\phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}=\alpha_1\frac{\Phi(\mathbf{p}_1)}{\phi(\mathbf{p}_1)}+\ldots+\alpha_m\frac{\Phi(\mathbf{p}_m)}{\phi(\mathbf{p}_m)}. \tag{13}$$

(ii) If Φ is $(k,h;m,\prec_1)$-convex and ϕ is $(k,h;m)$-affine then

$$\frac{\Phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}{\phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}\prec_1\alpha_1\frac{\Phi(\mathbf{p}_1)}{\phi(\mathbf{p}_1)}+\ldots+\alpha_m\frac{\Phi(\mathbf{p}_m)}{\phi(\mathbf{p}_m)}. \tag{14}$$

(iii) If Φ is $(k,h;m,\prec_1)$-convex and ϕ is $(k,h;m)$-concave, then

$$\frac{\Phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}{\phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)} \prec_K \beta\frac{\Phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}{\phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)} \prec_1 \alpha_1\frac{\Phi(\mathbf{p}_1)}{\phi(\mathbf{p}_1)}+\ldots+\alpha_m\frac{\Phi(\mathbf{p}_m)}{\phi(\mathbf{p}_m)}, \tag{15}$$

where

$$\beta = \frac{\phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}{h(t_1)\phi(\mathbf{p}_1)+\ldots+h(t_m)\phi(\mathbf{p}_m)}. \tag{16}$$

Proof. (i). Since Φ and ϕ are $(k,h;m)$-affine, we have

$$\Phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m) = h(t_1)\Phi(\mathbf{p}_1)+\ldots+h(t_m)\Phi(\mathbf{p}_m),$$

$$\phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m) = h(t_1)\phi(\mathbf{p}_1)+\ldots+h(t_m)\phi(\mathbf{p}_m).$$

Now, it is routine to verify that identity (13) holds.

(ii). In this case, we have

$$\Phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m) \prec_1 h(t_1)\Phi(\mathbf{p}_1)+\ldots+h(t_m)\Phi(\mathbf{p}_m),$$

$$\phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m) = h(t_1)\phi(\mathbf{p}_1)+\ldots+h(t_m)\phi(\mathbf{p}_m).$$

Since ϕ takes positive values on $\mathscr{D}_\phi^0$ and $\prec_1$ is preserved under multiplying by positive scalars, we obtain

$$\begin{aligned}\frac{\Phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}{\phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)} &\prec_1 \frac{h(t_1)\Phi(\mathbf{p}_1)+\ldots+h(t_m)\Phi(\mathbf{p}_m)}{h(t_1)\phi(\mathbf{p}_1)+\ldots+h(t_m)\phi(\mathbf{p}_m)}\\ &= \alpha_1\frac{\Phi(\mathbf{p}_1)}{\phi(\mathbf{p}_1)}+\ldots+\alpha_m\frac{\Phi(\mathbf{p}_m)}{\phi(\mathbf{p}_m)},\end{aligned}$$

which ends the proof of inequality (14).

(iii). From the $(k,h;m,\prec_1)$-convexity of Φ and $(k,h;m)$-concavity of ϕ, we have

$$\Phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m) \prec_1 h(t_1)\Phi(\mathbf{p}_1)+\ldots+h(t_m)\Phi(\mathbf{p}_m),$$

$$\phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m) \geqslant h(t_1)\phi(\mathbf{p}_1)+\ldots+h(t_m)\phi(\mathbf{p}_m).$$

Therefore, by taking β defined by (16), we get $\beta \geqslant 1$.

For this reason, by (9), we conclude that

$$\begin{aligned}\frac{\Phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}{\phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)} &\prec_K \beta\frac{\Phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}{\phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}\\ &= \frac{\Phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}{h(t_1)\phi(\mathbf{p}_1)+\ldots+h(t_m)\phi(\mathbf{p}_m)}\\ &\prec_1 \frac{h(t_1)\Phi(\mathbf{p}_1)+\ldots+h(t_m)\Phi(\mathbf{p}_m)}{h(t_1)\phi(\mathbf{p}_1)+\ldots+h(t_m)\phi(\mathbf{p}_m)}\\ &= \alpha_1\frac{\Phi(\mathbf{p}_1)}{\phi(\mathbf{p}_1)}+\ldots+\alpha_m\frac{\Phi(\mathbf{p}_m)}{\phi(\mathbf{p}_m)}.\end{aligned}$$

Summarizing all of this, we see that (15) holds. □

In the forthcoming theorem we aim to extend Theorem C (see Section 1) and [14, Theorem 2.3] from sub-/superadditive functions to $(k,h;m)$-convex/$(k,h;m)$-concave mappings, respectively.

THEOREM 2.2. *Let $k:(0,1)\to\mathbb{R}$ and $h:(0,1)\to(0,\infty)$ be two given functions and $m\geqslant 2$ be a given positive integer. Let $\Phi:\mathscr{D}\to\mathscr{W}$ and $\phi:\mathscr{D}\to[0,\infty)$ with a $(k;m)$-convex set $\mathscr{D}\subset\mathscr{V}$. Assume that $\prec_1$ is a preorder on $\mathscr{W}$ preserved under multiplying by positive scalars. Let $f:\mathscr{W}\to\mathbb{R}$ be a convex function.*

Then the following three statements hold.

(i) If Φ is $(k,h;m)$-affine and ϕ is $(k,h;m)$-affine then the mapping

$$\mathbf{p}\to\phi(\mathbf{p})f\left(\frac{\Phi(\mathbf{p})}{\phi(\mathbf{p})}\right),\quad \mathbf{p}\in\mathscr{D}_\phi^0,$$

is $(k,h;m)$-convex, i.e., for $\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_m\in\mathscr{D}_\phi^0$ and $t_1,t_2,\ldots,t_m>0$ with $t_1+t_2+\ldots+t_m=1$,

$$\begin{aligned}&\phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)f\left(\frac{\Phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}{\phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}\right)\\&\leqslant h(t_1)\phi(\mathbf{p}_1)f\left(\frac{\Phi(\mathbf{p}_1)}{\phi(\mathbf{p}_1)}\right)+\ldots+h(t_m)\phi(\mathbf{p}_m)f\left(\frac{\Phi(\mathbf{p}_m)}{\phi(\mathbf{p}_m)}\right).\end{aligned}\tag{17}$$

(ii) If Φ is $(k,h;m,\prec_1)$-convex and ϕ is $(k,h;m)$-affine and f is $\prec_1$-increasing, then (17) holds.

(iii) If Φ is $(k,h;m,\prec_1)$-convex and ϕ is $(k,h;m)$-concave and f is $\prec_1$-increasing with $f(0)\leqslant 0$, then (17) holds.

Proof. (i). Using the convexity of f and eq. (13) in Lemma 2.1, it is easy to obtain the following inequality:

$$\begin{aligned}&f\left(\frac{\Phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}{\phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}\right)\\&\leqslant\frac{h(t_1)\phi(\mathbf{p}_1)}{h(t_1)\phi(\mathbf{p}_1)+\ldots+h(t_m)\phi(\mathbf{p}_m)}f\left(\frac{\Phi(\mathbf{p}_1)}{\phi(\mathbf{p}_1)}\right)+\ldots\\&\quad+\frac{h(t_m)\phi(\mathbf{p}_m)}{h(t_1)\phi(\mathbf{p}_1)+\ldots+h(t_m)\phi(\mathbf{p}_m)}f\left(\frac{\Phi(\mathbf{p}_m)}{\phi(\mathbf{p}_m)}\right).\end{aligned}\tag{18}$$

Multiplying both the sides of (18) by

$$\phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)=h(t_1)\phi(\mathbf{p}_1)+\ldots+h(t_m)\phi(\mathbf{p}_m)>0$$

yields (17).

(ii). Similarly as in the proof of (i), we employ eq. (14) in Lemma 2.1 and use the $\prec_1$-increasity and convexity of f, which gives (18) and (17).

(iii). From Lemma 2.1, eq. (15), we have

$$\beta\frac{\Phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}{\phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}\prec_1\alpha_1\frac{\Phi(\mathbf{p}_1)}{\phi(\mathbf{p}_1)}+\ldots+\alpha_m\frac{\Phi(\mathbf{p}_m)}{\phi(\mathbf{p}_m)},$$

where β is defined by (16). Hence, by the $\prec_1$-increasity and convexity of f, we derive

$$f\left(\beta\frac{\Phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}{\phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}\right)\leqslant\alpha_1 f\left(\frac{\Phi(\mathbf{p}_1)}{\phi(\mathbf{p}_1)}\right)+\ldots+\alpha_m f\left(\frac{\Phi(\mathbf{p}_m)}{\phi(\mathbf{p}_m)}\right). \quad (19)$$

Since ϕ is $(k,h;m)$-concave, we have $\beta\geqslant 1$ (see (16)). Simultaneously, $f(0)\leqslant 0$, so we infer that

$$\beta f\left(\frac{\Phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}{\phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}\right)\leqslant f\left(\beta\frac{\Phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}{\phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}\right). \quad (20)$$

Combining (20) and (19) yields

$$\beta f\left(\frac{\Phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}{\phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}\right)\leqslant\alpha_1 f\left(\frac{\Phi(\mathbf{p}_1)}{\phi(\mathbf{p}_1)}\right)+\ldots+\alpha_m f\left(\frac{\Phi(\mathbf{p}_m)}{\phi(\mathbf{p}_m)}\right).$$

Now, in order to get (17), it suffices to multiply the above inequality by $h(t_1)\phi(\mathbf{p}_1)+\ldots+h(t_m)\phi(\mathbf{p}_m)>0$. This completes the proof. □

REMARK 2.3. In the case

$$(\mathscr{W},\prec_1)=(\mathbb{R},\leqslant)\quad\text{and}\quad k(t)=1,\quad h(t)=1,\quad t\in(0,1),$$

Theorem 2.2 reduces to Theorem C (see Section 1).

REMARK 2.4. In Theorem 2.2, part (iii), if the condition $f(0)\leqslant 0$ is replaced by "f is $\prec_K$-increasing", then (17) becomes

$$\begin{aligned}&(h(t_1)\phi(\mathbf{p}_1)+\ldots+h(t_m)\phi(\mathbf{p}_m))f\left(\frac{\Phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}{\phi(k(t_1)\mathbf{p}_1+\ldots+k(t_m)\mathbf{p}_m)}\right)\\&\leqslant h(t_1)\phi(\mathbf{p}_1)f\left(\frac{\Phi(\mathbf{p}_1)}{\phi(\mathbf{p}_1)}\right)+\ldots+h(t_m)\phi(\mathbf{p}_m)f\left(\frac{\Phi(\mathbf{p}_m)}{\phi(\mathbf{p}_m)}\right).\end{aligned}$$

COROLLARY 2.5. *Under assumptions of Theorem 2.2, if in addition* $\mathbf{p}\to L(\mathbf{p})\in\mathbb{R}$, $\mathbf{p}\in\mathscr{D}$, *is a* $(k,h;m)$*-affine mapping on* $\mathscr{D}$, *then the mapping*

$$\mathbf{p}\to L(\mathbf{p})-\phi(\mathbf{p})f\left(\frac{\Phi(\mathbf{p})}{\phi(\mathbf{p})}\right),\quad\mathbf{p}\in\mathscr{D}^0_\phi\ , \quad (21)$$

is $(k,h;m)$*-concave.*

REMARK 2.6. In Corollary 2.5, the $(k,h;m)$-concavity of the mapping (21) corresponds to the superadditivity of the Jensen functional (2) (see Theorem A) and of the Jensen-Mercer functional (5) (see Theorem B).

We now give a specification of Theorem 2.2, part (iii), for

$$(\mathscr{W},\prec_1)=(\mathbb{R},\leqslant) \quad \text{and} \quad k(t)=t, \quad h(t)=t^s, \quad s\in\mathbb{R}, \quad t\in(0,1).$$

COROLLARY 2.7. *Let $\Phi:\mathscr{D}\to\mathbb{R}$ and $\phi:\mathscr{D}\to[0,\infty)$ with a convex set $\mathscr{D}\subset\mathscr{V}$. If Φ is s-convex and ϕ is s-concave and $f:\mathbb{R}\to\mathbb{R}$ is an increasing convex function with $f(0)\leqslant 0$, then the mapping*

$$\mathbf{p}\to\phi(\mathbf{p})f\left(\frac{\Phi(\mathbf{p})}{\phi(\mathbf{p})}\right), \quad \mathbf{p}\in\mathscr{D}_\phi^0 ,$$

is s-convex, i.e., for $\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_m\in\mathscr{D}_\phi^0$ and $t_1,t_2,\ldots,t_m>0$ with $t_1+t_2+\ldots+t_m=1$,

$$\phi(t_1\mathbf{p}_1+\ldots+t_m\mathbf{p}_m)f\left(\frac{\Phi(t_1\mathbf{p}_1+\ldots+t_m\mathbf{p}_m)}{\phi(t_1\mathbf{p}_1+\ldots+t_m\mathbf{p}_m)}\right)$$
$$\leqslant t_1^s\phi(\mathbf{p}_1)f\left(\frac{\Phi(\mathbf{p}_1)}{\phi(\mathbf{p}_1)}\right)+\ldots+t_m^s\phi(\mathbf{p}_m)f\left(\frac{\Phi(\mathbf{p}_m)}{\phi(\mathbf{p}_m)}\right).$$

3. Completely monotone and Bernstein functions

In this section we give an interpretation of Theorem 2.2 for completely monotone and Bernstein functions.

The definitions and lemma below are quoted from [1].

DEFINITION 4. (See [1, Def. 1]) A function $\varphi:(0,\infty)\to\mathbb{R}$ *is completely monotone* if it is infinitely differentiable, non-negative, and $(-1)^n\varphi^{(n)}(x)\geqslant 0$ for $n=1,2,\ldots$ and $x>0$.

If in addition $\lim_{x\to\infty}\varphi(x)=0$ then φ is said to be a *bare completely monotone function*. The set of all bare completely monotone functions is denoted by $CM0$.

DEFINITION 5. (See [1, Def. 2]) A function $\varphi:(0,\infty)\to\mathbb{R}$ *is a Bernstein function* if it is infinitely differentiable, non-negative, and $(-1)^n\varphi^{(n)}(x)\leqslant 0$ for $n=1,2,\ldots$ and $x>0$.

If in addition $\lim_{x\to\infty}\frac{\varphi(x)}{x}=0$ and $\varphi(0^+)=0$. then φ is said to be a *bare Bernstein function*. The set of all bare Bernstein functions is denoted by $BF0$.

DEFINITION 6. (See [1, p. 604]) For positive integers l, a function $\varphi:(0,\infty)\to\mathbb{R}$ belongs to the class BFl if

$$\varphi(x)=\int_0^x\psi(t)\,dt, \quad x\in(0,\infty),$$

for some $\psi\in BF(l-1)$.

LEMMA 3.1. (See [1, Lemma 4]) *Let* $\varphi : (0,\infty) \to \mathbb{R}$. *If* $\varphi \in CM0 \cup BF0$ *then* φ *is subadditive on* $(0,\infty)$. *If* $\varphi \in CMl$, $l > 1$, *then* φ *is superadditive on* $(0,\infty)$.

By using Theorem 2.2 for $k \equiv 1$, $h \equiv 1$, and $\mathscr{V} = \mathscr{W} = \mathbb{R}$ and $\mathscr{D} = [0,\infty)$, one obtains the following.

COROLLARY 3.2. *Let* $\Phi : [0,\infty) \to \mathbb{R}$ *and* $\phi : [0,\infty) \to [0,\infty)$. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be an increasing convex function with* $f(0) \leqslant 0$.

If $\Phi \in CM0 \cup BF0$ *and* $\phi \in CMl$, $l > 1$, *then the mapping*

$$\mathbf{p} \to \phi(\mathbf{p}) f\left(\frac{\Phi(\mathbf{p})}{\phi(\mathbf{p})}\right) \quad \textit{for } \mathbf{p} \in [0,\infty) \textit{ such that } \phi(\mathbf{p}) > 0\ ,$$

is subadditive, i.e., for $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m \in [0,\infty)$ *such that* $\phi(\mathbf{p}_i) > 0$,

$$\phi(\mathbf{p}_1 + \ldots + \mathbf{p}_m) f\left(\frac{\Phi(\mathbf{p}_1 + \ldots + \mathbf{p}_m)}{\phi(\mathbf{p}_1 + \ldots + \mathbf{p}_m)}\right) \leqslant \phi(\mathbf{p}_1) f\left(\frac{\Phi(\mathbf{p}_1)}{\phi(\mathbf{p}_1)}\right) + \ldots + \phi(\mathbf{p}_m) f\left(\frac{\Phi(\mathbf{p}_m)}{\phi(\mathbf{p}_m)}\right).$$

4. Applications for group majorization

A vector $\mathbf{q} = (q_1, q_2, \ldots, q_n) \in \mathbb{R}^n$ is said to be *majorized* by a vector $\mathbf{p} = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$, written as $\mathbf{q} \prec_m \mathbf{p}$, if

$$\sum_{j=1}^{k} q_{[j]} \leqslant \sum_{j=1}^{k} p_{[j]} \quad \text{for all } k = 1, 2, \ldots, n$$

with equality for $k = n$ (see [12, p. 8]). Here the symbols $q_{[j]}$ and $p_{[j]}$ stand for the jth largest entry of $\mathbf{q}$ and $\mathbf{p}$, respectively.

It is well-known (see [12, pp. 10, 34]) that for $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$,

$$\mathbf{q} \prec_m \mathbf{p} \text{ if and only if } \mathbf{q} \in \operatorname{conv} \mathbf{p}\mathbb{P}_n, \tag{22}$$

where $\mathbb{P}_n$ denotes the group of $n \times n$ permutation matrices, and $\operatorname{conv} \mathbf{p}\mathbb{P}_n$ denotes the convex hull of the set $\mathbf{p}\mathbb{P}_n = \{\mathbf{p}g : g \in \mathbb{P}_n\}$.

Remind that throughout $\mathscr{V}$ and $\mathscr{W}$ are real linear spaces.

Let $G \subset GL(\mathscr{V})$ be a subgroup of the group $GL(\mathscr{V})$ of all invertible linear operators acting on $\mathscr{V}$. By the *group majorization* induced by G we mean the preorder $\prec_G$ on $\mathscr{V}$ defined by: for $\mathbf{p}, \mathbf{q} \in \mathscr{V}$,

$$\mathbf{q} \prec_G \mathbf{p} \quad \text{if and only if} \quad \mathbf{q} \in \operatorname{conv} G\mathbf{p},$$

where $\operatorname{conv} G\mathbf{p}$ denotes the convex hull of the set $G\mathbf{p} = \{g\mathbf{p} : g \in G\}$ (see [7, p. 112], cf. (22)).

For a given group $G \subset GL(\mathscr{V})$, a set $\mathscr{D} \subset \mathscr{V}$ is said to be *G-invariant* if $g\mathbf{p} \in \mathscr{D}$ for all $g \in G$ and $\mathbf{p} \in \mathscr{D}$. A mapping F defined on a G-invariant set $\mathscr{D} \subset \mathscr{V}$ is said to be *G-invariant* if

$$F(g\mathbf{p}) = F(\mathbf{p}) \quad \text{for } \mathbf{p} \in \mathscr{D} \text{ and } g \in G. \tag{23}$$

Here we apply Lemma 2.1 for the identity function $k(t) = t$ for $t \in (0,1)$.

THEOREM 4.1. *Let $G \subset GL(\mathscr{V})$ be a group inducing group majorization $\prec_G$ on $\mathscr{V}$. Let $\mathscr{D}$ be a G-invariant convex set in $\mathscr{V}$ and $\prec_1$ be a cone preorder on $\mathscr{W}$.*

Let $h:(0,1)\to(0,\infty)$ be a given multiplicative function. Suppose that $\Phi:\mathscr{D}\to\mathscr{W}$ and $\phi:\mathscr{D}\to[0,\infty)$ are G-invariant mappings.

Then for $\mathbf{p},\mathbf{q}\in\mathscr{D}_\phi^0$ the following three statements hold.

(i) If Φ is (id,h)-affine and ϕ is (id,h)-affine then

$$\mathbf{q}\prec_G\mathbf{p} \quad \textit{implies} \quad \frac{\Phi(\mathbf{q})}{\phi(\mathbf{q})}=\frac{\Phi(\mathbf{p})}{\phi(\mathbf{p})}. \tag{24}$$

(ii) If Φ is $(\mathrm{id},h;\prec_1)$-convex and ϕ is (id,h)-affine then

$$\mathbf{q}\prec_G\mathbf{p} \quad \textit{implies} \quad \frac{\Phi(\mathbf{q})}{\phi(\mathbf{q})}\prec_1\frac{\Phi(\mathbf{p})}{\phi(\mathbf{p})}. \tag{25}$$

(iii) If Φ is $(\mathrm{id},h;\prec_1)$-convex and ϕ is (id,h)-concave then

$$\mathbf{q}\prec_G\mathbf{p} \quad \textit{implies} \quad \frac{\Phi(\mathbf{q})}{\phi(\mathbf{q})}\prec_{K,1}\frac{\Phi(\mathbf{p})}{\phi(\mathbf{p})}, \tag{26}$$

where $\prec_{K,1}$ is defined by (10).

Proof. Let $\mathbf{p},\mathbf{q}\in\mathscr{D}_\phi^0$ such that $\mathbf{q}\prec_G\mathbf{p}$ be taken arbitrarily. According to (22), we can write that

$$\mathbf{q}=t_1g_1\mathbf{p}+\ldots+t_mg_m\mathbf{p} \tag{27}$$

for some positive integer m, linear operators $g_1,\ldots,g_m\in G$ and numbers $t_1,\ldots,t_m>0$ such that $\sum_{i=1}^m t_i=1$. If $m=1$ then $\mathbf{q}=g_1\mathbf{p}$ and the theorem follows directly by the G-invariance of Φ and ϕ.

So, assume $m\geqslant 2$. Then by (27) we find that

$$\frac{\Phi(\mathbf{q})}{\phi(\mathbf{q})}=\frac{\Phi(t_1g_1\mathbf{p}+\ldots+t_mg_m\mathbf{p})}{\phi(t_1g_1\mathbf{p}+\ldots+t_mg_m\mathbf{p})}. \tag{28}$$

By denoting

$$\alpha_i=\frac{h(t_i)\phi(g_i\mathbf{p})}{h(t_1)\phi(g_1\mathbf{p})+\ldots+h(t_m)\phi(g_m\mathbf{p})} \quad \text{for } i=1,2,\ldots,m,$$

we have $\sum_{i=1}^m\alpha_i=1$ with $\alpha_i>0$.

It follows that $g_i\mathbf{p}\in\mathscr{D}$ for all i, since $\mathscr{D}$ is G-invariant. From Lemma 2.1 applied for $\mathbf{p}_i=g_i\mathbf{p}$, we obtain

$$\frac{\Phi(t_1g_1\mathbf{p}+\ldots+t_mg_m\mathbf{p})}{\phi(t_1g_1\mathbf{p}+\ldots+t_mg_m\mathbf{p})}\prec_0\alpha_1\frac{\Phi(g_1\mathbf{p})}{\phi(g_1\mathbf{p})}+\ldots+\alpha_m\frac{\Phi(g_m\mathbf{p})}{\phi(g_m\mathbf{p})}, \tag{29}$$

where the symbol $\prec_0$ is defined as follows. For $\mathbf{u},\mathbf{w}\in\mathscr{W}$,

$$\mathbf{u}\prec_0\mathbf{w}\quad\text{iff}\quad\begin{cases}\mathbf{u}=\mathbf{w} & \text{if } \Phi \text{ is } (\mathrm{id},h)\text{-affine and } \phi \text{ is } (\mathrm{id},h)\text{-affine},\\ \mathbf{u}\prec_1\mathbf{w} & \text{if } \Phi \text{ is } (\mathrm{id},h;\prec_1)\text{-convex and } \phi \text{ is } (\mathrm{id},h)\text{-affine},\\ \mathbf{u}\prec_{K,1}\mathbf{w} & \text{if } \Phi \text{ is } (\mathrm{id},h;\prec_1)\text{-convex and } \phi \text{ is } (\mathrm{id},h)\text{-concave}.\end{cases}\tag{30}$$

From (28), (29), (23) and by the G-invariance of ϕ we get

$$\frac{\Phi(\mathbf{q})}{\phi(\mathbf{q})}\prec_0\alpha_1\frac{\Phi(\mathbf{p})}{\phi(\mathbf{p})}+\ldots+\alpha_m\frac{\Phi(\mathbf{p})}{\phi(\mathbf{p})}=\frac{\Phi(\mathbf{p})}{\phi(\mathbf{p})}.\tag{31}$$

Finally, by (31) and (30), we conclude that the required inequalities in (24)-(26) hold, which is our claim. □

COROLLARY 4.2. *Under assumptions of Theorem 4.1, let $f:\mathscr{W}\to\mathbb{R}$ be a function.*

Then for $\mathbf{p},\mathbf{q}\in\mathscr{D}_\phi^0$,

$$\mathbf{q}\prec_G\mathbf{p}\quad\textit{implies}\quad f\left(\frac{\Phi(\mathbf{q})}{\phi(\mathbf{q})}\right)\leqslant f\left(\frac{\Phi(\mathbf{p})}{\phi(\mathbf{p})}\right),$$

(*with equality in case* (**i**)) *with the additional assumption that f is $\prec_1$-increasing in case* (**ii**)*, or f is both $\prec_1$-increasing and $\prec_K$-increasing in case* (**iii**)*.*

In light of (8), for $\mathbf{x},\mathbf{y}\in\mathscr{W}$, we write

$$\mathbf{x}\prec_{1,H}\mathbf{y}\quad\text{if and only if}\quad\mathbf{x}\prec_1\mathbf{z}\prec_H\mathbf{y}\quad\text{for some } \mathbf{z}\in\mathscr{W}$$

(see e.g. (34)). Likewise,

$$\mathbf{x}\prec_{K,1,H}\mathbf{y}\quad\text{if and only if}\quad\mathbf{x}\prec_K\mathbf{z}\prec_1\mathbf{w}\prec_H\mathbf{y}\quad\text{for some } \mathbf{z},\mathbf{w}\in\mathscr{W}$$

(see e.g. (35)).

THEOREM 4.3. *Let $G\subset GL(\mathscr{V})$ be a group inducing group majorization $\prec_G$ on $\mathscr{V}$, and $H\subset GL(\mathscr{W})$ be a group inducing group majorization $\prec_H$ on $\mathscr{W}$. Let $\mathscr{D}$ be a G-invariant convex set in $\mathscr{V}$ and $\prec_1$ be a cone preorder on $\mathscr{W}$.*

Let $h:(0,1)\to(0,\infty)$ be a given multiplicative function. Suppose that $\phi:\mathscr{D}\to[0,\infty)$ is a G-invariant mapping on $\mathscr{D}$. Assume that for any $\mathbf{p}\in\mathscr{D}$, a mapping $\Phi:\mathscr{D}\to\mathscr{W}$ attains $\prec_H$-maximum on the set $\mathscr{D}_\phi^0\cap G\mathbf{p}$ at some point $\mathbf{p}_0\in\mathscr{D}_\phi^0\cap G\mathbf{p}$, i.e.,

$$\Phi(g\mathbf{p})\prec_H\Phi(\mathbf{p}_0)\quad\textit{for all } \mathbf{p}\in\mathscr{D} \textit{ and } g\in G.\tag{32}$$

Then for $\mathbf{p},\mathbf{q}\in\mathscr{D}_\phi^0$ the following three statements hold.

(i) If Φ is (id,h)-affine and ϕ is (id,h)-affine then

$$\mathbf{q}\prec_G\mathbf{p}\quad\textit{implies}\quad\frac{\Phi(\mathbf{q})}{\phi(\mathbf{q})}\prec_H\frac{\Phi(\mathbf{p}_0)}{\phi(\mathbf{p}_0)}.\tag{33}$$

(ii) If Φ *is* $(\mathrm{id},h;\prec_1)$*-convex and* ϕ *is* (id,h)*-affine then*

$$\mathbf{q} \prec_G \mathbf{p} \quad \textit{implies} \quad \frac{\Phi(\mathbf{q})}{\phi(\mathbf{q})} \prec_{1,H} \frac{\Phi(\mathbf{p}_0)}{\phi(\mathbf{p}_0)}. \tag{34}$$

(iii) If Φ *is* $(\mathrm{id},h;\prec_1)$*-convex and* ϕ *is* (id,h)*-concave then*

$$\mathbf{q} \prec_G \mathbf{p} \quad \textit{implies} \quad \frac{\Phi(\mathbf{q})}{\phi(\mathbf{q})} \prec_{K,1,H} \frac{\Phi(\mathbf{p}_0)}{\phi(\mathbf{p}_0)}. \tag{35}$$

Proof. As in the proof of Theorem 4.1, by applying Lemma 2.1 we get

$$\frac{\Phi(\mathbf{q})}{\phi(\mathbf{q})} = \frac{\Phi(t_1 g_1 \mathbf{p} + \ldots + t_m g_m \mathbf{p})}{\phi(t_1 g_1 \mathbf{p} + \ldots + t_m g_m \mathbf{p})} \prec_0 \alpha_1 \frac{\Phi(g_1 \mathbf{p})}{\phi(g_1 \mathbf{p})} + \ldots + \alpha_m \frac{\Phi(g_m \mathbf{p})}{\phi(g_m \mathbf{p})} \tag{36}$$

with $\prec_0$ defined by (30) and $g_1,\ldots,g_m$ and $t_1,\ldots,t_m$ given by (27).

On account of (36), (32) and the G-invariance of ϕ, we have

$$\frac{\Phi(\mathbf{q})}{\phi(\mathbf{q})} \prec_0 \alpha_1 \frac{\Phi(g_1 \mathbf{p})}{\phi(\mathbf{p}_0)} + \ldots + \alpha_m \frac{\Phi(g_m \mathbf{p})}{\phi(\mathbf{p}_0)} \prec_H \alpha_1 \frac{\Phi(\mathbf{p}_0)}{\phi(\mathbf{p}_0)} + \ldots + \alpha_m \frac{\Phi(\mathbf{p}_0)}{\phi(\mathbf{p}_0)} = \frac{\Phi(\mathbf{p}_0)}{\phi(\mathbf{p}_0)}. \tag{37}$$

Now, it follows from (37) and (30) that inequalities in (33)-(35) are valid. □

COROLLARY 4.4. *Under assumptions of Theorem 4.3, let* $f : \mathscr{W} \to \mathbb{R}$ *be an* $\prec_H$*-increasing function.*

Then for $\mathbf{p},\mathbf{q} \in \mathscr{D}^0_\phi$*,*

$$\mathbf{q} \prec_G \mathbf{p} \quad \textit{implies} \quad f\left(\frac{\Phi(\mathbf{q})}{\phi(\mathbf{q})}\right) \leqslant f\left(\frac{\Phi(\mathbf{p}_0)}{\phi(\mathbf{p}_0)}\right),$$

with the additional assumption that f *is* $\prec_1$*-increasing in case* **(ii)***, or* f *is both* $\prec_1$*-increasing and* $\prec_K$*-increasing in case* **(iii)***.*

COROLLARY 4.5. *Under assumptions of Theorem 4.3, let* $h(t) = t^s$*,* $t \in (0,1)$*,* $s \in \mathbb{R}$*.*

If Φ *is* $(s,\prec_1)$*-convex and* ϕ *is* s*-concave then for* $\mathbf{p},\mathbf{q} \in \mathscr{D}^0_\phi$ *statement (35) is met.*

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