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EIGENVALUE DECAY OF INTEGRAL OPERATORS GENERATED BY POWER SERIES–LIKE KERNELS

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Abstract. We deduce decay rates for eigenvalues of integral operators generated by power series-like kernels on a subset X of either $\mathbb{R}^q$ or $\mathbb{C}^q$. A power series-like kernel is a Mercer kernel having a series expansion based on an orthogonal family $\{f_\alpha\}_{\alpha\in\mathbb{Z}_+^q}$ in $L^2(X,\mu)$, in which μ is a complete measure on X. As so, we show that the eigenvalues of the integral operators are given by an explicit formula defined by the coefficients in the series expansion of the kernel and the elements of the orthogonal family. The inequalities and, in particular, the decay rates for the sequence of eigenvalues are obtained from decay assumptions on the sequence of coefficients in the expansion of the kernel and on the sequence $\{\|f_\alpha\|\}_{\alpha\in\mathbb{Z}_+^q}$.

1. Introduction

Let X be a nonempty subset of either $\mathbb{R}^q$ or $\mathbb{C}^q$ endowed with a complete measure μ and consider the space $L^2(X) := L^2(X,\mu)$. If K is a convenient positive definite kernel on X then the formula

$$\mathscr{K}(f) = \int_X K(\cdot,w)f(w)d\mu(w), \quad f \in L^2(X), \tag{1}$$

defines a compact, positive and self-adjoint operator $\mathscr{K} : L^2(X) \to L^2(X)$. If we order the eigenvalues of $\mathscr{K}$ such as

$$\lambda_1(\mathscr{K}) \geqslant \lambda_2(\mathscr{K}) \geqslant \cdots \geqslant 0,$$

taking into account multiplicities, the basic decay $\lambda_n(\mathscr{K}) = o(n^{-1})$, as $n \to \infty$, holds. A common problem in Applied Functional Analysis is then to search for improved decay rates for the sequence $\{\lambda_n(\mathscr{K})\}$ under additional assumptions on either K or $\mathscr{K}$ itself. Usually, these assumptions depend on the additional structure the set X may carry. We refer the reader to [1, 3, 5, 6, 9, 12] and references therein for decay rates in some specific cases. Related material on the same topic can be found in [4, 14] while general information is available in the classical references [8, 10]. We observe that in some cases, it is also quite natural to analyze the sharpness of the rates obtained within

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Paper MIA-17-51

the setting considered, but that will be not an issue here. In any case, we advise the reader that our estimates along the paper are sharp.

In this paper, we will consider measurable kernels defined by expansions of the form

$$K(z,w) = \sum_{\alpha \in \mathbb{Z}_+^q} a_\alpha f_\alpha(z)\overline{f_\alpha(w)}, \quad z,w \in X, \tag{2}$$

in which $\{a_\alpha\} \subset (0,\infty)$ and the sequence $\{f_\alpha : \alpha \in \mathbb{Z}_+^q\}$ is orthogonal (but not orthonormal) in $L^2(X)$, that is,

$$\langle f_\alpha, f_\beta \rangle_2 := \int_X f_\alpha(z)\overline{f_\beta(z)}d\mu(z) = 0, \quad \alpha \neq \beta,$$

and $0 \neq \langle f_\alpha, f_\alpha \rangle_2 \neq 1$, $\alpha \in \mathbb{Z}_+^q$. The introduction of a setting as above, using an orthogonal family in the representation (2), instead of an orthonormal one, has some reasons as we now explain. Depending on the set X, one can consider kernels defined by monomial expansions, that is,

$$f_\alpha(z) = z^\alpha, \quad z \in X, \quad \alpha \in \mathbb{Z}_+^q.$$

The computations with sequences of this type may be easier to handle than those with a general orthonormal sequence, which may include more complicated functions. For instance, if the setting is carefully defined or chosen, the eigenvalues of the integral operator can be computed through simple formulas involving elements of the sequences $\{a_\alpha\}$ and $\{\|f_\alpha\|_2\}$, where $\|\cdot\|_2$ denotes the usual norm in $L^2(X)$. As a consequence, the decay for the sequence of eigenvalues can be obtained from decays for the sequences $\{a_\alpha\}$ and $\{\|f_\alpha\|_2\}$ directly. A prominent example is the case in which X is a sphere or a disk in $\mathbb{C}^q$ and μ is the Lebesgue measure on X. The setting we will adopt contemplates the features mentioned above and that justifies the use of the terminology "power series-like kernels" in the title of the paper.

If the integral operator has a series representation that agrees with that provided by the spectral theorem for compact and self-adjoint operators on a Hilbert space, then the eigenvalues can be somehow computed. In Section 2, we describe two paths one can follow in order to achieve such a representation in the setting we have chosen. Keeping the setting described in Section 2, we will use Sections 3 and 4 to deduce decay rates for the sequence of eigenvalues of $\mathscr{K}$ from decay assumptions on $\{a_\alpha\}$ and $\{\|f_\alpha\|_2\}$. In Section 5, we discuss upon two relevant examples that fit into the setting considered in the paper, a case we intended to cover from the beginning.

2. Matching the spectral theorem

Let K and $\mathscr{K}$ be as in (2) and (1) respectively. A first critical step is to define reasonable assumptions in order that $\mathscr{K}$ be a well defined compact and positive operator on $L^2(X)$ possessing a series representation agreeing with that provided by the well-know spectral theorem for compact and self-adjoint operators on Hilbert spaces. In this section, we assume $\mathbb{Z}_+^q$ is partially ordered and describe a general way of achieving that. All convergence and summation appearing ahead need to be in consonance with the partial order adopted in $\mathbb{Z}_+^q$.

Compactness and positivity of $\mathscr{K}$ as an operator on $L^2(X)$ can be reached as follows.

PROPOSITION 1. *If* $\sum_\alpha a_\alpha \|f_\alpha\|_2^2 < \infty$ *then* $\mathscr{K}$ *is a well-defined compact, positive and self-adjoint operator on* $L^2(X)$.

Proof. Assume $\sum_\alpha a_\alpha \|f_\alpha\|_2^2 < \infty$. An application of [7, p. 55] reveals that

$$\int_X \sum_\alpha a_\alpha |f_\alpha(z)|^2 d\mu(z) \int_X \sum_\alpha a_\alpha |f_\alpha(w)|^2 d\mu(w) = \left(\sum_\alpha a_\alpha \|f_\alpha\|_2^2\right)^2.$$

Coupling this information with an application of the Cauchy-Schwarz inequality leads to

$$\int_X \int_X \left(\sum_\alpha a_\alpha^{1/2}|f_\alpha(z)| a_\alpha^{1/2}|f_\alpha(w)|\right)^2 d\mu(z)d\mu(w) \leqslant \left(\sum_\alpha a_\alpha \|f_\alpha\|_2^2\right)^2.$$

Using the same symbol $\|\cdot\|_2$ to denote the usual norm in $L^2(X\times X) := L^2(X\times X, \mu\times\mu)$, the Fubini-Tonelli Theorem authorizes us to write

$$\begin{aligned}||K||_2^2 &= \int_{X\times X} \left|\sum_\alpha a_\alpha f_\alpha(z)\overline{f_\alpha(w)}\right|^2 d(\mu\times\mu)(z,w)\\ &\leqslant \int_X \int_X \left(\sum_\alpha a_\alpha^{1/2}|f_\alpha(z)| a_\alpha^{1/2}|f_\alpha(w)|\right)^2 d\mu(z)d\mu(w).\end{aligned}$$

Therefore, $||K||_2^2 < \infty$ and a well-known result from functional analysis implies that $\mathscr{K}$ is a compact operator from $L^2(X)$ into itself ([2, p. 86]). As for the positivity, Hölder's inequality validates the inequality

$$\sum_\alpha a_\alpha \int_{X\times X} \left|f_\alpha(z)\overline{f_\alpha(w)} f(z)\overline{f(w)}\right| d(\mu\times\mu)(z,w) \leqslant \|f\|_2^2 \sum_\alpha a_\alpha \|f_\alpha\|_2^2,$$

for $f \in L^2(X)$. As so, we can interchange the integral and summation symbols to deduce that

$$\langle \mathscr{K}(f), f\rangle_2 = \sum_\alpha a_\alpha \int_{X\times X} f_\alpha(z) f_\alpha(w) f(z)\overline{f(w)} d(\mu\times\mu)(z,w), \quad f\in L^2(X).$$

Iterating once again leads to

$$\langle \mathscr{K}(f), f\rangle_2 = \sum_\alpha a_\alpha \int_X f_\alpha(z) f(z) d\mu(z) \int_X f_\alpha(w)\overline{f(w)}\, d\mu(w), \quad f\in L^2(X),$$

that is,

$$\langle \mathscr{K}(f), f\rangle_2 = \sum_\alpha a_\alpha |\langle f, f_\alpha\rangle_2|^2 \geqslant 0, \quad f\in L^2(X).$$

The proof is complete. □

REMARK 1. The equality

$$\int_X K(x,x)\,d\mu(x) = \int_X \sum_\alpha a_\alpha f_\alpha(x)\overline{f_\alpha(x)}\,d\mu(x) = \sum_\alpha a_\alpha \|f_\alpha\|_2^2,$$

shows that the assumption $\sum_\alpha a_\alpha \|f_\alpha\|_2^2 < \infty$ used in the previous theorem is equivalent to the integrability of the function $x \in X \to K(x,x)$. In particular, the operator $\mathscr{K}$ is also trace-class.

Let us move to a series representation for $\mathscr{K}$. A sole representation can be reached as follows.

PROPOSITION 2. *Assume the following assumptions hold:*
(i) $\sup_\alpha |f_\alpha(z)| < \infty$ *for every* z *in* X*;*
(ii) $\sum_\alpha a_\alpha \|f_\alpha\|_2 < \infty$.
Then $\mathscr{K}$ *has a series expansion in the form*

$$\mathscr{K}(f) = \sum_\alpha a_\alpha \langle f, f_\alpha\rangle_2 f_\alpha, \quad f \in L^2(X).$$

Proof. Fix $z \in X$. If $\sup_\alpha |f_\alpha(z)| < \infty$, Hölder's inequality implies that

$$\sum_\alpha a_\alpha \int_X |f_\alpha(z)||\overline{f_\alpha(w)}||f(w)|d\mu(w) \leqslant \|f\|_2 \sup_\alpha |f_\alpha(z)| \sum_\alpha a_\alpha \|f_\alpha\|_2 < \infty.$$

So, we can interchange the summation and integral symbols to deduce that

$$\begin{aligned}\mathscr{K}(f)(z) &= \int_X \sum_\alpha a_\alpha f_\alpha(z)\overline{f_\alpha(w)} f(w) d\mu(w)\\ &= \sum_\alpha a_\alpha \langle f, f_\alpha\rangle_2 f_\alpha(z), \quad f \in L^2(X).\end{aligned}$$

The proposition is proved. □

REMARK 2. If $\mu(X) < \infty$, assumptions (i) and (ii) can be replaced with the sole assumption $\sum_\alpha a_\alpha ||f_\alpha||_2^2 < \infty$. Indeed, due to the inequality

$$\begin{aligned}\int_X F(z)d\mu(z) &= \sum_\alpha a_\alpha \int_X |f_\alpha(z)|d\mu(z) \int_X |\overline{f_\alpha(w)}||f(w)|d\mu(w)\\ &\leqslant \mu(X)^{1/2} ||f||_2 \sum_\alpha a_\alpha ||f_\alpha||_2^2,\end{aligned}$$

the function F given by the formula

$$F(z) = \sum_\alpha a_\alpha \int_X |f_\alpha(z)||\overline{f_\alpha(w)}||f(w)|\,d\mu(w), \quad z \in X,$$

would be integrable. Consequently, it would be finite a.e. and, therefore,

$$\sum_{\alpha} a_\alpha \int_X |f_\alpha(z)||\overline{f_\alpha(w)}||f(w)|d\mu(w) < \infty, \quad z \in X \text{ a.e..}$$

But, Proposition 2.20 in [7] reveals that $\{z \in X : |f_\alpha(z)| = \infty\}$ has measure 0, as long as $f_\alpha \in L^1(X)$.

A basic representation for $\mathscr{K}$ implied by the assumption used in Proposition 1 is as follows. Here, it is convenient to assume that the partial order $\preceq$ in $\mathbb{Z}_+^q$ does not deviate from the expected in the following sense: $\alpha \preceq \beta$ whenever $|\alpha| \leqslant |\beta|$.

PROPOSITION 3. *If $\sum_\alpha a_\alpha \|f_\alpha\|_2^2 < \infty$ then $\mathscr{K}$ has a series expansion in the form*

$$\mathscr{K}(f) = \sum_{|\alpha|=0}^{\infty} a_\alpha \langle f, f_\alpha\rangle_2 f_\alpha, \quad f \in L^2(X).$$

Proof. If $\sum_\alpha a_\alpha \|f_\alpha\|_2^2 < \infty$, we can imitate the proof of Proposition 1 to show that the formula

$$\mathscr{K}_n(f)(x) = \int_X \left(\sum_{|\alpha|=0}^{n-1} a_\alpha f_\alpha(x)\overline{f_\alpha(y)} \right) f(y)d\mu(y), \quad x \in X, \quad f \in L^2(X),$$

generates a well-defined integral operator $\mathscr{K}_n : L^2(X) \to L^2(X)$. Hölder's inequality implies that

$$\|\mathscr{K}(f) - \mathscr{K}_n(f)\|_2^2 \leqslant \|f\|_2^2 \left(\sum_{|\alpha|=n}^{\infty} a_\alpha \|f_\alpha\|_2^2 \right)^2, \quad f \in L^2(X).$$

But,

$$\mathscr{K}_n(f)(x) = \sum_{|\alpha|=0}^{n-1} a_\alpha \langle f, f_\alpha\rangle_2 f_\alpha(x), \quad x \in X, \quad f \in L^2(X),$$

and the result follows.

The convenience of the assumption from Proposition 1 is justified by the following result.

COROLLARY 1. *If $\sum_\alpha a_\alpha \|f_\alpha\|_2^2 < \infty$ then the series representation for $\mathscr{K}$ provided by Proposition 1 holds.*

Proof. The convergence $\sum_\alpha a_\alpha \|f_\alpha\|_2^2 < \infty$ implies that each one of the series $\sum_\alpha a_\alpha \langle f, f_\alpha\rangle_2 f_\alpha$ is absolutely convergent. In particular, the series is unconditionally convergent in $L^2(X)$. As so, its convergence in $L^2(X)$ occurs in any order. □

If the outcomes in either Propositions 1 and 2 hold then each number

$$\lambda_\alpha(\mathscr{K}) = a_\alpha \|f_\alpha\|_2^2, \quad \alpha \in \mathbb{Z}_+^q,$$

is an eigenvalue of $\mathscr{K}$ with corresponding eigenfunction f_α. Since $\{f_\alpha : \alpha \in \mathbb{Z}_+^q\}$ is orthogonal, these are the only nonzero eigenvalues of $\mathscr{K}$. Indeed, if $\mathscr{K}(g) = \lambda g$ for some $g \neq 0$ and $0 \neq \lambda \notin \{a_\alpha \langle f_\alpha, f_\alpha \rangle_2 : \alpha \in \mathbb{Z}_+^q\}$ then g is orthogonal to every f_α and, consequently,

$$\lambda g = \mathscr{K}(g) = \sum_\alpha a_\alpha \langle g, f_\alpha \rangle_2 f_\alpha = 0,$$

a contradiction.

3. Eigenvalue estimates

From now on, we will assume $\mathbb{Z}_+^q$ is partially ordered as described in the paragraph preceding Proposition 3.

Throughout the section we will consider the integral operator (1) generated by a kernel as described in (2) and fulfilling the following assumptions: $\sup_\alpha |f_\alpha(z)| < \infty$ for every z in X, $\sum_\alpha a_\alpha \|f_\alpha\|_2 < \infty$ and $\{\|f_\alpha\|_2\}$ is bounded. As so, the representation in Proposition 3 holds and the convergence of the series $\sum_\alpha a_\alpha \|f_\alpha\|_2^2$ implies that $\lim_{|\alpha| \to \infty} \lambda_\alpha(\mathscr{K}) = 0$. The intention here is to describe decay rates for $\{\lambda_\alpha(\mathscr{K})\}$ as $|\alpha| \to \infty$ from decay rates for both $\{a_\alpha\}$ and $\{\|f_\alpha\|_2\}$. The boundedness assumption on $\{\|f_\alpha\|_2\}$ is plainly justified, since it is an immediate consequence of any basic decay assumption on $\{\|f_\alpha\|_2\}$ as $|\alpha| \to \infty$.

The following result from calculus, not easily found in the literature, plays an important role in some of our proofs. An elementary proof of such result is included for the convenience of the reader while a more elaborated one can be found in [11].

LEMMA 1. *Let $\{d_n\}$ be a decreasing sequence of positive real numbers and r and s fixed positive real numbers. If $\sum_{n=1}^\infty n^r d_n^s$ converges then*

$$d_n = o(n^{-(r+1)/s}) \quad as \quad n \to \infty.$$

Proof. Let $\varepsilon > 0$. If the sequence $\{n^r d_n^s\}$ is summable then it converges to 0. As so, we can select $N > 0$ so that

$$\sum_{n=k+1}^{2k} n^r d_n^s < \frac{\varepsilon}{2^{2r+2}}, \quad k > N.$$

It is now clear that

$$k^{r+1} d_{2k}^s = \sum_{n=k+1}^{2k} k^r d_{2k}^s \leqslant \sum_{n=k+1}^{2k} n^r d_n^s < \frac{\varepsilon}{2^{2r+2}}, \quad k > N,$$

that is,

$$(2k)^{r+1} d_{2k}^s < \frac{\varepsilon}{2^{r+1}} < \varepsilon, \quad k > N.$$

On the other hand, since

$$(2k+1)^{r+1} d_{2k+1}^s = \left(1 + \frac{1}{2k}\right)^{r+1} (2k)^{r+1} d_{2k+1}^s \leqslant 2^{r+1} (2k)^{r+1} d_{2k}^s,$$

then

$$(2k+1)^{r+1}d_{2k+1}^s < \varepsilon, \quad k > N.$$

Thus,

$$k^{r+1}d_k^s < \varepsilon, \quad k > 2N+1.$$

We have proved that $\{n^{r+1}d_n^s\}$ converges to 0 as $n \to \infty$ and that implies the assertion of the lemma. □

The basic setting adopted in this section provides the convergence of $\{\sum_{|\alpha|=n}\lambda_\alpha(\mathscr{K})\}$ to 0 but not the monotonicity of the sequence. In the next two results, we will assume that $\{\sum_{|\alpha|=n}\lambda_\alpha(\mathscr{K})\}$ decreases to 0 but will not mention that in the statements of the results.

PROPOSITION 4. *If there exists a nonnegative number r such that $\|f_\alpha\|_2 = O(|\alpha|^{-r})$ as $|\alpha| \to \infty$ then*

$$\lambda_\alpha(\mathscr{K}) = o(|\alpha|^{-1-r}) \quad as \quad |\alpha| \to \infty.$$

Proof. If $\|f_\alpha\|_2 = O(|\alpha|^{-r})$ as $|\alpha| \to \infty$ then there exists $C > 0$ so that

$$\|f_\alpha\|_2 \leqslant C|\alpha|^{-r}, \quad \alpha \in \mathbb{Z}_+^q.$$

This leads to the obvious inequality

$$n^r \sum_{|\alpha|=n} \lambda_\alpha(\mathscr{K}) \leqslant C \sum_{|\alpha|=n} a_\alpha \|f_\alpha\|_2, \quad n = 1,2,\ldots, \tag{3}$$

and that implies

$$\sum_{n=0}^{\infty} n^r \sum_{|\alpha|=n} \lambda_\alpha(\mathscr{K}) \leqslant C \sum_{n=0}^{\infty} \sum_{|\alpha|=n} a_\alpha \|f_\alpha\|_2 = C \sum_\alpha a_\alpha \|f_\alpha\|_2 < \infty. \tag{4}$$

Applying the lemma, we deduce that

$$\sum_{|\alpha|=n} \lambda_\alpha(\mathscr{K}) = o(n^{-1-r}) \quad \text{as} \quad n \to \infty.$$

The assertion of the proposition follows. □

If a decay for the sequence $\{a_\alpha\}$ is to be taken into account, then the procedure used in the proof of the previous proposition can be adapted as follows.

PROPOSITION 5. *If there exists a nonnegative number r such that $\|f_\alpha\|_2 = O(|\alpha|^{-r})$ as $|\alpha| \to \infty$ and also a positive number s so that $a_\alpha = O(|\alpha|^{-s})$ as $|\alpha| \to \infty$ then*

$$\lambda_\alpha(\mathscr{K}) = O(|\alpha|^{-2r-s}) \quad as \quad |\alpha| \to \infty$$

and

$$\sum_{|\alpha|=n} \lambda_\alpha(\mathscr{K}) = O(|\alpha|^{-1-2r-s+q}) \quad as \quad |\alpha| \to \infty.$$

Proof. The first decay is obvious. As for the other, following the steps of the proof of Proposition 4 leads to

$$n^{2r+s}\sum_{|\alpha|=n}\lambda_\alpha(\mathscr{K})\leqslant C'b_n^q,\quad n=1,2,\ldots,$$

for some $C'>0$, in which b_n^q is the cardinality of $B_n^q:=\{\alpha\in\mathbb{Z}_+^q:|\alpha|=n\}$. Clearly,

$$b_n^q:=\binom{q-1+n}{q-1}=O(n^{q-1}),\quad \text{as}\quad n\to\infty,$$

and the previous inequality implies that

$$n^{2r+s-q+1}\sum_{|\alpha|=n}\lambda_\alpha(\mathscr{K})\leqslant C'',\quad n=1,2,\ldots,$$

for some $C''>0$. The assertion of the proposition follows. □

The second decay provided by the previous proposition is meaningful whenever $2r+s\geqslant q$, even though we did not enforce that in the statement of the result.

4. Eigenvalue estimates taking into account the order in $\mathbb{Z}_+^q$

In this section, the basic assumptions introduced in previous section will be kept. Since $\sum_\alpha a_\alpha\|f_\alpha\|_2^2<\infty$, the sequence $\{\lambda_\alpha(\mathscr{K})\}$ decreases to 0 with respect to the order $\preceq$. We will re-index the sequence as

$$\lambda_1(\mathscr{K})\geqslant\lambda_2(\mathscr{K})\geqslant\cdots\geqslant 0,$$

abandoning the multi-index counter appearing in the previous notation, and deduce decay rates for $\{\lambda_n(\mathscr{K})\}$, as $n\to\infty$.

The first result in this new notation is as follows.

THEOREM 1. *If there exists a real number r so that $\|f_\alpha\|_2=O(|\alpha|^{-r})$ as $|\alpha|\to\infty$ then*

$$\lambda_n(\mathscr{K})=o(n^{-1-r/q})\quad as\quad n\to\infty.$$

Proof. It suffices to modify the proof of Proposition 5 in accordance with the re-ordering mentioned above. First of all, inequality (4) can be upgraded to

$$\sum_{n=0}^{\infty}b_n^q n^r\lambda_{b_0^q+\cdots+b_n^q}(\mathscr{K})<\infty.$$

Since $\sum_{j=0}^n b_j^q=b_n^{q+1}$ and $n^{q-1}=O(b_n^q)$, as $n\to\infty$, we obtain

$$\sum_{n=0}^{\infty}n^{r+q-1}\lambda_{b_n^{q+1}}(\mathscr{K})<\infty.$$

Since $b_n^{q+1} = O(n^q)$ as $n \to \infty$, we can select a positive integer l so that

$$b_n^{q+1} \leqslant (l\,n)^q, \quad n = 1, 2, \ldots.$$

Returning to the previous sum, we may conclude that

$$\sum_{n=0}^{\infty} n^{r+q-1} \lambda_{(l\,n)^q}(\mathscr{K}) < \infty.$$

In particular,

$$\sum_{n=0}^{\infty} (l\,n)^{r+q-1} \lambda_{(l\,n)^q}(\mathscr{K}) < \infty.$$

The fact that $\{\lambda_n(\mathscr{K})\}$ decreases now implies that

$$\sum_{n=0}^{\infty} (l\,n + j)^{r+q-1} \lambda_{(l\,n+j)^q}(\mathscr{K}) < \infty, \quad j = 0, 1, \ldots, l-1.$$

Hence,

$$\sum_{n=0}^{\infty} n^{r+q-1} \lambda_{n^q}(\mathscr{K}) < \infty.$$

An elementary manipulation of the series leads to

$$\sum_{n=0}^{\infty} n^{r/q} \lambda_n(\mathscr{K}) < \infty.$$

An application of Lemma 1 provides the decay in the statement of the theorem. □

Our second result requires the following lemma.

LEMMA 2. *Let $\{c_n\}$ be a sequence of nonnegative reals decreasing to 0. If there exist positive integers l and m and a real number t so that*

$$c_{(l\,n)^m} \leqslant \frac{C}{n^t}, \quad n \geqslant n_0 \tag{5}$$

for some n_o then

$$c_n \leqslant \frac{C'}{n^{t/m}}, \quad n \geqslant n_1$$

for some $C' > 0$ and some n_1.

Proof. Let l, m and t be as in the statement of the lemma. The inequality (5) implies the following one:

$$c_{n^m} \leqslant \frac{Cl^t}{n^t}, \quad n \geqslant ln_0.$$

Define $C_1 = Cl^t$ and pick N in the set $\{ln_0, ln_0+1, \ldots\}$. Since $c_{N^m} \leqslant C_1 N^{-t}$ and the sequence $\{c_n\}$ decreases, then

$$c_{N^m+1} \leqslant \frac{C_1}{(N^m)^{t/m}} = \frac{C_1}{(N^m+1)^{t/m}} \left(\frac{N^m+1}{N^m}\right)^{t/m}.$$

Inductively,

$$c_{N^m+j} \leqslant \frac{C_1}{(N^m+j)^{t/m}} \left(\frac{N^m+j}{N^m}\right)^{t/m}, \quad j = 1, 2, \ldots, (N+1)^m - N^m - 1.$$

However, since

$$(N+1)^m - N^m - 1 \leqslant N^m \left[\left(1+\frac{1}{N}\right)^m - 1\right] \leqslant (2^m - 1)N^m$$

it follows that

$$\frac{N^m+j}{N^m} \leqslant 2^m.$$

Hence,

$$c_{N^m+j} \leqslant \frac{C_1 2^t}{(N^m+j)^{t/m}}, \quad j = 1, 2, \ldots, (N+1)^m - N^m - 1.$$

Therefore, the desired inequality follows with $C' = C_1 2^t$ and $n_1 = N^m$. □

THEOREM 2. *If there exists a nonnegative number r such that $\|f_\alpha\|_2 = O(|\alpha|^{-r})$ as $|\alpha| \to \infty$ and also a positive number s so that $a_\alpha = O(|\alpha|^{-s})$ as $|\alpha| \to \infty$ then*

$$\lambda_n(\mathscr{K}) = O(n^{-(2r+s)/q}) \quad as \quad n \to \infty.$$

Proof. Here, inequality (3) can be put into the form

$$n^{2r+s} b_n^q \lambda_{b_n^{q+1}}(\mathscr{K}) \leqslant C b_n^q, \quad n = 1, 2, \ldots,$$

for some $C > 0$, and that leads to

$$n^{2r+s} \lambda_{(ln)^q}(\mathscr{K}) \leqslant C', \quad n = 1, 2, \ldots,$$

for some $C' > 0$, in which l is as described in the proof of the previous theorem. Applying Lemma 2, we conclude that

$$n^{(2r+s)/q} \lambda_n(\mathscr{K}) \leqslant C'', \quad n = 1, 2, \ldots,$$

for some $C'' > 0$. The proof is complete. □

5. Two concrete examples

The first example we would like to discuss is the one that served as motivation to us: X is the unit sphere Ω_{2q} in $\mathbb{C}^q$, $q \geqslant 2$, μ_q is the usual probability measure on X while

$$f_\alpha(z) = z^\alpha, \quad \alpha \in \mathbb{Z}_+^q.$$

Here, the order $\preceq$ in $\mathbb{Z}_+^q$ needs to match the following requirement: if $\alpha \preceq \beta$ then either $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$ and $\alpha! \leqslant \beta!$.

It is well-known ([13, p. 16], [15, p. 13]) that

$$\int_{\Omega_{2q}} z^\alpha \bar{z}^\beta \, d\mu_q(z) = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \dfrac{(q-1)!\alpha!}{(q-1+|\alpha|)!} & \text{if } \alpha = \beta. \end{cases}$$

Hence, the family $\{f_\alpha\}$ is orthogonal,

$$\|f_\alpha\|_2 \leqslant 1, \quad \alpha \in \mathbb{Z}_+^q,$$

and

$$\sup_z \sup_\alpha |f_\alpha(z)| \leqslant 1.$$

In particular, the setting adopted in the previous sections is a reality in this case. Next, let us determine how the sequence $\{\|f_\alpha\|\}$ decays. First, observe that

$$\frac{(q-1)!\alpha!}{(q-1+|\alpha|)!} = \binom{q-1+|\alpha|}{|\alpha|}^{-1} \frac{\alpha!}{|\alpha|!} = \left(b^q_{|\alpha|}\right)^{-1} \frac{\alpha!}{|\alpha|!}, \quad \alpha \in \mathbb{Z}_+^q.$$

On the other hand, we know that $|\alpha|^{q-1} = O(b^q_{|\alpha|})$ as $|\alpha| \to \infty$ while an induction argument on the dimension q reveals that $\alpha! \leqslant |\alpha|!$, $\alpha \in \mathbb{Z}_+^q$. Thus,

$$\|f_\alpha\|_2 = O(|\alpha|^{-(q-1)/2}), \quad \text{as} \quad |\alpha| \to \infty.$$

The decay in Proposition 4 becomes

$$\lambda_\alpha(\mathscr{K}) = o(|\alpha|^{-1-(q-1)/2}) \quad \text{as} \quad |\alpha| \to \infty$$

while the one in Proposition 5 takes the form

$$\lambda_\alpha(\mathscr{K}) = O(|\alpha|^{-s}) \quad \text{as} \quad |\alpha| \to \infty.$$

Moving to Section 4, the decay in Theorem 1 is

$$\lambda_n(\mathscr{K}) = o(n^{-1-(q-1)/2q}), \quad \text{as} \quad n \to \infty,$$

while that in Theorem 2 is

$$\lambda_n(\mathscr{K}) = O(n^{-1-(s-1)/q}), \quad \text{as} \quad n \to \infty.$$

The previous setting can be extended to the case in which X is the unit disk $\Delta_q := \{z \in \mathbb{C}^q : z\overline{z} \leqslant 1\}$ in $\mathbb{C}^q$ and ν_q is the probability measure on Δ_q obtained from the usual Lebesgue measure on $\mathbb{C}^q$. Since the measures ν_q and μ_q are related to each other by the formula ([13, p. 13], [15, p. 9])

$$\int_{\Delta_q} f d\nu_q = 2q \int_0^1 r^{2q-1} dr \int_{\Omega_{2q}} f(rz) d\mu_q(z),$$

we deduce that

$$\begin{aligned}\int_{\Delta_q} \xi^\alpha \overline{\xi}^\beta d\nu_q(\xi) &= 2q \int_0^1 r^{2q-1+|\alpha|+|\beta|} dr \int_{\Omega_{2q}} z^\alpha \overline{z^\beta} d\mu_q(z) \\ &= \frac{2q}{2q+|\alpha|+|\beta|} \int_{\Omega_{2q}} z^\alpha \overline{z^\beta} d\mu_q(z).\end{aligned}$$

Therefore,

$$\int_{\Delta_q} \xi^\alpha \overline{\xi}^\beta d\nu_q(z) = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \dfrac{q!\alpha!}{(q+|\alpha|)!} & \text{if } \alpha = \beta, \end{cases}$$

and the same decays described above hold in the present case.

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