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SOME MAXIMAL TYPE INEQUALITIES FOR N–DEMIMARTINGALES AND RELATED RESULTS

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Abstract. In this paper, we go on to investigate the concave Young function inequalities for N-demimartingales and obtain some maximal type inequalities for these stochastic process. As some specific concave Young functions, some related inequalities for N-demimartingales are presented. Meanwhile, some convex function inequalities for nonnegative N-demimartingales are also obtained, including the classical Doob type inequalities. In addition, the Marshall type inequalities and other maximal type inequalities for N-demimartingales are studied too.

1. Introduction

Assume that X_1, $X_2, \cdots$ or S_1, $S_2, \cdots$ is a sequence of random variables defined on a probability space $(\Omega, \mathscr{F}, P)$. Let $S_0 = 0$ and I_A be the indicator function of the set A.

DEFINITION 1.1. Let $S_1, S_2, \cdots$ be an L^1 sequence of random variables. Assume that for $j = 1, 2, \cdots$

$$E\{(S_{j+1} - S_j) f(S_1, \cdots, S_j)\} \geqslant 0 \tag{1.1}$$

for all coordinatewise nondecreasing functions f such that the expectation is defined. Then $\{S_j\}_{j\geqslant 1}$ is called a demimartingale. If in addition the function f is assumed to be nonnegative, the sequence $\{S_j\}_{j\geqslant 1}$ is called a demisubmartingale.

DEFINITION 1.2. Let $S_1, S_2, \cdots$ be an L^1 sequence of random variables. Assume that for $j = 1, 2, \cdots$

$$E\{(S_{j+1} - S_j) f(S_1, \cdots, S_j)\} \leqslant 0 \tag{1.2}$$

for all coordinatewise nondecreasing functions f such that the expectation is defined. Then $\{S_j\}_{j\geqslant 1}$ is called an N-demimartingale. If in addition the function f is assumed to be nonnegative, the sequence $\{S_j\}_{j\geqslant 1}$ is called an N-demisupermartingale.

The concept of demimartingales and demisubmartingales was due to Newman and Wright [19]. It can be checked that a submartingale with the natural choice of

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σ-algebras is a demisubmartingale but the converse statement cannot always be true. Newman and Wright [19] proved that the partial sum of mean zero associated random variables forms a demimartingale. For the examples of associated random variables, limit theorems of associated random fields and related systems, one can refer to the book by Bulinski and Shaskin [2].

Similarly, the notion of N-demimartingales and N-demisupermartingales can be found in Christofides [4]. It is trivial to verify that the partial sum of mean zero negatively associated random variables forms an N-demimartingale and a supermartingale with the natural choice of σ-algebras is an N-demisupermartingale, but the converse statement cannot always be true (see Christofides [4]).

It is worth mentioning that a martingale with the natural choice of σ-algebras is both a demimartingale and an N-demimartingale, since it satisfies (1.1) and (1.2) respectively. Various results and examples of demimartingales and N-demimartingales can be found in Newman and Wright [19], Wood [32, 33], Christofides [3, 4, 5], Wang [26], Prakasa Rao [22, 23], Christofides and Hadjikyriakou [6, 7], Wang et al. [27], Wang et al. [28], Wang et al. [29], Hadjikyriakou [10, 11], the book by Prakasa Rao [24] and the references therein. For the related works on martingales, one can refer to Osękowski [20] and Wang et al. [30], etc.

By investigating the concave Young function inequalities for nonnegative supermartingales and submartingales, Agbeko [1] obtained some maximal inequalities for these stochastic processes. Christofides [4] extended the result of Agbeko [1] for supermartingales to the case of N-demisupermartingales. Recently, Wang et al. [31] also extended some results of Agbeko [1] to the cases of demimartingales and N-demimartingales. Meanwhile, the Marshall type inequalities for demimartingales and the convex function type inequalities for demimartingales and N-demimartingales were presented in Hu et al. [13], who extended some results of Mu and Miao [18].

Inspired by Agbeko [1], Christofides [4], Prakasa Rao [22, 23], Mu and Miao [18], Wang et al. [31], etc., we go on to study the concave Young function inequalities for N-demimartingales, and obtain some similar results of Agbeko [1]. As some specific concave Young functions, some related inequalities for N-demimartingales are presented. For the details, please see the results and remarks in Section 2. Based on the convex functions, we also have some convex function inequalities for nonnegative N-demimartingales, which include the classical Doob type inequalities. For the details, one can refer to the corresponding results in Section 3. Furthermore, by using the maximal inequality for N-demimartingales, we study the Marshall type inequalities for N-demimartingales (see the results in Section 4). Finally, some other maximal type inequalities of nonnegative N-demimartingales are also presented in Section 5.

Now, we give a preliminary, which is a key technique to obtain the results in this paper.

LEMMA 1.1. (cf. Prakasa Rao [22], Theorem 3.1) *Assume that* $\{S_n\}_{n\geqslant 1}$ *is an N-demimartingale and* $m(\cdot)$ *is a nonnegative nondecreasing function on* $\mathbb{R}$ *with* $m(0) = 0$. *Let* $g(\cdot)$ *be a function on* $\mathbb{R}$ *with* $g(0) = 0$ *and suppose that*

$$g(x) - g(y) \geqslant (y - x)h(y) \tag{1.3}$$

for all x,y*, where* $h(\cdot)$ *is a nonnegative and nondecreasing function. Further assume that* $\{c_k, 1 \leqslant k \leqslant n\}$ *is a sequence of positive numbers such that* $(c_k - c_{k+1})g(S_k) \geqslant 0$ *for* $1 \leqslant k \leqslant n-1$*. Define* $Y_k = \max_{1 \leqslant j \leqslant k} c_j g(S_j)$*,* $k \geqslant 1$*,* $Y_0 = 0$*. Then*

$$E\Big(\int_0^{Y_n} u dm(u)\Big) \leqslant \sum_{i=1}^{n} c_i E[(g(S_i) - g(S_{i-1}))m(Y_n)].$$

Particularly, for every $\varepsilon > 0$*,*

$$\varepsilon P\big(\max_{1 \leqslant k \leqslant n} c_k g(S_k) \geqslant \varepsilon\big) \leqslant \sum_{i=1}^{n} c_i E[(g(S_i) - g(S_{i-1}))I(\max_{1 \leqslant k \leqslant n} c_k g(S_k) \geqslant \varepsilon)]. \tag{1.4}$$

REMARK 1.1. It can be seen that $g(x) = -\alpha x$, $\alpha \geqslant 0$ and $g(x) = -\alpha x^+$, $\alpha \geqslant 0$, satisfy the condition of (1.3) (see Prakasa Rao [22, 24]). Here $x^+ = x$ if $x \geqslant 0$ and $x^+ = 0$ if $x < 0$.

By using Lemma 1.1, Hadjikyriakou [10] got the following maximal inequality for N-demimartingales.

COROLLARY 1.1. (Hadjikyriakou [10], Theorem 3.2.1) *Assume that* $\{S_n\}_{n \geqslant 1}$ *is an N-demimartingale. Then for every* $\varepsilon > 0$*,*

$$\varepsilon P\big(\max_{1 \leqslant k \leqslant n} S_k \geqslant \varepsilon\big) \leqslant E\big(S_n I(\max_{1 \leqslant k \leqslant n} S_k \geqslant \varepsilon)\big). \tag{1.5}$$

Similar to Corollary 1.1, we apply Lemma 1.1 and get the following result.

COROLLARY 1.2. *Let* $\{S_n\}_{n \geqslant 1}$ *be a nonnegative N-demimartingale and* $\{c_n\}_{n \geqslant 1}$ *be a nonincreasing sequence of positive numbers. Then for every* $\varepsilon > 0$

$$\varepsilon P\big(\max_{1 \leqslant k \leqslant n} c_k S_k \geqslant \varepsilon\big) \leqslant \sum_{i=1}^{n} c_i E[(S_i - S_{i-1})I(\max_{1 \leqslant k \leqslant n} c_k S_k \geqslant \varepsilon)]. \tag{1.6}$$

Proof. The proof is inspired by the proof of Theorem 3.2.1 of Hadjikyriakou [10]. It is a fact that if $\{S_n\}_{n \geqslant 1}$ is an N-demimartingale, then $\{-S_n\}_{n \geqslant 1}$ is also an N-demimartingale (see Christofides [4] or Prakasa Rao [24]). If $g(x) = -x$ and $\{S_n\}$ is replace by $\{-S_n\}$, then it can be seen that

$$(c_k - c_{k+1})g(S_k) = (c_k - c_{k+1})S_k \geqslant 0, \ \ k \geqslant 1.$$

Therefore, by (1.4), one has (1.6) finally. □

2. Concave function inequalities for nonnegative N-demimartingales

Let $\varphi(t)$ be a right continuous decreasing function defined on $(0,\infty)$ satisfying the condition

$$\varphi(\infty)=\lim_{t\to\infty}\varphi(t)=0.$$

Suppose that φ is also integrable with respect to the Lebesgue measure on any finite interval $(0,x)$. Let

$$\Phi(x)=\int_0^x\varphi(t)dt,\quad x\geqslant 0.$$

Then Φ is a nonnegative increasing concave function such that $\Phi(0)=0$. We further assume that $\Phi(\infty)=\infty$. Then Φ is called a concave Young function.

For more details and properties of concave Young function, one can refer to Agbeko [1]. Examples of such functions are $\Phi(x)=x^p$, $0<p<1$, $x\geqslant 0$ and $\Phi(x)=\ln(1+x)$, $x\geqslant 0$. One can also refer to Long [16] for the properties of concave Young function. Agbeko [1] obtained the following maximal inequalities based on the class of concave Young functions for supermartingales (see Theorem 2.1 of Agbeko [1]).

THEOREM 2.1. *Let Φ be a concave Young function. Denote $\xi(x)=\Phi(x)-x\varphi(x)$. Then for any nonnegative supermartingale $(X_n,\mathscr{F}_n)$ we have:*

(i)

$$E\xi(\max_{1\leqslant k\leqslant n}X_k)\leqslant E\Phi(X_1).$$

(ii) The inequality

$$(1-b)E\Phi(\max_{1\leqslant k\leqslant n}X_k)-a\leqslant E\xi(\max_{1\leqslant k\leqslant n}X_k)$$

is valid for some constants $a\geqslant 0$ and $0<b<1$, if and only if

$$\limsup_{x\to\infty}\frac{x\varphi(x)}{\Phi(x)}<1. \tag{2.1}$$

(iii) If (2.1) satisfies, then

$$E\Phi(\max_{1\leqslant k\leqslant n}X_k)\leqslant K_\Phi[1+E\Phi(X_1)]$$

for some constant $K_\Phi>0$ depending only on Φ.

By studying the properties of N-demimartingales, Christofides [4] extended Theorem 2.1 to the case of N-demisupermartingales. Wang et al. [31] also obtained some concave Young function inequalities for demimartingales and N-demimartingales.

In this section, we also investigate the concave Young function inequalities for N-demimartingales, and get the following theorem.

THEOREM 2.2. *Assume that $\{S_n\}_{n\geqslant 1}$ is a nonnegative N-demimartingale and $\{c_n\}_{n\geqslant 1}$ is a nonincreasing sequence of positive numbers. Let Φ be a concave Young function. Denote $\xi(x)=\Phi(x)-x\varphi(x)$. Then we have*

(i)

$$E\xi(\max_{1\leqslant k\leqslant n} c_k S_k) \leqslant \Phi(c_1 ES_1). \tag{2.2}$$

(ii) The inequality

$$(1-b)E\Phi(\max_{1\leqslant k\leqslant n} c_k S_k) - a \leqslant E\xi(\max_{1\leqslant k\leqslant n} c_k S_k) \tag{2.3}$$

is valid for some constants $a \geqslant 0$ *and* $0 < b < 1$ *if and only if (2.1) satisfies.*

(iii) If (2.1) satisfies, then

$$E\Phi(\max_{1\leqslant k\leqslant n} c_k S_k) \leqslant K_\Phi[1+\Phi(c_1 ES_1)] \tag{2.4}$$

for some constant $K_\Phi > 0$ *depending only on* Φ.

Proof. The proof is inspired by the proof of Theorem 2.1 of Agbeko [1]. Denote $T_n = \sum_{i=1}^{n} c_i(S_i - S_{i-1})$, $n \geqslant 1$. Under the conditions of Theorem 2.2, it is easy to see that

$$T_n = \sum_{i=1}^{n} c_i(S_i - S_{i-1}) = \sum_{i=1}^{n-1}(c_i - c_{i+1})S_i + c_n S_n \geqslant 0, \quad n \geqslant 1.$$

By the property of N-demimartingale that $ES_n = \cdots = ES_1$, $n \geqslant 1$, it can be seen that $ET_n = c_1 ES_1$. Consequently, by (1.6) in Corollary 1.2, we can find that

$$xP(\max_{1\leqslant k\leqslant n} c_k S_k \geqslant x) \leqslant \min\{ET_n, x\} = \min\{c_1 ES_1, x\}, \quad x > 0. \tag{2.5}$$

By the properties of φ and Φ with $\Phi(0) = 0$, it can be seen that

$$\lim_{x\to 0} x\varphi(x) \leqslant \lim_{x\to 0} 2(\Phi(x) - \Phi(x/2)) = 0.$$

So, integrate both sides of (2.5) on $[0,\infty)$ with respect to the measure $d(-\varphi(x))$, we have

$$\begin{aligned}\int_0^\infty \int_{(\max_{1\leqslant k\leqslant n} c_k S_k \geqslant x)} xdPd(-\varphi(x)) &\leqslant \int_0^\infty \min\{c_1 ES_1, x\} d(-\varphi(x)) \\ &= \int_0^{c_1 ES_1} xd(-\varphi(x)) + c_1 ES_1 \int_{c_1 ES_1}^\infty d(-\varphi(x)) \\ &= -c_1 ES_1 \varphi(c_1 ES_1) + \Phi(c_1 ES_1) + c_1 ES_1 \varphi(c_1 ES_1) \\ &= \Phi(c_1 ES_1),\end{aligned}$$

which implies

$$E\int_0^{\max_{1\leqslant k\leqslant n} c_k S_k} xd(-\varphi(x)) \leqslant \Phi(c_1 ES_1). \tag{2.6}$$

On the other hand, integration by parts, it follows

$$\begin{aligned}E\int_0^{\max_{1\leqslant k\leqslant n} c_k S_k} xd(-\varphi(x)) &= -E(\max_{1\leqslant k\leqslant n} c_k S_k \varphi(\max_{1\leqslant k\leqslant n} c_k S_k)) + E\Phi(\max_{1\leqslant k\leqslant n} c_k S_k) \\ &= E\xi(\max_{1\leqslant k\leqslant n} c_k S_k).\end{aligned} \tag{2.7}$$

Combining (2.6) with (2.7), one has (2.2) immediately.

Meanwhile, similar to the proof of (ii) of Theorem 2.1 of Agbeko [1], we can get the result of (ii) of Theorem 2.2.

Finally, by (2.2), (2.3) and the proof of (iii) of Theorem 2.1 of Agbeko [1], it is easy to have the result of (2.4). □

As the specific concave Young functions such as $\Phi(x) = x^p$, $0 < p < 1$, $x \geqslant 0$ and $\Phi(x) = \ln(1+x)$, $x \geqslant 0$, we get some related inequalities for N-demimartingales in this section.

COROLLARY 2.1. *Assume that* $\{S_n\}_{n\geq 1}$ *is a nonnegative N-demimartingale and* $\{c_n\}_{n\geq 1}$ *is a nonincreasing sequence of positive numbers. Then for any* $0 < p < 1$,

$$E(\max_{1\leqslant k\leqslant n} c_k S_k)^p \leqslant \frac{1}{1-p}(c_1 ES_1)^p, \tag{2.8}$$

$$E(\max_{1\leqslant k\leqslant n} c_k S_k)^\nu \leqslant \left(\frac{1}{1-p}\right)^{\nu/p}(c_1 ES_1)^\nu, \quad 0 < \nu < p < 1. \tag{2.9}$$

Particularly, if $c_n \equiv 1$, $n \geqslant 1$, *then for any* $0 < p < 1$, *it has*

$$ES_1^p \leqslant E(\max_{1\leqslant k\leqslant n} S_k)^p \leqslant \frac{1}{1-p}(ES_1)^p, \tag{2.10}$$

$$ES_1^\nu \leqslant E(\max_{1\leqslant k\leqslant n} S_k)^\nu \leqslant \left(\frac{1}{1-p}\right)^{\nu/p}(ES_1)^\nu, \quad 0 < \nu < p < 1. \tag{2.11}$$

Proof. If $\Phi(x) = x^p$, $0 < p < 1$, then $\varphi(x) = px^{p-1}$ and $\xi(x) = \Phi(x) - x\varphi(x) = (1-p)x^p$. So, by (2.2), it follows

$$E\xi(\max_{1\leqslant k\leqslant n} c_k S_k) = (1-p)E(\max_{1\leqslant k\leqslant n} c_k S_k)^p \leqslant (c_1 ES_1)^p,$$

which implies (2.8) immediately.

On the other hand, if $0 < \nu < p < 1$, by Hölder inequality and (2.8), it can be checked that

$$E(\max_{1\leqslant k\leqslant n} c_k S_k)^\nu \leqslant [E(\max_{1\leqslant k\leqslant n} c_k S_k)^p]^{\nu/p} \leqslant \left[\frac{1}{1-p}(c_1 ES_1)^p\right]^{\nu/p} = \left(\frac{1}{1-p}\right)^{\nu/p}(c_1 ES_1)^\nu,$$

i.e. (2.9) holds.

Obviously, it has $ES_1^p \leqslant E(\max_{1\leqslant k\leqslant n} S_k)^p$ and $ES_1^\nu \leqslant E(\max_{1\leqslant k\leqslant n} S_k)^\nu$. Meanwhile, by taking $c_n \equiv 1$, $n \geqslant 1$ in (2.8) and (2.9), we can get (2.10) and (2.11) finally. □

Similar to Corollary 2.1, by taking $\Phi(x) = \ln(1+x)$ in Theorem 2.2, we will get the following result.

COROLLARY 2.2. *Assume that* $\{S_n\}_{n\geqslant 1}$ *is a nonnegative N-demimartingale and* $\{c_n\}_{n\geqslant 1}$ *is a nonincreasing sequence of positive numbers. Then*

$$E\ln(1+\max_{1\leqslant k\leqslant n} c_kS_k)\leqslant 1+\ln(1+c_1ES_1). \tag{2.12}$$

Particularly, if $c_n\equiv 1,\ n\geqslant 1$, *then*

$$E\ln(1+S_1)\leqslant E\ln(1+\max_{1\leqslant k\leqslant n} S_k)\leqslant 1+\ln(1+ES_1). \tag{2.13}$$

Proof. If $\Phi(x)=\ln(1+x)$, $x\geqslant 0$, then $\varphi(x)=\frac{1}{1+x}$ and

$$\xi(x)=\Phi(x)-x\varphi(x)=\ln(1+x)-\frac{x}{1+x},\quad x\geqslant 0.$$

So, by (2.2), it has

$$E\xi(\max_{1\leqslant k\leqslant n} c_kS_k)=E\ln(1+\max_{1\leqslant k\leqslant n} c_kS_k)-E\frac{\max_{1\leqslant k\leqslant n} c_kS_k}{1+\max_{1\leqslant k\leqslant n} c_kS_k}\leqslant\ln(1+c_1ES_1).$$

By the inequality above, (2.12) holds immediately.

On the other hand, it is easy to see that $E\ln(1+S_1)\leqslant E\ln(1+\max_{1\leqslant k\leqslant n} S_k)$. Last, let $c_n\equiv 1$, $n\geqslant 1$ in (2.12), we obtain (2.13) immediately. □

REMARK 2.1. Similar to our Theorem 2.2, some concave function inequalities for N-demimartingales are obtained in Wang et al. [31] (see Theorem 3.1, Corollary 3.1 and Corollary 3.2 of Wang et al. [31]). It is pointed out that $g(x)=-x$ cannot be taken in Theorem 3.1 of Wang et al. [31]. But we take $g(x)=-x$ to prove Corollary 1.2, which is used to get our Theorem 2.2. So our Theorem 2.2, Corollary 2.1 and Corollary 2.2 cannot be obtained by Theorem 3.1 of Wang et al. [31].

Similar to the Theorem 3.2 of Agbeko [1], we continue to study the estimate of $E\Phi(\max_{1\leqslant k\leqslant n} c_kS_k)$ under a different assumption from the one of (2.1).

THEOREM 2.3. *Assume that* $\{S_n\}_{n\geqslant 1}$ *is a nonnegative N-demimartingale and* $\{c_n\}_{n\geqslant 1}$ *is a nonincreasing sequence of positive numbers. Let* Φ *be a concave Young function satisfying that*

$$\int_1^\infty\frac{\varphi(t)}{t}dt=K_\varphi<\infty, \tag{2.14}$$

where K_φ *is a positive constant depending only on* φ. *Then*

$$E\Phi(\max_{1\leqslant k\leqslant n} c_kS_k)\leqslant\Phi(1)+K_\varphi c_1ES_1. \tag{2.15}$$

Proof. The proof is inspired by the proof of Theorem 3.2 of Agbeko [1]. Integrate inequality (1.6) with $\varepsilon=x>0$ on $[1,\infty)$, with respect to the measure generated by the nondecreasing function

$$\int_1^x\frac{\varphi(t)}{t}dt.$$

By Fubini theorem, we have that

$$\begin{aligned}\int_1^\infty P(\max_{1\leqslant k\leqslant n} c_kS_k \geqslant x)\varphi(x)dx &\leqslant \sum_{i=1}^n c_i \int_1^\infty E[(S_i - S_{i-1})I(\max_{1\leqslant k\leqslant n} c_kS_k \geqslant x)]\frac{\varphi(x)}{x}dx \\ &= \sum_{i=1}^n c_iE\left[(S_i - S_{i-1})\int_1^{\max\{\max_{1\leqslant k\leqslant n} c_kS_k,1\}} \frac{\varphi(x)}{x}dx\right] \\ &= \sum_{i=1}^{n-1}(c_i - c_{i+1})E\left[S_i\int_1^{\max\{\max_{1\leqslant k\leqslant n} c_kS_k,1\}} \frac{\varphi(x)}{x}dx\right] \\ &\quad + c_nE\left[S_n\int_1^{\max\{\max_{1\leqslant k\leqslant n} c_kS_k,1\}} \frac{\varphi(x)}{x}dx\right] \\ &\leqslant K_\varphi \sum_{i=1}^{n-1}(c_i - c_{i+1})ES_i + K_\varphi c_nES_n \\ &= K_\varphi c_1ES_1, \end{aligned} \tag{2.16}$$

since $ES_n = \cdots = ES_1$, $n \geqslant 1$.

Meanwhile, by Fubini theorem again, one has that

$$\begin{aligned}\int_1^\infty P(\max_{1\leqslant k\leqslant n} c_kS_k \geqslant x)\varphi(x)dx &= E\left[\int_1^{\max\{\max_{1\leqslant k\leqslant n} c_kS_k,1\}} \varphi(x)dx\right] \\ &= E\Phi(\max\{\max_{1\leqslant k\leqslant n} c_kS_k, 1\}) - \Phi(1) \\ &\geqslant E\Phi(\max_{1\leqslant k\leqslant n} c_kS_k) - \Phi(1). \end{aligned} \tag{2.17}$$

Combining (2.16) with (2.17), we obtain (2.15) finally. □

REMARK 2.2. Similar to our Theorem 2.3, Wang et al. [31] extended Theorem 3.2 of Agbeko [1] for nonnegative submartingales to the case of demimartingales (see Theorem 2.2 of Wang et al. [31]). In this paper, we also extend Theorem 3.2 of Agbeko [1] to the case of nonnegative N-demimartingales. On the other hand, if $\Phi(x) = x^p$, $0 < p < 1$, $x \geqslant 0$, then $\varphi(x) = px^{p-1}$ and $K_\varphi = \int_1^\infty \frac{\varphi(t)}{t}dt = \frac{p}{1-p}$. So, it follows from (2.15) that

$$E(\max_{1\leqslant k\leqslant n} c_kS_k)^p \leqslant 1 + \frac{p}{1-p}c_1ES_1.$$

3. Convex functions inequalities for nonnegative N-demimartingales

First, we introduce some results for convex function (see Garsia [9]) in this section. Assume that the convex function $\Phi(u)$ is of the type

$$\Phi(u) = \int_0^u \varphi(t)dt$$

with, $\varphi(t)$ strictly increasing and nonnegative in $[0,\infty)$.

To such a $\Phi(u)$ we can associate a convex function $\Psi(v)$ of the same type (the conjugate of Φ in the sense of Young) such that

$$\Psi(v) = \int_0^v \psi(t)dt,$$

where $\varphi(t)$ and $\psi(t)$ are inverses of each other.

It can be checked that

$$u\varphi(u) = \Phi(u) + \Psi(\varphi(u)),$$

$$\int_0^v t d\psi(t) = \Phi(\psi(v)), \tag{3.1}$$

$$uv \leqslant \Phi(u) + \Psi(v) \qquad \text{(Young's inequality)}, \tag{3.2}$$

$$\Phi\left(\frac{u}{\lambda}\right) \leqslant \frac{1}{\lambda}\Phi(u), \quad \forall\, \lambda \geqslant 1. \tag{3.3}$$

Furthermore, when

$$\Phi(2u) \leqslant c\Phi(u), \tag{3.4}$$

then, setting

$$p = \sup_{u>0} \frac{u\varphi(u)}{\Phi(u)}, \tag{3.5}$$

one can easily get $1 < p \leqslant c - 1 < \infty$.

Finally, one also gets

$$\Phi(\rho u) \leqslant \rho^p \Phi(u), \quad \forall\, \rho > 1,$$

$$\Psi(v) \leqslant (p-1)\Phi(\psi(v)). \tag{3.6}$$

The proofs of these assertions can be found in Krasnoselkii and Rutickii [15] or Long [16]. The following pairs of functions (see Krasnoselkii and Rutickii [15]):

$$\Phi_1(u) = \frac{u^p}{p}, \quad u \geqslant 0, \quad \Psi_1(v) = \frac{v^q}{q}, \quad v \geqslant 0, \quad p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

$$\Phi_2(u) = e^u - u - 1, \quad u \geqslant 0, \quad \Psi_2(v) = (1+v)\ln(1+v) - v, \quad v \geqslant 0$$

can serve as examples of Φ and Ψ functions above.

If $\varphi(0) = 0$, then $\psi(0) = 0$, since $\varphi(\cdot)$ strictly increases and $\varphi(\cdot)$ and $\psi(\cdot)$ are inverses of each other.

Assume that $\{X_n, \mathscr{F}_n\}_{n\geqslant 1}$ is a nonnegative submartingale and $\Psi(v)$ is a convex function (the conjugate of Φ in the sense of Young). Garsia [9] obtained that

$$E[\Psi(\max_{1\leqslant k\leqslant n} X_k)] \leqslant pE[\Psi(pX_n)], \tag{3.7}$$

where p satisfies (3.4) and (3.5).

Hu et al. [13] extended the similar inequality (3.7) for demimartingales and N-demimartingales. In this section, we also study the inequality (3.7) for nonnegative N-demimartingales.

THEOREM 3.1. *Assume that* $\{S_n\}_{n\geqslant 1}$ *is a nonnegative N-demimartingale and* $\{c_n\}_{n\geqslant 1}$ *is a nonincreasing sequence of positive numbers. Let*

$$S_n^{\max} = \max_{1\leqslant k\leqslant n} c_k S_k, \quad T_n = \sum_{i=1}^{n} c_i(S_i - S_{i-1}), \quad n \geqslant 1, \tag{3.8}$$

$\Psi(v)$ *be a convex function (the conjugate of* Φ *in the sense of Young) with* $\psi(0) = 0$. *Then,*

$$E[\Psi(S_n^{\max})] \leqslant pE[\Psi(pT_n)], \tag{3.9}$$

where p *satisfies (3.4) and (3.5).*

Proof. By (1.6) and (3.8), it follows

$$tP(S_n^{\max} \geqslant t) \leqslant E[T_n I(S_n^{\max} \geqslant t)], \quad t > 0.$$

Integrate both sides of this inequality on $[0,\infty)$ with respect to the measure $d\psi(t)$, we get

$$\int_0^\infty \int_{(S_n^{\max} \geqslant t)} t\,dP\,d\psi(t) \leqslant \int_0^\infty \int_{(S_n^{\max} \geqslant t)} T_n\,dP\,d\psi(t).$$

From Fubini theorem, it has

$$\int_\Omega \int_0^{S_n^{\max}} t\,d\psi(t)\,dP \leqslant \int_\Omega T_n \int_0^{S_n^{\max}} d\psi(t)\,dP,$$

which implies

$$E[\Phi(\psi(S_n^{\max}))] \leqslant E[T_n \psi(S_n^{\max})],$$

following from relation (3.1) and $\psi(0) = 0$. Thus, by Young's inequality (3.2), we have

$$E[\Phi(\psi(S_n^{\max}))] \leqslant E\Big[\Phi\Big(\frac{\psi(S_n^{\max})}{p}\Big)\Big] + E[\Psi(pT_n)],$$

and by (3.3) we also have

$$\frac{p-1}{p} E[\Phi(\psi(S_n^{\max}))] \leqslant E[\Psi(pT_n)].$$

Finally, by (3.6) and the two inequalities above, one can get the desired result (3.9). □

If $c_n \equiv 1$ for each $n \geqslant 1$ in Theorem 3.1, one will get the following result.

COROLLARY 3.1. *Let* $\{S_n\}_{n\geqslant 1}$ *be a nonnegative N-demimartingale and* $\Psi(v)$ *be a convex function (the conjugate of* Φ *in the sense of Young) with* $\psi(0) = 0$. *Then,*

$$E[\Psi(\max_{1\leqslant k\leqslant n} S_k)] \leqslant pE[\Psi(pS_n)], \tag{3.10}$$

where p *satisfies (3.4) and (3.5).*

REMARK 3.1. Inequalities of this type of Theorem 3.1 and Corollaries 3.1, especially in the case $\Psi(v) = \frac{v^q}{q} (q > 1)$, are classical Doob type inequalities for N-demimartingales. For example, the Doob type inequality of the nonnegative N-demimartingales was presented in Corollary 3.2.4 of Hadjikyriakou [10]. On the other hand, the convex function inequalities are also obtained in Theorem 3.2 and Corollary 3.3 of Hu et al. [13]. But similar to Remark 2.1, $g(x) = -x$ cannot be taken in Theorem 3.2 of Hu et al. [13], so our Theorem 3.1 and Corollary 3.1 cannot be obtained by Theorem 3.2 of Hu et al. [13].

4. Marshall type inequalities for N-demimartingales

Let

$$EX_1 = 0,\ E(X_i|X_1, X_2, \cdots, X_{i-1}) = 0, \quad a.s., \quad 2 \leqslant i \leqslant n$$

and $S_k = \sum_{j=1}^{k} X_j$, $1 \leqslant k \leqslant n$. Assume that $E|X_i|^p < \infty$, $p \geqslant 2$, $i = 1, 2, \cdots, n$, Mu and Miao [18] generalized the Marshall inequality to the form:

$$P\{\max_{1\leqslant k\leqslant n} S_k \geqslant \varepsilon\} \leqslant \frac{E|S_n|^p}{\alpha^{1-p}\varepsilon^p + E|S_n|^p}, \quad \forall\, \varepsilon > 0, \tag{4.1}$$

where α is the maximum value of the function

$$h(x) = 1 - x + (1-x)^{2-q}x^{q-1}, \quad x \in [0,1]$$

and $1/p + 1/q = 1$. In particular, when $p = 2$, inequality (4.1) is the Marshall's inequality.

Hu et al. [13] generalized some results of Mu and Miao [18] for martingales to the case of demimartingales. For the more details of Marshall inequality, one can refer to Uspensky [25], Marshall [17], Mu and Miao [18], Hu et al. [13] and the references therein.

In this section, we also investigate the Marshall type inequalities for N-demimartingales and get some similar results of Hu et al. [13]. The following lemma is useful to prove our results in this section.

LEMMA 4.1. *Let $\{S_n\}_{n\geqslant 1}$ be an N-demimartingale with $ES_1 \leqslant 0$ and assume that there exists $p > 1$ such that $E|S_i|^p < \infty$ for $1 \leqslant i \leqslant n$. Let $1/p + 1/q = 1$. Denote $\Lambda = \{\max_{1\leqslant k\leqslant n} S_k \geqslant \varepsilon\}$. Then for every $\varepsilon > 0$,*

$$[P(\Lambda)(1-P(\Lambda))^q + (1-P(\Lambda))P(\Lambda)^q]^{1/q}(E|S_n|^p)^{1/p} \geqslant \varepsilon P(\Lambda). \tag{4.2}$$

Proof. It is similar to the proof of Lemma 2.1 of Mu and Miao (2011) and the proof of Lemma 2.4 of Hu et al. (2012). For an N-demimartingale, it has a property that $ES_n = \cdots = ES_1$ $n \geqslant 1$. Therefore, by Hölder's inequality, inequaliy (1.5) and $ES_1 \leqslant 0$, we have that

$$\begin{aligned}(E|Y - EY|^q)^{1/q}(E|S_n|^p)^{1/p} &\geqslant E[(Y - EY)S_n] = E(YS_n) - ES_nEY \\ &\geqslant E(S_nI_\Lambda) \geqslant \varepsilon P(\Lambda),\end{aligned} \tag{4.3}$$

where $Y = I_\Lambda$. Meanwhile, it has

$$E|Y - EY|^q = P(\Lambda)(1-P(\Lambda))^q + (1-P(\Lambda))P(\Lambda)^q. \tag{4.4}$$

Thus, (4.2) follows from (4.3) and (4.4) finally. □

THEOREM 4.1. *Let $\{S_n\}_{n\geqslant 1}$ be an N-demimartingale with $ES_1 \leqslant 0$. Assume that there exists $p > 1$ such that $0 < E|S_i|^p < \infty$ for $1 \leqslant i \leqslant n$. Let $1/p+1/q=1$. Then for every $\varepsilon > 0$,*

$$P(\max_{1\leqslant k\leqslant n} S_k \geqslant \varepsilon) \leqslant \frac{1}{1+M}, \tag{4.5}$$

where M is the positive solution of the following equation:

$$x^q = (\beta-1)x+\beta, \ \ x\in(0,\infty),$$

and $\beta = \varepsilon^q/(E|S_n|^p)^{q/p}$.

Proof. Similar to the proofs of Theorem 2.1 of Mu and Miao [18] and Theorem 2.1 of Hu et al. [13], by (4.2) in Lemma 4.1, one will get (4.5) immediately. We omit its proof here. □

THEOREM 4.2. *Let $\{S_n\}_{n\geqslant 1}$ be an N-demimartingale with $ES_1 \leqslant 0$. Assume that there exists $p \geqslant 2$ such that $E|S_i|^p < \infty$ for $1 \leqslant i \leqslant n$. Let $1/p+1/q=1$. Then for every $\varepsilon > 0$,*

$$P(\max_{1\leqslant k\leqslant n} S_k \geqslant \varepsilon) \leqslant \frac{E|S_n|^p}{\alpha^{1-p}\varepsilon^p + E|S_n|^p}, \tag{4.6}$$

where α is the maximum value of function

$$h(x) = 1-x+(1-x)^{2-q}x^{q-1}, \quad x\in[0,1].$$

In particular, inequality (4.6) is the Marshall's inequality when $p=2$.

Proof. Similar to the proofs of Theorem 3.1 of Mu and Miao [18], by (4.2), one can easily get (4.6) finally. □

COROLLARY 4.1. *If $\{S_n\}_{n\geqslant 1}$ is an N-demimartingale with $ES_1 \leqslant 0$ and $E|S_i|^3 < \infty$ for $1 \leqslant i \leqslant n$, then for every $\varepsilon > 0$,*

$$P(\max_{1\leqslant k\leqslant n} S_k \geqslant \varepsilon) \leqslant \frac{E|S_n|^3}{4\varepsilon^3/(3+2\sqrt{2}) + E|S_n|^3}.$$

Proof. As an application of Theorem 4.2, similar to the proof of Corollary 3.2 of Mu and Miao [18], it is easy to have the desired result. □

COROLLARY 4.2. *Let $\{S_n\}_{n\geqslant 1}$ be an N-demimartingale with $ES_1 \leqslant 0$. Assume that there exists $p \geqslant 2$ such that $E|S_i|^p < \infty$ for $1 \leqslant i \leqslant n$. Then for every $\varepsilon > 0$,*

$$P(\max_{1\leqslant k\leqslant n} S_k \geqslant \varepsilon) \leqslant \frac{E|S_n|^p}{4(\varepsilon/2)^p + E|S_n|^p}.$$

If $p=2$, then the Marshall's inequality also holds true.

Proof. Since the proof is similar to the proof of Theorem 3.2 of Mu and Miao [18], we omit its proof here. □

THEOREM 4.3. *Let* $\{S_n\}_{n\geqslant 1}$ *be an N-demimartingale with* $ES_1 \leqslant 0$ *and assume that there exists* $\delta > 0$ *such that* $E|S_i|^{1+\delta} < \infty$ *for* $1 \leqslant i \leqslant n$. *Then for every* $\varepsilon \geqslant E|S_n|$,

$$P(\max_{1\leqslant k\leqslant n} S_k \geqslant \varepsilon) \leqslant \frac{E|S_n|}{\varepsilon + E|S_n|}.$$

Proof. Similar to the proofs of Theorem 3.3 of Mu and Miao [18] and Theorem 2.4 of Hu et al. [13], we will have the desired result by (1.5) and Lemma 4.1 finally. □

REMARK 4.1. Based on the paper of Mu and Miao [18], Hu et al. [13] not only obtained the maximal Marshall type inequalities for demimartingales, but also had the minimal Marshall type inequalities for nonnegative demimartingales. Under some conditions, we can find that the maximal Marshall type inequalities for N-demimartingales also hold true. The partial sum of mean zero negatively associated random variables as well as a martingale with the natural choice of σ-algebras are N-demimartingales, the results obtained in this section also hold true for these random variables.

5. Some other maximal type inequalities for nonnegative N-demimartingales

In this section, we go on to investigate some other maximal type inequalities for nonnegative N-demimartingales. Let $\ln^+ x \doteq \ln(\max(x,1))$. Pakes [21] proved a limit superior of maximal inequality for the nonnegative submartingales. As a slight generalization of Pakes [21], Iksanov and Marynych [14] extended their result to the case of nonnegative martingales and got the following theorem.

THEOREM 5.1. *Let* $\{Z_n, n \in \mathbb{N}_0\}$ *be a nonnegative martingale with* $Z_0 = a > 0$, $E(Z_n \ln^+ Z_n) < \infty$, $n \in \mathbb{N}$, *and* $\lim_{n\to\infty} E(Z_n \ln^+ Z_n) = \infty$, *then*

$$\limsup_{n\to\infty} \frac{E(\max_{0\leqslant k\leqslant n} Z_k)}{E(Z_n \ln^+ Z_n)} \leqslant a.$$

In this section, we generalize Theorem 5.1 to the case of nonnegative N-demimartingales.

THEOREM 5.2. *Let* $\{S_n, n \geqslant 1\}$ *be a nonnegative N-demimartingale with* $S_1 = a > 0$. *Assume that* $E(S_n \ln^+ S_n) < \infty$, $n \in \mathbb{N}$, *and* $\lim_{n\to\infty} E(S_n \ln^+ S_n) = \infty$. *Then*

$$\limsup_{n\to\infty} \frac{E(\max_{1\leqslant k\leqslant n} S_k)}{E(S_n \ln^+ S_n)} \leqslant a. \tag{5.1}$$

Proof. It can be studied the N-demimartingale S_n/a instead of S_n. Without loss of generality, we assume that $S_1 = 1$ in the proof. It is known that for any $a, b > 0$ and $x_0 > e$, it has

$$b \ln^+ a \leqslant b \ln^+ b + a x_0^{-1} + b(\ln x_0 - 1). \tag{5.2}$$

Observe that $\max_{1\leqslant k\leqslant n} S_k \geqslant S_1 = 1$ and hence we have by (1.5) and (5.2) with $a = \max_{1\leqslant k\leqslant n} S_k$ and $b = S_n$ that for any $x_0 > e$,

$$\begin{aligned} E(\max_{1\leqslant k\leqslant n} S_k) - 1 &= \int_0^\infty P(\max_{1\leqslant k\leqslant n} S_k \geqslant t)dt - 1 = \int_1^\infty P(\max_{1\leqslant k\leqslant n} S_k \geqslant t)dt \\ &\leqslant \int_1^\infty \frac{1}{t} E(S_n I(\max_{1\leqslant k\leqslant n} S_k \geqslant t))dt \\ &= E[S_n \int_1^{\max_{1\leqslant k\leqslant n} S_k} \frac{1}{t}dt] = E(S_n \ln^+(\max_{1\leqslant k\leqslant n} S_k)) \qquad (5.3) \\ &\leqslant E(S_n \ln^+ S_n) + x_0^{-1} E(\max_{1\leqslant k\leqslant n} S_k) + ES_n(\ln x_0 - 1). \end{aligned}$$

Combining $ES_n = \cdots = ES_1$ with $ES_1 = 1$, we get that

$$\limsup_{n\to\infty} \frac{E(\max_{1\leqslant k\leqslant n} S_k)}{E(S_n \ln^+ S_n)} \leqslant \frac{x_0}{x_0 - 1}. \qquad (5.4)$$

To prove (5.1), we make $x_0 \to \infty$ in (5.4) to complete the proof finally. □

REMARK 5.1. Wang et al. [29] extended Theorem 5.1 for nonnegative martingales to the case of nonnegative demimartingales (see Theorem 3.1 of Wang et al. [29]). Similar to Wang et al. [29], we extend Theorem 5.1 to the case of nonnegative N-demimartingales.

Harremoës [12] obtained the following inequalities for nonnegative martingales (see Theorem 3 of Harremoës [12]).

THEOREM 5.3. *Let* $(S_1, \mathscr{F}_1)$, $(S_2, \mathscr{F}_2)$, $\cdots$, $(S_n, \mathscr{F}_n)$ *be a nonnegative martingale. If* $S_1 = 1$, *then*

$$\gamma(E(\max_{1\leqslant k\leqslant n} S_k)) \leqslant E(S_n \ln S_n) \qquad (5.5)$$

and

$$\gamma(E(\min_{1\leqslant k\leqslant n} S_k)) \leqslant E(S_n \ln S_n), \qquad (5.6)$$

where $\gamma(x) = x - 1 - \ln x$ *for* $x > 0$.

It can be seen that $\gamma(x)$ is a strictly convex function with minimum $\gamma(1) = 0$. Prakasa Rao [23] extended (5.5) and (5.6) for nonnegative martingales to the case of nonnegative demimartingales (see Theorem 2.10 of Prakasa Rao [23]).

Inspired by Harremoës [12] and Prakasa Rao [23], we extend (5.5) to the case of nonnegative N-demimartingales.

THEOREM 5.4. *Let* $\{S_n, n \geqslant 1\}$ *be a nonnegative N-demimartingale. If* $S_1 = 1$, *then (5.5) holds.*

Proof. Since $\max_{1\leqslant k\leqslant n} S_k \geqslant S_1 = 1$, by the proof of (5.3), it has

$$E(\max_{1\leqslant k\leqslant n} S_k) - 1 \leqslant E(S_n \ln^+(\max_{1\leqslant k\leqslant n} S_k)) = E(S_n \ln(\max_{1\leqslant k\leqslant n} S_k)).$$

Since γ is nonnegative and $ES_n = \cdots = ES_1 = 1$, we have

$$
\begin{aligned}
E(\max_{1\leqslant k\leqslant n} S_k) - 1 &\leqslant E\left[S_n\left(\ln(\max_{1\leqslant k\leqslant n} S_k) + \gamma(\frac{\max_{1\leqslant k\leqslant n} S_k}{S_n E(\max_{1\leqslant k\leqslant n} S_k)})\right)\right] \\
&= 1 - ES_n + E(S_n \ln S_n) + E(S_n \ln(E \max_{1\leqslant k\leqslant n} S_k)) \\
&= E(S_n \ln S_n) + \ln E(\max_{1\leqslant k\leqslant n} S_k). \qquad (5.7)
\end{aligned}
$$

According to the definition of γ, we obtain (5.5) by reorganizing (5.7) immediately. □

REMARK 5.2. By using maximal inequality for demisubmaringales, Prakasa Rao [23] investigated the Orlicz functions and obtained some maximal ϕ-inequalities for nonnegative demisubmartingales (see Theorem 3.1, Theorem 3.2, Theorem 3.4-Theorem 3.8 of Prakasa Rao [23]). Combining the maximal inequality (1.5) with Prakasa Rao [23], one can have some similar ϕ-inequalities for nonnegative N-demi(super)martingales. We omit them here.

By taking $f \equiv 1$ and $f \equiv -1$ in Definition 1.1 and Definition 1.2, we can check that demimartingales and N-demimartingales have the same property that $ES_n = \cdots = ES_1$, $n \geqslant 1$. Furthermore, it can be found that demimartingales and N-demimartingales have some similar inequalities such as various maximal type inequalities as well as martingales. On the other hand, there are some differences between demimartingales and N-demimartingales. For example, like to martingales, demimartingales have some minimal type inequalities (see Theorem 2.8-Theorem 2.10 of Prakasa Rao [23], Lemma 2.5 and Theorem 2.2 of Hu et al. [13]), but these minimal type inequalities for N-demimartingales have not been obtained up to now. Meanwhile, some exponential inequalities and Marcinkiewicz-Zygmund type inequalities for N-demimartingales have been presented in Christofides and Hadjikyriakou [6] and Hadjikyriakou [11], respectively. But the moment inequalities for demimartingales are presented in Christofides and Hadjikyriakou [7], which are different from the ones of N-demimartingales obtained by Hadjikyriakou [11]. So it is also interesting to investigate the properties of demimartingales and N-demimartingales. By the way, some related definitions such as conditional association, conditional negatively associated, conditional demimartingales and conditional N-demimartingales have been received more attention. One can refer to Christofides and Hadjikyriakou [8], Hadjikyriakou [10] and Prakasa Rao [24] for the details.

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