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AN APPLICATION OF JENSEN'S INEQUALITY IN DETERMINING THE ORDER OF MAGNITUDE OF MULTIPLE FOURIER COEFFICIENTS OF FUNCTIONS OF BOUNDED ϕ–VARIATION

BHIKHA LILA GHODADRA

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Abstract. For a Lebesgue integrable complex-valued function f defined over the n-dimensional torus $\mathbb{T}^n := [0,2\pi)^n$, $n \in \mathbb{N}$, let $\hat{f}(\mathbf{k})$ denote the Fourier coefficient of f, where $\mathbf{k} = (k_1,\ldots,k_n) \in \mathbb{Z}^n$. The Riemann-Lebesgue lemma shows that $\hat{f}(\mathbf{k}) = o(1)$ as $|\mathbf{k}| \to 0$ for any $f \in L^1(\mathbb{T}^n)$. However, it is known that, these Fourier coefficients can tend to zero as slowly as we wish. The definitive results are due to V. Fülöp and F. Móricz for functions of bounded variation, and due to B. L. Ghodadra for functions of bounded p-variation. In this paper, defining the notion of bounded ϕ-variation for a function from $[0,2\pi]^n$ to $\mathbb{C}$ in two different ways, we prove that this is the case for Fourier coefficients of such functions also. Interestingly, in proving our main results we use the famous Jensen's inequality for integrals. Our new results with $\phi(x) = x^p$ $(p \geqslant 1)$ gives our earlier results [Acta Math. Hungar, 128 (4) (2010), 328–343].

1. Introduction

For a function of two variables several definitions of bounded variation are given and various properties are studied (see, for example, [5, 1]). In 2002 F. Móricz [6] studied the order of magnitude of double Fourier coefficients with the help of Riemann-Stieltjes integral of functions of two variables and in 2004 V. Fülöp and F. Móricz [4] studied the order of magnitude of multiple Fourier coefficients of functions of bounded variation in the sense of Vitali and, Hardy and Krause (see [2]) in a straightforward way without using Riemann-Stieltjes integral. In [3] we have defined the concept of *bounded* p*-variation* $(p \geqslant 1)$ for a function of several variables in two different ways and studied the order of magnitude of Fourier coefficients for such functions. Here, we define the concept of *bounded* ϕ*-variation* and study the order of magnitude of Fourier coefficients for such functions. Interestingly, in proving our main results (Theorems 1 and 2 below) we use the famous Jensen's inequality for integrals. Our new results with $\phi(x) = x^p$ $(p \geqslant 1)$ gives our earlier results [3].

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2. Notation and Definitions

Let R be the rectangle $R = [a_1, b_1] \times \ldots \times [a_n, b_n]$. By a (finite) partition $\mathscr{P}$ of R we mean the set $\mathscr{P} = \{R_1, \ldots, R_m\}$, in which R_i's are pairwise disjoint (no two have common interior) subrectangles of R having their sides (faces) parallel to the standard coordinate hyperplanes and whose union is R. Let $f = f(x_1, \ldots, x_n)$ be a real or complex-valued function on R. For any subrectangle $R' = [\alpha_1, \beta_1] \times \ldots \times [\alpha_n, \beta_n]$ of R with $a_i \leqslant \alpha_i < \beta_i \leqslant b_i$ for all $i = 1, 2, \ldots, n$, we define $\Delta f(R')$ as follows: When $n = 2$ we put

$$\begin{aligned}\Delta f(R') :&= \Delta f([\alpha_1, \beta_1] \times [\alpha_2, \beta_2]) \\ &= f(\beta_1, \beta_2) - f(\beta_1, \alpha_2) - f(\alpha_1, \beta_2) + f(\alpha_1, \alpha_2);\end{aligned}$$

for $n = 3$

$$\begin{aligned}\Delta f(R') :&= \Delta f([\alpha_1, \beta_1] \times \ldots \times [\alpha_3, \beta_3]) \\ &= [f(\beta_1, \beta_2, \beta_3) - f(\beta_1, \alpha_2, \beta_3) - f(\alpha_1, \beta_2, \beta_3) + f(\alpha_1, \alpha_2, \beta_3)] \\ &\quad - [f(\beta_1, \beta_2, \alpha_3) - f(\beta_1, \alpha_2, \alpha_3) - f(\alpha_1, \beta_2, \alpha_3) + f(\alpha_1, \alpha_2, \alpha_3)] \\ &= \Delta_{[\alpha_3, \beta_3]} \Delta f([\alpha_1, \beta_1] \times [\alpha_2, \beta_2]), say;\end{aligned}$$

and successively for any $n \geqslant 3$

$$\begin{aligned}\Delta f(R') :&= \Delta f([\alpha_1, \beta_1] \times \ldots \times [\alpha_n, \beta_n]) \\ &= \Delta_{[\alpha_n, \beta_n]} \Delta f([\alpha_1, \beta_1] \times \ldots \times [\alpha_{n-1}, \beta_{n-1}]).\end{aligned}$$

In what follows, we consider $\phi : [0, \infty) \to \mathbb{R}$ a strictly increasing convex function with $\phi(0) = 0$. The function ϕ is said to be a Δ_2-function if there is a constant $d \geqslant 2$ such that $\phi(2x) \leqslant d\phi(x)$ for all $x \geqslant 0$.

DEFINITION 1. We say that f is of bounded ϕ-variation over R in the sense of Vitali (written as $f \in \phi\mathrm{BV_V}(R)$) if $V_\phi(f; R)$, the total ϕ-variation of f over R, is finite, where

$$V_\phi(f; R) := \sup \left\{ \sum_{i=1}^{m} \phi\left(|\Delta f(R_i)|\right) \right\}, \tag{1}$$

in which the supremum is taken over all partitions $\{R_1, \ldots, R_m\}$ of R.

REMARK 1. Note that for $\phi(x) = x^p$ $(p \geqslant 1)$ above definition is same as the definition of a function of bounded p-variation (see [3, Definition V]) and hence for $\phi(x) = x$ above definition is equivalent to that of Vitali (see, for example, [2, 4] and [3, Remark 1]).

As noted by V. Fülöp and F. Móricz [4, p. 96], in this case also, when $n \geqslant 2$, a function f in the class $\phi\mathrm{BV_V}(R)$ is not necessarily measurable in the sense of Lebesgue. This is a consequence of the fact that if a function $f = f(x_1, \ldots, x_n)$ does not depend on at least one of the $x_1, \ldots, x_n$, then for any subrectangle R' of R we have $\Delta f(R') = 0$, so that $V_\phi(f;R)=0$. Consequently, the class $\phi\mathrm{BV_V}(R)$ contains functions for which the n-dimensional Lebesgue integral over R fails to exist. Following definition is motivated by this fact.

DEFINITION 2. In case $n=2$, we say that a function $f=f(x_1,x_2)$ is of bounded ϕ-variation over $R:=[a_1,b_1]\times[a_2,b_2]$ in the sense of Hardy, in symbol: $f\in\phi\mathrm{BV_H}(R)$, if it is in the class $\phi\mathrm{BV_V}(R)$ and if the marginal functions $f(x_1,a_2)$ and $f(a_1,x_2)$ are of bounded ϕ-variation on the intervals $I_1:=[a_1,b_1]$ and $I_2:=[a_2,b_2]$, respectively in the sense of Young [8].

In case $n\geqslant 3$, the notion of bounded ϕ-variation in the sense of Hardy over a rectangle R can naturally be defined by the following recurrence: $f\in\phi\mathrm{BV_H}(R)$ if $f\in\phi\mathrm{BV_V}(R)$ and each of the marginal functions $f(x_1,\dots,a_k,\dots,x_n)$ is in the class $\phi\mathrm{BV_H}(R(a_k))$, where $k=1,\dots,n$ and

$$R(a_k)=\{(x_1,\dots,x_{k-1},x_{k+1},\dots,x_n)\in\mathbb{R}^{n-1}:\ a_j\leqslant x_j\leqslant b_j \quad \text{for } j=1,\dots,k-1,k+1,\dots,n\}.$$

This definition can be equivalently reformulated as follows: $f\in\phi\mathrm{BV_H}(R)$ if and only if $f\in\phi\mathrm{BV_V}(R)$ and for any choice of $(1\leqslant)\ j_1<\ldots<j_m\ (\leqslant n)$, $1\leqslant m<n$, the function $f(x_1,\dots,a_{j_1},\dots,a_{j_m},\dots,x_n)$ is in the class $\phi\mathrm{BV_V}(R(a_{j_1},\dots,a_{j_m}))$, where

$$R(a_{j_1},\dots,a_{j_m}):=\{(x_{\ell_1},\dots,x_{\ell_{n-m}})\in\mathbb{R}^{n-m}:\ a_j\leqslant x_j\leqslant b_j \quad \text{for } j=\ell_1,\dots,\ell_{n-m}\}$$

and $\{\ell_1,\dots,\ell_{n-m}\}$ is the complementary set of $\{j_1,\dots,j_m\}$ with respect to $\{1,\dots,n\}$.

REMARK 2. When $\phi(x)=x^p$ $(p\geqslant 1)$ our Definition 2 is same as our earlier definition of a function of bounded p-variation (see [3, Definition H]) and hence when $\phi(x)=x$ above definition is equivalent that given by Hardy and Krause (see, for example, [2, 4])(refer Lemma 2 below).

Next let n be a positive integer, $\mathbb{T}^n$ the n-dimensional torus identified with $\mathbf{Q}=[-\pi,\pi]^n$ and let its dual be identified with $\mathbb{Z}^n$. The points $(x_1,\dots,x_n)$ of $\mathbf{Q}$ and $(k_1,\dots,k_n)$ of $\mathbb{Z}^n$ are denoted by $\mathbf{x}$ and $\mathbf{k}$ respectively; $\mathbf{k}\cdot\mathbf{x}$ denotes the scalar product given by $\mathbf{k}\cdot\mathbf{x}=k_1x_1+\ldots+k_nx_n$ and $|\mathbf{x}|$ denotes the number $\sqrt{|x_1|^2+\ldots+|x_n|^2}$. For $f\in\mathrm{L}^1(\mathbb{T}^n)$ its formal Fourier series is given by

$$f(\mathbf{x})\sim\sum_{\mathbf{k}\in\mathbb{Z}^n}\hat f(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x})},$$

where $\hat f(\mathbf{k})$ denotes the $\mathbf{k}^{th}$ Fourier coefficient of $f(\mathbf{x})$ given by

$$\hat f(\mathbf{k})=\frac{1}{(2\pi)^n}\int_{\mathbf{Q}}f(\mathbf{x})e^{-i(\mathbf{k}\cdot\mathbf{x})}d\mathbf{x}.$$

3. Statements of Results

We prove the following results.

LEMMA 1. *If* $f \in \phi\mathrm{BV_H}(R)$ *then* f *is bounded over* R.

LEMMA 2. *If* ϕ *is* Δ_2 *and* $f \in \phi\mathrm{BV_H}(R)$ *then for any arbitrary fixed values* $c_{j_1} \in [a_{j_1}, b_{j_1}], \ldots, c_{j_m} \in [a_{j_m}, b_{j_m}]$, $(1 \leqslant)\ j_1 < \ldots < j_m\ (\leqslant n)$, *and* $1 \leqslant m < n$, *the function* $f(\cdot, \ldots, c_{j_1}, \ldots, c_{j_m}, \ldots, \cdot)$ *is in the class* $\phi\mathrm{BV_H}(R(a_{j_1}, \ldots, a_{j_m}))$ *and that*

$$\begin{aligned}
&V_\phi(f(\cdot, \ldots, c_{j_1}, \ldots, c_{j_m}, \ldots, \cdot); R(a_{j_1}, \ldots, a_{j_m})) \\
&\quad \leqslant d^m \left\{ V_\phi(f;R) + \sum_{k=1}^{m} \sum_{\substack{s_1 < \ldots < s_k, \\ s_1, \ldots, s_k \in \{j_1, \ldots, j_m\}}} V_\phi\left(f(\cdot, \ldots, a_{s_1}, \ldots, a_{s_k}, \ldots, \cdot); R(a_{s_1}, \ldots, a_{s_k})\right) \right\}.
\end{aligned}$$

LEMMA 3. *Let* $f \in \phi\mathrm{BV_V}(R)$, *where* $R = [a_1, b_1] \times \ldots \times [a_n, b_n]$. *Let* $\{R_1, \ldots, R_m\}$ *be a partition of* R. *Then* $f \in \phi\mathrm{BV_V}(R_i)$ *for each* $i = 1, \ldots, m$, *and that*

$$\sum_{i=1}^{m} V_\phi(f; R_i) \leqslant V_\phi(f;R).$$

LEMMA 4. *Let* $f \in \phi\mathrm{BV_V}(R)$, *where* $R = [a_1, b_1] \times [a_2, b_2]$. *If* $f(x, \bar{y})$ *(respectively* $f(\bar{x}, y)$*) for some* $\bar{y}$ *(respectively* $\bar{x}$*) has only a denumerable number of discontinuities in* x *(respectively* y*), the discontinuities in* x *(respectively* y*) of* $f(x,y)$ *are located on a denumerable number of parallels to the* y*-axis (respectively* x*-axis).*

LEMMA 5. *Let* $f \in \phi\mathrm{BV_V}(R)$, *where* $R = [a_1, b_1] \times [a_2, b_2]$. *Then the set of all points* $(\bar{x}, \bar{y}) \in R$ *for which* $f(x,y)$ *is discontinuous at* $(\bar{x}, \bar{y})$, *but* $f(x, \bar{y})$ *is continuous at* $\bar{x}$ *and* $f(\bar{x}, y)$ *is continuous at* $\bar{y}$, *is denumerable.*

LEMMA 6. *Let* $f \in \phi\mathrm{BV_H}(R)$, *where* $R = [a_1, b_1] \times [a_2, b_2]$. *Then the discontinuities of* $f(x,y)$ *are located on a countable number of parallels to the axes.*

LEMMA 7. *Let* $f \in \phi\mathrm{BV_H}(R)$, *where* $R = [a_1, b_1] \times \ldots \times [a_n, b_n]$. *Then the discontinuities of* f *are located on a countable number of* $(n-1)$*-dimensional hyperplanes parallel to some of the coordinate hyperplanes.*

THEOREM 1. *Let* $f : \mathbb{R}^n \to \mathbb{C}$ *be* 2π*-periodic in each variable. If* ϕ *is* Δ_2, $f \in \phi\mathrm{BV_V}([0, 2\pi]^n) \cap \mathrm{L}^1(\mathbb{T}^n)$ *and* $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ *is such that* $k_j \neq 0$ *for each* j, *then*

$$\hat{f}(\mathbf{k}) = O\left(\phi^{-1}\left(\frac{1}{|\prod_{j=1}^{n} k_j|}\right)\right).$$

THEOREM 2. *Let $f:\mathbb{R}^n\to\mathbb{C}$ be 2π-periodic in each variable. If ϕ is Δ_2 and $f\in\phi\mathrm{BV_H}([0,2\pi]^n)$ then for any $\mathbf{0}\neq\mathbf{k}=(k_1,\ldots,k_n)\in\mathbb{Z}^n$,*

$$\hat{f}(\mathbf{k})=O\left(\phi^{-1}\left(\frac{1}{|\prod_{j=1,k_j\neq 0}^{n}k_j|}\right)\right).$$

REMARK 3. Taking $\phi(x)=x^p$ $(p\geqslant 1)$, we get our earlier results (see [3, Theorems 1 and 2]) and hence our new results generalize our earlier results.

4. Proof of the results

Proof of Lemma 1. Observe that when $n=2$, for any $(x_1,x_2)\in R=I_1\times I_2$ we have

$$\begin{aligned}|f(x_1,x_2)|&\leqslant|f(x_1,x_2)-f(x_1,a_2)-f(a_1,x_2)+f(a_1,a_2)|\\&\qquad+|f(x_1,a_2)-f(a_1,a_2)|+|f(a_1,x_2)-f(a_1,a_2)|+|f(a_1,a_2)|\\&\leqslant\phi^{-1}(V_\phi(f;R))+\phi^{-1}(V_\phi(f(\cdot,a_2);I_1))+\phi^{-1}(V_\phi(f(a_1,\cdot);I_2))+|f(a_1,a_2)|.\end{aligned}$$

Similarly when $n\geqslant 2$, for any $\mathbf{x}\in R=[a_1,b_1]\times\ldots\times[a_n,b_n]$ we have

$$\begin{aligned}|f(\mathbf{x})|&\leqslant\phi^{-1}(V_\phi(f;R))+|f(\mathbf{a})|\\&\quad+\left\{\sum_{m=1}^{n-1}\sum_{1\leqslant j_1<\ldots<j_m\leqslant n}\phi^{-1}\left(V_\phi\left(f\left(\cdot,\ldots,a_{j_1},\ldots,a_{j_m},\ldots,\cdot\right);R(a_{j_1},\ldots,a_{j_m})\right)\right)\right\}.\end{aligned}$$

This completes the proof of Lemma 1. $\square$

Proof of Lemma 2. Note that for any $x,y\geqslant 0$ we have

$$\phi(x+y)\leqslant\phi(2\max\{x,y\})\leqslant d\phi(\max\{x,y\})\leqslant d\left(\phi(x)+\phi(y)\right),\tag{2}$$

since ϕ is increasing and Δ_2.

First we will prove the lemma for $n=2$. We must show that for any $a_2<c_2\leqslant b_2$ and $a_1<c_1\leqslant b_1$ we have

$$V_\phi(f(\cdot,c_2);I_1)\leqslant d\left\{V_\phi(f;R)+V_\phi(f(\cdot,a_2);I_1)\right\};\tag{3}$$

$$V_\phi(f(c_1,\cdot);I_2)\leqslant d\left\{V_\phi(f;R)+V_\phi(f(a_1,\cdot);I_2)\right\}.\tag{4}$$

Fix $a_2<c_2\leqslant b_2$. Then for any partition $\{a_1=x_1^0,\,x_1^1,\,\ldots,\,x_1^m=b_1\}$ of I_1, in view of

(2), we have

$$\begin{aligned}
&\sum_{i=1}^{m}\phi\left(|f(x_1^i,c_2)-f(x_1^{i-1},c_2)|\right)\\
&\quad=\sum_{i=1}^{m}\phi\left(|\Delta f\left([x_1^{i-1},x_1^i]\times[a_2,c_2]\right)+\{f(x_1^i,a_2)-f(x_1^{i-1},a_2)\}|\right)\\
&\quad\leqslant d\left\{\sum_{i=1}^{m}\phi\left(|\Delta f\left([x_1^{i-1},x_1^i]\times[a_2,c_2]\right)|\right)+\sum_{i=1}^{m}\phi\left(|f(x_1^i,a_2)-f(x_1^{i-1},a_2)|\right)\right\}\\
&\quad\leqslant d\left\{V_\phi(f;R)+V_\phi(f(\cdot,a_2);I_1)\right\}.
\end{aligned}$$

Taking supremum over all partitions of I_1 we get (3). The proof of (4) is similar to that of (3).

Now we will show the lemma for $n=3$. By symmetry in the variables x_1,x_2,x_3, it is enough to show the following:

(i) For any $a_3<c_3\leqslant b_3$

$$V_\phi(f(\cdot,\cdot,c_3);R(a_3))\leqslant d\left\{V_\phi(f;R)+V_\phi(f(\cdot,\cdot,a_3);R(a_3))\right\}.$$

(ii) For any $a_2<c_2\leqslant b_2$ and $a_3<c_3\leqslant b_3$

$$\begin{aligned}
V_\phi(f(\cdot,c_2,c_3);R(a_2,a_3))\leqslant d^2\{&V_\phi(f;R)+V_\phi(f(\cdot,a_2,\cdot);R(a_2))\\
&+V_\phi(f(\cdot,\cdot,a_3);R(a_3))+V_\phi(f(\cdot,a_2,a_3);R(a_2,a_3))\}.
\end{aligned}$$

To prove (i), consider a partition $\{R_i\}_{i=1}^s$ of $R(a_3)$. Then $\{R_i\times[a_3,c_3]\}_{i=1}^s$ is a collection of disjoint subrectangles of R. Therefore in view of (2)

$$\begin{aligned}
\sum_{i=1}^{s}\phi\left(|\Delta f(\cdot,\cdot,c_3)(R_i)|\right)&=\sum_{i=1}^{s}\phi\left(|\Delta f(R_i\times[a_3,c_3])+\Delta f(\cdot,\cdot,a_3)(R_i)|\right)\\
&\leqslant d\left\{\sum_{i=1}^{s}\phi\left(|\Delta f(R_i\times[a_3,c_3])|\right)+\sum_{i=1}^{s}\phi\left(|\Delta f(\cdot,\cdot,a_3)(R_i)|\right)\right\}\\
&\leqslant d\left\{V_\phi(f;R)+V_\phi(f(\cdot,\cdot,a_3);R(a_3))\right\}.
\end{aligned}$$

Taking supremum over all partitions of $R(a_3)$ we get (i).

Next, to prove (ii), consider a partition $\{a_1=x_1^0,x_1^1,\dots,x_1^s=b_1\}$ of $R(a_2,a_3)$.

Then using (2) twice we get

$$\begin{aligned}
&\sum_{i=1}^{s} \phi(|f(x_1^i,c_2,c_3)-f(x_1^{i-1},c_2,c_3)|)\\
&=\sum_{i=1}^{s} \phi\big(|\Delta f([x_1^{i-1},x_1^i]\times[a_2,c_2]\times[a_3,c_3])+\Delta f(\cdot,a_2,\cdot)([x_1^{i-1},x_1^i]\times[a_3,c_3])\\
&\qquad+\Delta f(\cdot,\cdot,a_3)([x_1^{i-1},x_1^i]\times[a_2,c_2])+\{f(x_1^i,a_2,a_3)-f(x_1^{i-1},a_2,a_3)\}|\big)\\
&\leqslant d^2\sum_{i=1}^{s}\Big\{\phi\big(|\Delta f([x_1^{i-1},x_1^i]\times[a_2,c_2]\times[a_3,c_3])|\big)+\phi\big(|\Delta f(\cdot,a_2,\cdot)([x_1^{i-1},x_1^i]\times[a_3,c_3])|\big)\\
&\qquad+\phi\big(|\Delta f(\cdot,\cdot,a_3)([x_1^{i-1},x_1^i]\times[a_2,c_2])|\big)+\phi\big(|f(x_1^i,a_2,a_3)-f(x_1^{i-1},a_2,a_3)|\big)\Big\}\\
&\leqslant d^2\big\{V_\phi(f;R)+V_\phi(f(\cdot,a_2,\cdot);R(a_2))+V_\phi(f(\cdot,\cdot,a_3);R(a_3))\\
&\qquad\qquad+V_\phi(f(\cdot,a_2,a_3);R(a_2,a_3))\big\}.
\end{aligned}$$

This proves the lemma for $n=3$. A similar argument proves the lemma for any n. □

Proof of Lemma 3. Let $\{R_{ij}: j=1,\dots,n_i\}$ be any partition of R_i, for each $i=1,\dots,m$. Then $\{R_{ij}: j=1,\dots,n_i; i=1,\dots,m\}$ is clearly a partition of R and since $f\in\phi\mathrm{BV_V}(R)$,

$$\sum_{i=1}^{m}\sum_{j=1}^{n_i}\phi\left(|\Delta f(R_{ij})|\right)\leqslant V_\phi(f;R).$$

Taking supremum over all partitions $\{R_{1j}: j=1,\dots,n_1\}$ of R_1 (keeping the partitions of $R_2,\dots,R_m$ fixed) we get

$$V_\phi(f;R_1)+\sum_{i=2}^{m}\sum_{j=1}^{n_i}\phi\left(|\Delta f(R_{ij})|\right)\leqslant V_\phi(f;R).$$

Similarly taking supremum over all partitions of R_2 (keeping the partitions of $R_3,\dots,R_m$ fixed), and continuing in this way for $R_3,\dots,R_m$ we get the lemma. □

Proof of Lemma 4. Let $E=\{(x,y)\in R: f \text{ has a discontinuity in } x\}$ and $E_{\bar y}=\{(\bar x,\bar y)\in R: f(x,\bar y) \text{ is discontinuous at } \bar x\}$. Then $E_{\bar y}\subset E$ and by our assumption $E_{\bar y}$ is denumerable.

If possible suppose there is a non-denumerable set S of vertical lines each containing at least one point of E. Since $E_{\bar y}$ is denumerable, clearly only a denumerable subset of S made up wholly of points of $E_{\bar y}$. Let the remaining lines of S constitute the subset S_1; then each line of S_1 contains at least one point of E and no point of $E_{\bar y}$, and S_1 is non-denumerable. On each line of S_1 (which lie interior to R) choose a point of E; at this point the saltus, say, s of f in x is positive and hence $\phi(s/4)$ is also positive. This non-denumerable set of positive values $\phi(s/4)$ contains a subset whose elements are the terms of a divergent series. Thus there is a sequence $\{(x_i,y_i)\}_{i=1}^{\infty}$ of distinct points in E, which lie interior to R and on different lines in S_1, such that

$$\sum_{i=1}^{\infty} \phi\left(\frac{s_i}{4}\right) = \infty,$$

where $s_i =$ the saltus in x at $(x_i, y_i) = |f(x_i+, y_i) - f(x_i-, y_i)|$. Now by arguing as in the proof of Lemma 4 in [3], for $\varepsilon_i = \frac{s_i}{4}$, we can choose x_i', x_i'' such that $x_i' < x_i < x_i''$ for each i, the intervals $\{[x_i', x_i'']\}_{i=1}^{\infty}$ are pairwise disjoint, and

$$|f(x_i'', y_i) - f(x_i', y_i) - f(x_i'', \bar{y}) + f(x_i', \bar{y})| \geqslant \varepsilon_i.$$

Thus if R_i denotes the rectangle with vertices (x_i'', y_i), (x_i', y_i), $(x_i'', \bar{y})$ and $(x_i', \bar{y})$ for each i then, since ϕ is non-decreasing, we get

$$\sum_{i=1}^{\infty} \phi\left(|\Delta f(R_i)|\right) \geqslant \sum_{i=1}^{\infty} \phi(\varepsilon_i) = \sum_{i=1}^{\infty} \phi\left(\frac{s_i}{4}\right) = \infty.$$

This shows that $V_\phi(f;R) = \infty$; from this contradiction lemma follows. $\square$

Proof of Lemma 5. Let $(\bar{x}, \bar{y})$ be such a discontinuity. Then there exists $\varepsilon > 0$ such that for every $\delta > 0$ there is a point, say, (x', y') (depending on δ) such that

$$\sqrt{(x' - \bar{x})^2 + (y' - \bar{y})^2} < \delta \quad \text{but} \quad |f(x', y') - f(\bar{x}, \bar{y})| \geqslant \varepsilon. \tag{5}$$

Also, by the continuity of $f(\cdot, \bar{y})$ and $f(\bar{x}, \cdot)$ at $\bar{x}$ and $\bar{y}$ respectively, there is a $\delta > 0$ such that

$$|x - \bar{x}| < \delta \Rightarrow |f(x, \bar{y}) - f(\bar{x}, \bar{y})| < \frac{\varepsilon}{4} \quad \text{and} \quad |y - \bar{y}| < \delta \Rightarrow |f(\bar{x}, y) - f(\bar{x}, \bar{y})| < \frac{\varepsilon}{4}.$$

For this δ, as above, there is a point (x', y') such that (5) holds. Since

$$\sqrt{(x' - \bar{x})^2 + (y' - \bar{y})^2} \geqslant |x' - \bar{x}| \quad \text{and} \quad \sqrt{(x' - \bar{x})^2 + (y' - \bar{y})^2} \geqslant |y' - \bar{y}|$$

we get

$$|f(x', \bar{y}) - f(\bar{x}, \bar{y})| < \frac{\varepsilon}{4}, \quad |f(\bar{x}, y') - f(\bar{x}, \bar{y})| < \frac{\varepsilon}{4};$$

which shows that

$$|f(x', \bar{y}) + f(\bar{x}, y') - 2f(\bar{x}, \bar{y})| < \frac{\varepsilon}{2}.$$

Thus for the rectangle R' with sides parallel to the axes and whose two vertices are $(\bar{x}, \bar{y})$ and (x', y'), we have

$$\begin{aligned}
\phi\left(|\Delta f(R')|\right) &= \phi\left(|f(x', y') - f(x', \bar{y}) - f(\bar{x}, y') + f(\bar{x}, \bar{y})|\right) \\
&\geqslant \phi\left(|f(x', y') - f(\bar{x}, \bar{y})| - |f(x', \bar{y}) + f(\bar{x}, y') - 2f(\bar{x}, \bar{y})|\right) \\
&> \phi\left(\varepsilon - \frac{\varepsilon}{2}\right) = \phi\left(\frac{\varepsilon}{2}\right).
\end{aligned}$$

The assumption that the set of such discontinuities is non-denumerable then leads to a contradiction just as in the case of Lemma 4. □

Proof of Lemma 6. Since $f \in \phi\mathrm{BV_H}(R)$, $f \in \phi\mathrm{BV_V}(R)$ and the marginal functions $f(x,a_2)$ and $f(a_1,y)$ are of bounded ϕ-variation on I_1 and I_2 respectively. Therefore $f(x,a_2)$ has only a denumerable number of discontinuities in x and $f(a_1,y)$ has only a denumerable number of discontinuities in y (see, for example, [7, p. 51]). So, in view of Lemma 4, the discontinuities in x or y of $f(x,y)$ are located on a countable number of parallels to the coordinate axes. Now the lemma follows from Lemma 5. □

Proof of Lemma 7. In view of Lemmas 5 and 6, the proof of this lemma is similar to that of Lemma 7 in [3] and we shall omit it. □

Proof of Theorem 1. For the sake of simplicity in writing, we carry out the proof for $n=2$, and we write (x,y) and (k,ℓ) in place of (x_1,x_2) and (k_1,k_2) respectively.

Let $\mathbf{k}=(k,\ell)\in\mathbb{Z}^2$ be such that $k\neq 0$, $\ell\neq 0$. Then the functions e^{-ikx} and $e^{-i\ell y}$ are periodic functions of periods $\frac{2\pi}{|k|}$ and $\frac{2\pi}{|\ell|}$ respectively. Thus by putting

$$a_r = r\cdot\frac{2\pi}{|k|}\ \ (r=0,1,\ldots,|k|);\quad b_s = s\cdot\frac{2\pi}{|\ell|}\ \ (s=0,1,\ldots,|\ell|)$$

we get

$$\int_{a_{r-1}}^{a_r} e^{-ikx}dx = 0\ \ (r=1,2,\ldots,|k|);\quad \int_{b_{s-1}}^{b_s} e^{-i\ell y}dy = 0\ \ (s=1,2,\ldots,|\ell|). \tag{6}$$

Define three functions f_1, f_2, f_3 on $\mathbb{T}^2$ by setting

$$f_1(x,y) = f(x,b_{s-1})\ \ (0\leqslant x<2\pi;\ b_{s-1}\leqslant y<b_s)\ \text{for}\ s=1,\ldots,|\ell|;$$

$$f_2(x,y) = f(a_{r-1},y)\ \ (a_{r-1}\leqslant x<a_r;\ 0\leqslant y<2\pi)\ \text{for}\ r=1,\ldots,|k|;$$

and

$$f_3(x,y) = f(a_{r-1},b_{s-1})\ \ (a_{r-1}\leqslant x<a_r;\ b_{s-1}\leqslant y<b_s)$$

for $r=1,\ldots,|k|;\ s=1,\ldots,|\ell|$.

Since $f\in\phi\mathrm{BV_V}([0,2\pi]^2)\cap \mathrm{L}^1(\mathbb{T}^2)$, each $f_i\in\phi\mathrm{BV_V}([0,2\pi]^2)\cap \mathrm{L}^1(\mathbb{T}^2)$. Since ϕ is Δ_2, in view of (2), $f-f_1-f_2+f_3\in\phi\mathrm{BV_V}([0,2\pi]^2)\cap \mathrm{L}^1(\mathbb{T}^2)$. Further in view of Fubini's theorem and relations (6) we have

$$\int_0^{2\pi}\int_0^{2\pi} f_1(x,y)e^{-ikx}e^{-i\ell y}dxdy = \int_0^{2\pi}\left[\sum_{s=1}^{|\ell|}\int_{b_{s-1}}^{b_s} f(x,b_{s-1})e^{-i\ell y}dy\right]e^{-ikx}dx = 0,$$

$$\int_0^{2\pi}\int_0^{2\pi} f_2(x,y)e^{-ikx}e^{-i\ell y}dxdy = \int_0^{2\pi}\left[\sum_{r=1}^{|k|}\int_{a_{r-1}}^{a_r} f(a_{r-1},y)e^{-ikx}dx\right]e^{-i\ell y}dy = 0$$

and

$$\int_0^{2\pi}\int_0^{2\pi} f_3(x,y)e^{-ikx}e^{-i\ell y}dxdy = \sum_{r=1}^{|k|}\sum_{s=1}^{|\ell|}\int_{a_{r-1}}^{a_r}\int_{b_{s-1}}^{b_s} f(a_{r-1},b_{s-1})e^{-ikx}e^{-i\ell y}dxdy = 0.$$

Using these equations in the definition of $\hat{f}(\mathbf{k})$ we get

$$\begin{aligned}
|\hat{f}(\mathbf{k})| &= \left|\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} f(x,y)e^{-ikx}e^{-i\ell y}dxdy\right| \\
&= \left|\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} (f-f_1-f_2+f_3)(x,y)e^{-ikx}e^{-i\ell y}dxdy\right| \\
&\leqslant \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} |(f-f_1-f_2+f_3)(x,y)|dxdy.
\end{aligned}$$

Thus for any $c>0$, using Jensen's inequality for integrals, in view of Lemma 3, we get

$$\begin{aligned}
\phi(c|\hat{f}(\mathbf{k})|) &\leqslant \phi\left(\frac{c}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} |(f-f_1-f_2+f_3)(x,y)|dxdy\right) \\
&\leqslant \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \phi\left(c|(f-f_1-f_2+f_3)(x,y)|\right)dxdy \\
&= \frac{1}{(2\pi)^2}\sum_{r=1}^{|k|}\sum_{s=1}^{|\ell|}\int_{a_{r-1}}^{a_r}\int_{b_{s-1}}^{b_s} \phi\left(c|(f-f_1-f_2+f_3)(x,y)|\right)dxdy \\
&= \frac{1}{(2\pi)^2}\sum_{r=1}^{|k|}\sum_{s=1}^{|\ell|}\int_{a_{r-1}}^{a_r}\int_{b_{s-1}}^{b_s} \phi(c|f(x,y)-f(x,b_{s-1}) \\
&\qquad\qquad -f(a_{r-1},y)+f(a_{r-1},b_{s-1})|)dxdy \\
&\leqslant \frac{1}{(2\pi)^2}\sum_{r=1}^{|k|}\sum_{s=1}^{|\ell|} V_\phi(cf;[a_{r-1},a_r]\times[b_{s-1},b_s])(a_r-a_{r-1})(b_s-b_{s-1}) \\
&\leqslant \frac{1}{(2\pi)^2}\cdot\frac{(2\pi)^2}{|k\ell|}\cdot V_\phi(cf;[0,2\pi]^2).
\end{aligned} \tag{7}$$

Since ϕ is convex and $\phi(0)=0$, for $c\in(0,1)$ we have $\phi(cx)\leqslant c\phi(x)$ and hence we can choose sufficiently small $c\in(0,1)$ such that $V_\phi(cf;[0,2\pi]^2)\leqslant 1$. Thus, in view of (7), we get

$$|\hat{f}(\mathbf{k})| \leqslant \frac{1}{c}\phi^{-1}\left(\frac{1}{|k\ell|}\right).$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. Here also we will carry out the proof for $n=2$ and use notations as in the proof of Theorem 1. Since $f\in\phi\mathrm{BV_H}([0,2\pi]^2)$, in view of Lemma 6 (use Lemma 7 for general case), the discontinuities of f lie on countable number of parallels to the axes and hence f is measurable over $\mathbb{T}^2$ in the sense of Lebesgue. Further, by

Lemma 1, f is bounded over $[0,2\pi]^2$ and hence $f \in \mathrm{L}^1(\mathbb{T}^2)$. As $\phi\mathrm{BV_H}([0,2\pi]^2) \subset \mathrm{BV_V}([0,2\pi]^2)$, $f \in \mathrm{L}^1(\mathbb{T}^2) \cap \phi\mathrm{BV_V}([0,2\pi]^2)$. Therefore if $\mathbf{k} = (k,\ell) \in \mathbb{Z}^2$ is such that $k \neq 0, \ell \neq 0$, by Theorem 1, $\hat{f}(\mathbf{k}) = O\left(\phi^{-1}\left(1/|k\ell|\right)\right)$. Next, let $\mathbf{k} = (k,\ell) \in \mathbb{Z}^2$ be such that $k \neq 0$, $\ell = 0$ and let a_r's and f_2 be as defined in the proof of Theorem 1. Then we have

$$\int_0^{2\pi}\int_0^{2\pi} f_2(x,y)e^{-ikx}dxdy = \int_0^{2\pi}\left\{\sum_{r=1}^{|k|} f(a_{r-1},y)\left[\int_{a_{r-1}}^{a_r} e^{-ikx}dx\right]\right\}dy = 0,$$

in view of Fubini's theorem and (6); and,

$$\begin{aligned}|\hat{f}(\mathbf{k})| &= \left|\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}(f-f_2)(x,y)e^{-ikx}dxdy\right| \\ &\leqslant \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}|(f-f_2)(x,y)|dxdy.\end{aligned}$$

Thus for any $c > 0$, by Jensen's inequality, we have

$$\begin{aligned}\phi\left(c|\hat{f}(\mathbf{k})|\right) &\leqslant \phi\left(\frac{c}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}|(f-f_2)(x,y)|dxdy\right) \\ &\leqslant \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\phi\left(c|(f-f_2)(x,y)|\right)dxdy \\ &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\left[\sum_{r=1}^{|k|}\int_{a_{r-1}}^{a_r}\phi\left(c|f(x,y)-f(a_{r-1},y)|\right)dx\right]dy \\ &\leqslant \frac{1}{(2\pi)^2}\int_0^{2\pi}\left[\sum_{r=1}^{|k|}V_\phi(cf(\cdot,y);[a_{r-1},a_r])(a_r-a_{r-1})\right]dy \\ &\leqslant \frac{1}{(2\pi)^2}\cdot\frac{2\pi}{|k|}\int_0^{2\pi}V_\phi(cf(\cdot,y);[0,2\pi])dy \\ &\leqslant \frac{1}{2\pi|k|}\int_0^{2\pi}d[V_\phi(cf;[0,2\pi]^2)+V_\phi(cf(\cdot,0);[0,2\pi])]dy \\ &= \frac{d[V_\phi(cf;[0,2\pi]^2)+V_\phi(cf(\cdot,0);[0,2\pi])]}{|k|},\end{aligned}$$

in view of Lemma 3 (for a function of one variable) and Lemma 2. Since ϕ is convex and $\phi(0) = 0$, now we can choose $c \in (0,1)$ so small such that $V_\phi(cf;[0,2\pi]^2) \leqslant \frac{1}{2d}$ and $V_\phi(cf(\cdot,0);[0,2\pi]) \leqslant \frac{1}{2d}$. Thus by above inequality we get

$$\hat{f}(\mathbf{k}) = \hat{f}(k,0) = O\left(\phi^{-1}\left(\frac{1}{|k|}\right)\right).$$

The case $k = 0$, $\ell \neq 0$, is similar to the above case and in this case we get

$$\hat{f}(0,\ell) = O\left(\phi^{-1}\left(\frac{1}{|\ell|}\right)\right).$$

This completes the proof of the Theorem 2. □

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Bhikha Lila Ghodadra
Department of Mathematics, Faculty of Science
The Maharaja Sayajirao University of Baroda
Vadodara – 390 002 (Gujarat)
India
e-mail: bhikhu_ghodadra@yahoo.com