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APPLICATIONS OF HÖLDER'S AND JENSEN'S INEQUALITIES IN STUDYING THE β–ABSOLUTE CONVERGENCE OF VILENKIN–FOURIER SERIES

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Abstract. In this paper the β-absolute convergence ($0<\beta\leqslant 2$) of Vilenkin-Fourier series for the functions of various classes of functions of generalized bounded fluctuation is studied. In proving our main results we use famous Hölder's inequality and Jensen's inequality for integrals. As a particular case our results give bounded Vilenkin group analogue of the corresponding circle group results of Schramm and Waterman [Acta. Math. Acad. Sci. Hungar 40 (3–4) (1982), 273–276]. One of our results generalizes the earlier result of Uno [Sci. Rep. Kanazawa Univ. 29 (2) (1984), 97–102]. It also generalizes the results of Onneweer [Duke Math. J. 39 (4) (1972), 599–609; Corollary 3 and Corollary 4].

1. Introduction

Let G be a Vilenkin group, that is, a compact metrizable zero-dimensional (infinite) abelian group. Then the dual group X of G is a discrete, countable, torsion, abelian group (see [4, Theorems 24.15 and 24.26]). In 1947, N. J. Vilenkin [16] developed part of the Fourier theory on G and proved an analogue of Bernstein's theorem [1, Vol. II, p. 154] concerning the absolute convergence of Vilenkin-Fourier series for a primary group G [16, Theorem 5]. Later Onneweer and Waterman [5]–[8] introduced various classes of functions of bounded fluctuations and studied the convergence problems for functions of these classes. Interestingly, Onneweer [5, Corollary 2] proved an analogue of Bernstein's theorem for any bounded Vilenkin group and an analogue of Zygmund's theorem [1, Vol. II, p. 161] for functions of p-generalized bounded fluctuation ($1\leqslant p<2$) defined on any bounded Vilenkin group [5, Corollary 3]. Onneweer continued study further and in his second paper he obtained a sufficiency condition in terms of n-th integral modulus of continuity of order p of a function $f\in \mathrm{L^p}(G)$ to be in $\mathrm{A}(\beta)$ [6, Theorem 1] and derived an analogue of Szász's theorem [14] from it. Vilenkin and Rubinstěin [17] proved an analogue of a well-known theorem Stečhkin [13]. Quek and Yap [10] then extended above results of Onneweer to arbitrary Vilenkin groups and Uno [15] proved an analogue of a circle group result of Schramm and Waterman [11, Theorem 1] for any Vilenkin group. In this paper, for any bounded Vilenkin

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group, we first generalize the result of Uno (see Theorem 1) and then prove a similar result for functions in the class of ϕ-Λ-generalized bounded fluctuation (see Theorem 2). In proving our main results Theorem 1 and Theorem 2 we use famous Hölder's and Jensen's inequalities for integrals respectively. Our result (see Theorem 1) also generalizes the results of Onneweer [5, Corollary 3 and Corollary 4]. Further, our results contains as special cases Vilenkin group analogues of both the circle group results [11, Theorem 1 and Theorem 2] of Schramm and Waterman. As noted by Schramm and Waterman [11, p. 273], here also we observe that though the result for $\phi\Lambda GBF$, Theorem 2, is more general in the sense that it is more widely applicable, but unfortunately it does not contain Theorem 1 as a special case.

2. Notation and Definitions

For G and X as above, Vilenkin [16, Sections 1.1, 1.2] proved the existence of a sequence $\{X_n\}$ of finite subgroups of X and of a sequence $\{\varphi_n\}$ in X such that the following hold:

(i) $X_0 = \{\chi_0\}$, where χ_0 is the identity character on G;

(ii) $X_0 \subset X_1 \subset X_2 \subset \ldots$;

(iii) for each $n \geqslant 1$, the quotient group X_n/X_{n-1} is of prime order p_n;

(iv) $X = \bigcup_{n=0}^{\infty} X_n$;

(v) $\varphi_n \in X_{n+1} \setminus X_n$ for all $n \geqslant 0$;

(v) $\varphi_n^{p_{n+1}} \in X_n$ for all $n \geqslant 0$.

The group G is bounded if

$$p_0 = \sup_{i=1,2,\ldots} p_i < \infty;$$

otherwise, G is said to be unbounded. Using the φ_n's, we can enumerate X as follows. Let $m_0 = 1$, and let $m_n = \prod_{i=1}^{n} p_i$ for $n \in \mathbb{N}$. Then each $k \in \mathbb{N}$ can be uniquely represented as $k = \sum_{i=0}^{s} a_i m_i$ with $0 \leqslant a_i < p_{i+1}$ for $0 \leqslant i \leqslant s$; we define χ_k by the formula $\chi_k = \varphi_0^{a_0} \cdot \ldots \cdot \varphi_s^{a_s}$.

$G = \prod_{n=1}^{\infty} \mathbb{Z}_{p_n}$, $\{p_n\}$ – a sequence of prime numbers, is a standard example. If $p_n = 2$ for all n, X is the group of Walsh functions ψ_n, $n = 0, 1, 2, \ldots$, and $X_n = \{\psi_0, \psi_1, \ldots, \psi_{2^n-1}\}$ (using Payley enumeration; see [9]) described by N. J. Fine [3]. If $p_n = p$ for all n, X is the group of generalized Walsh functions [2].

Let dx or m denote the normalized Haar measure on G. For $f \in L^1(G)$, the Vilenkin-Fourier series of f is given by

$$S[f](x) = \sum_{n=0}^{\infty} \hat{f}(n)\chi_n(x), \quad \hat{f}(n) = \int_G f(x)\bar{\chi}_n(x)dx,$$

where $\hat{f}(n)$ $(n = 0, 1, 2, \ldots)$ is the nth Vilenkin-Fourier coefficient of f. It is said to be β*-absolutely convergent*, where β is a positive real number, if $\sum_{n=0}^{\infty} |\hat{f}(n)|^{\beta} < \infty$. In this case we write $f \in \mathrm{A}(\beta)$ and we shall denote $\mathrm{A}(1)$ by A.

Observe that for each n, $X_n = \{\chi_k : 0 \leqslant k < m_n\}$. Let G_n be the annihilator of X_n, that is,

$$G_n = \{x \in G : \chi(x) = 1, \chi \in X_n\} = \{x \in G : \chi_k(x) = 1,\ 0 \leqslant k < m_n\}.$$

Then obviously, $G = G_0 \supset G_1 \supset G_2 \supset \dots$, $\bigcap_{n=0}^{\infty} G_n = \{0\}$, and the G_n's form a fundamental system of neighborhoods of zero in G which are compact, open and closed subgroups of G. Further, the index of G_n in G is m_n and since the Haar measure is translation invariant with $m(G) = 1$, one has $m(G_n) = 1/m_n$. In [16, Section 3.2] Vilenkin proved that for each $n \geqslant 0$ there exists $x_n \in G_n \setminus G_{n+1}$ such that $\chi_{m_n}(x_n) = \exp(2\pi i/p_{n+1})$ and observed that each $x \in G$ has a unique representation $x = \sum_{i=0}^{\infty} b_i x_i$, with $0 \leqslant b_i < p_{i+1}$ for all $i \geqslant 0$. This representation of the elements of G enables one to order them by means of the lexicographic ordering of the corresponding sequence $\{b_n\}$ and one observes that for each $n = 1, 2, \dots$,

$$G_n = \left\{x \in G : x = \sum_{i=0}^{\infty} b_i x_i,\ b_0 = \dots = b_{n-1} = 0\right\}.$$

Consequently, each coset of G_n in G has a representation of the form $z + G_n$, where $z = \sum_{i=0}^{n-1} b_i x_i$ for some choice of the b_i with $0 \leqslant b_i < p_{i+1}$. These z, ordered lexicographically, are denoted by $\{z_{\alpha}^{(n)}\}$ $(0 \leqslant \alpha < m_n)$.

It may be noted that the choice of $\varphi_n \in X_{n+1} \setminus X_n$ and of the $x_n \in G_n \setminus G_{n+1}$ is not uniquely determined by the groups X and G. In the following, it is assumed that a particular choice has been made.

Let f be a complex function defined on G, let $p \geqslant 1$ be a real number, let $\Lambda = \{\lambda_n\}$ be a non-decreasing sequence of positive real numbers such that $\sum_{n=1}^{\infty}(1/\lambda_n)$ diverges, and let $\phi : [0,\infty) \to [0,\infty)$ be a strictly increasing function. Customarily ϕ is considered to be a convex function such that

$$\phi(0) = 0, \quad \frac{\phi(x)}{x} \to 0 \ \ (x \to 0_+), \quad \frac{\phi(x)}{x} \to \infty \ \ (x \to \infty).$$

Such a function is called an N-function. It is necessarily continuous and strictly increasing on $[0,\infty)$. For $H \subset G$, the *oscillation* of f on H is defined as

$$\operatorname{osc}(f;H) = \sup\{|f(x_1) - f(x_2)| : x_1, x_2 \in H\}.$$

We need the following definitions.

DEFINITION 1. For $n \in \mathbb{N} \cup \{0\}$, the *$n$-th modulus of continuity* [5, Definition 2] of f on G is defined as

$$\omega_n(f) = \sup\{|(T_h f - f)(x)| : x \in G, h \in G_n\},$$

where $(T_h f)(x) = f(x+h)$, for all $x \in G$.

DEFINITION 2. For $\alpha > 0$ the function f is said to satisfy the Lipschitz condition of order α on G (written as $f \in \operatorname{Lip}\alpha$) [5, Definition 3] if $\omega_n(f) = O(m_n^{-\alpha})$.

DEFINITION 3. For $n \in \mathbb{N} \cup \{0\}$ and $1 \leqslant p < \infty$, the *n-th integral modulus of continuity of order p* for a function f in $\mathrm{L^p}(G)$ [6, Defintion 1] is defined as

$$\omega^{(p)}(f,n) = \sup\{||T_h f - f||_p : h \in G_n\}.$$

When $p = \infty$ we put, $\omega^{(\infty)}(f,n) = \omega_n(f)$, where $\omega_n(f)$ is as in Definition 1.

DEFINITION 4. We say f is of

(a) *p-Λ-bounded fluctuation* ($f \in \Lambda\mathrm{BF}^{(p)}$) if the *total p-Λ-fluctuation* of f on G, given by

$$F_{p\Lambda}(f;G) = \sup\left\{\left(\sum_{n=1}^{\infty}\frac{(\mathrm{osc}(f;I_n))^p}{\lambda_n}\right)^{1/p}\right\}$$

is finite, where the supremum is taken over all sequences $\{I_n\}$ of disjoint cosets in G;

(b) *ϕ-Λ-bounded fluctuation* ($f \in \phi\Lambda\mathrm{BF}$) if the *total ϕ-Λ-fluctuation* of f on G, given by

$$F_{\phi\Lambda}(f;G) = \sup\left\{\sum_{n=1}^{\infty}\frac{\phi(\mathrm{osc}(f;I_n))}{\lambda_n}\right\}$$

is finite, where the supremum is taken over all sequences $\{I_n\}$ of disjoint cosets in G.

DEFINITION 5. We say f is of

(a) *p-Λ-generalized bounded fluctuation* ($f \in \Lambda\mathrm{GBF}^{(p)}$) if the *total generalized p-Λ-fluctuation* of f on G, given by

$$\Lambda GF_p(f;G) = \sup_n \sup_\alpha \left(\sum_{j=0}^{m_n-1}\frac{\left(\mathrm{osc}(f;z_\alpha^{(n)}+G_n)\right)^p}{\lambda_{j+1}}\right)^{1/p}$$

is finite, where $\sup_\alpha$ denotes the supremum over all permutations of $\{0,1,...,m_n-1\}$;

(b) *ϕ-Λ-generalized bounded fluctuation* ($f \in \phi\Lambda\mathrm{GBF}$) if the *total generalized ϕ-Λ-fluctuation* of f on G, given by

$$\Lambda GF_\phi(f;G) = \sup_n \sup_\alpha \sum_{j=0}^{m_n-1}\frac{\phi\left(\mathrm{osc}(f;z_\alpha^{(n)}+G_n)\right)}{\lambda_{j+1}}$$

is finite, where $\sup_\alpha$ is as in (a) above.

We observe that when $p = 1$, the class $\Lambda\mathrm{BF}^{(p)}$ is same as the class $\Lambda\mathrm{BF}$ of functions of Λ-bounded fluctuation on G (see [7, Definition 2]). Also, if $\phi(x) = x^p$ ($p \geqslant 1$), then $\phi\Lambda\mathrm{BF} = \Lambda\mathrm{BF}^{(p)}$ and $\phi\Lambda\mathrm{GBF} = \Lambda\mathrm{GBF}^{(p)}$; we shall omit writing the superscript (p) when $p = 1$. Further, from definitions it is clear that $\Lambda\mathrm{BF}^{(p)} \subset \Lambda\mathrm{GBF}^{(p)}$ and $\phi\Lambda\mathrm{BF} \subset \phi\Lambda\mathrm{GBF}$.

3. Statements of Results

Let G be bounded and $f \in \mathrm{L}^1(G)$. We prove the following results.

THEOREM 1. *If* $f \in \Lambda\mathrm{GBF}^{(p)}$, $1 \leqslant p < 2r$, $1 \leqslant r < \infty$ *and*

$$\sum_{n=0}^{\infty}\left[\frac{(m_n)^{2/\beta-1}\left(\omega^{(p+(2-p)s)}(f,n)\right)^{2-p/r}}{\left(\sum_{j=1}^{m_n}\frac{1}{\lambda_j}\right)^{1/r}}\right]^{\beta/2} < \infty,$$

in which $\frac{1}{r}+\frac{1}{s}=1$, *then* $f \in \mathrm{A}(\beta)$ *for* $0 < \beta \leqslant 2$.

REMARK 1. Since $\Lambda\mathrm{BF}^{(p)} \subset \Lambda\mathrm{GBF}^{(p)}$, Theorem 1 holds for functions in $\Lambda\mathrm{BF}^{(p)}$ also. Taking $\beta = 1$ in Theorem 1 we obtain

COROLLARY 1. *Let* $f \in \Lambda\mathrm{GBF}^{(p)}$, $1 \leqslant r < \infty$, $\frac{1}{r}+\frac{1}{s}=1$ *and* $1 \leqslant p < 2r$. *Then* $f \in \mathrm{A}$ *if*

$$\sum_{n=0}^{\infty}\frac{(m_n)^{1/2}\left(\omega^{(p+(2-p)s)}(f,n)\right)^{1-p/2r}}{\left(\sum_{j=1}^{m_n}\frac{1}{\lambda_j}\right)^{1/2r}} < \infty. \tag{1}$$

REMARK 2. Corollary 1 is a result equivalent to a result of Yoshikazu Uno [15] for a bounded Vilenkin group.

COROLLARY 2. *If* $f \in \mathrm{GBF}^{(p)}$ *with* $1 \leqslant p < 2$ *and if* $f \in \mathrm{Lip}\,\alpha$ *for some* $\alpha > 0$ *then* $f \in \mathrm{A}$.

COROLLARY 3. *If* $f \in \mathrm{Lip}\,\alpha$ *for some* $\alpha > 0$ *and if* $f \in \Lambda\mathrm{GBF}$ *for some sequence* $\Lambda = \{\lambda_n\}$ *such that* $\lambda_{m_n} = O(m_n^{\gamma})$, *with* $0 \leqslant \gamma < \alpha$ *then* $f \in \mathrm{A}$.

REMARK 3. Corollaries 2 and 3 are results of Onneweer [5, Corollary 3 and Corollary 4]. Thus our Theorem 1 generalizes these results of Onneweer.

THEOREM 2. *If* $f \in \phi\Lambda\mathrm{GBF}$, $1 \leqslant p < 2r$, $1 \leqslant r < \infty$ *and*

$$\sum_{n=0}^{\infty}\left[(m_n)^{2/\beta-1}\left\{\phi^{-1}\left(\frac{\left(\omega^{(p+(2-p)s)}(f,n)\right)^{2r-p}}{\sum_{j=1}^{m_n}\frac{1}{\lambda_j}}\right)\right\}^{1/r}\right]^{\beta/2} < \infty,$$

then $f \in \mathrm{A}(\beta)$ *for* $0 < \beta \leqslant 2$, *in which* $\frac{1}{r}+\frac{1}{s}=1$ *and* ϕ *is a* Δ_2-*function (that is, there is a constant* $d \geqslant 2$ *such that* $\phi(2x) \leqslant d\phi(x)$, $\forall x \geqslant 0$*).*

REMARK 4. Since $\phi\Lambda\mathrm{BF} \subset \phi\Lambda\mathrm{GBF}$, Theorem 2 holds for functions in $\phi\Lambda\mathrm{BF}$ also. With $\beta = 1$, Theorems 1 and 2 are bounded Vilenkin group analogues of the corresponding circle group results of Schramm and Waterman [11].

4. Proof of the Results

The following lemmas are needed.

LEMMA 1. ([17, p. 5]) *For each* $N = 0, 1, 2, \ldots$ *and* $k \geqslant m_N$ *we have*

(a) $\int_{G_N} \chi_k(h)dh = 0$;

(b) $\int_{G_N} |\chi_k(h) - 1|^2 dh = 2\int_{G_N}[1 - Re\chi_k(h)]dh = 2|G_N| = \frac{2}{m_N}$.

LEMMA 2. ([12, Lemma 2]) *If* $u_n \geqslant 0$ *for* $n \in \mathbb{N}$, $u_n \not\equiv 0$ *and a function* $F(u)$ *is concave, increasing, and* $F(0) = 0$, *then*

$$\sum_{n=1}^{\infty} F(u_n) \leqslant 2 \sum_{n=1}^{\infty} F\left(\frac{1}{n}\sum_{k=n}^{\infty} u_k\right).$$

LEMMA 3. ([11]) *If* $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n > 0$, $\sum_{i=1}^{n} a_i = 1$ *and* $b_1 \geqslant b_2 \geqslant \ldots \geqslant b_n$, *then*

$$\sum_{i=1}^{n} b_i \leqslant n \sum_{i=1}^{n} a_i b_i.$$

Proof of Theorem 1. Let $M \in \mathbb{N}$ be fixed and let $N \in \mathbb{N}$ be the integer such that $m_N \leqslant M < m_{N+1}$. For each $\alpha = 0, 1, \ldots, m_N - 1$ and $h \in G_N$ put

$$f_\alpha(x) = f\left(x + z_\alpha^{(N)} + h\right) - f\left(x + z_\alpha^{(N)}\right), \quad \forall x \in G.$$

Then for each $n \geqslant 0$ we have

$$\hat{f}_\alpha(n) = \hat{f}(n)\chi_n\left(z_\alpha^{(N)} + h\right) - \hat{f}(n)\chi_n\left(z_\alpha^{(N)}\right) = \hat{f}(n)\chi_n\left(z_\alpha^{(N)}\right)(\chi_n(h) - 1).$$

Since $f \in \Lambda\mathrm{GBF}^{(p)}$ for any $x \in G = G_0$ we see that

$$\begin{aligned}
|f(x)|^p &= |f(0) + f(x) - f(0)|^p \\
&\leqslant 2^p|f(0)|^p + 2^p|f(x) - f(0)|^p \\
&\leqslant 2^p|f(0)|^p + 2^p\left(\operatorname{osc}\left(f; z_0^{(0)} + G_0\right)\right)^p \\
&= 2^p|f(0)|^p + 2^p\lambda_1 \frac{\left(\operatorname{osc}\left(f; z_0^{(0)} + G_0\right)\right)^p}{\lambda_1} \\
&\leqslant 2^p|f(0)|^p + 2^p\lambda_1 (\Lambda GF_p(f;G))^p.
\end{aligned}$$

Thus f is bounded on G and hence $f \in L^2(G)$. As a result each $f_\alpha \in L^2(G)$ and so by Parseval's equality (since $|\chi_n(z_\alpha^{(N)})| = 1$) we have

$$B(h) \equiv \sum_{n=0}^{\infty} |\hat{f}(n)|^2 |\chi_n(h) - 1|^2 = ||f_\alpha||_2^2, \text{ for all } \alpha. \tag{2}$$

Now, suppose $r > 1$ and set $2 = \frac{p+(2-p)s}{s} + \frac{p}{r}$; then using the Hölder's inequality we get

$$\begin{aligned} ||f_\alpha||_2^2 &= \int_G |f_\alpha(x)|^2 dx \\ &= \int_G |f_\alpha(x)|^{\left(\frac{p+(2-p)s}{s}+\frac{p}{r}\right)} dx \\ &= \int_G \left(|f_\alpha(x)|^{(p+(2-p)s)}\right)^{1/s} (|f_\alpha(x)|^p)^{1/r} dx \\ &\leqslant \left\{\int_G |f_\alpha(x)|^{(p+(2-p)s)} dx\right\}^{1/s} \left\{\int_G |f_\alpha(x)|^p dx\right\}^{1/r} \\ &\leqslant (\Omega_N)^{1/r} \left(\int_G |f_\alpha(x)|^p dx\right)^{1/r}, \end{aligned}$$

since $h \in G_N$, where $\Omega_N = \left(\omega^{(p+(2-p)s)}(f,N)\right)^{2r-p}$. This together with (2) implies

$$(B(h))^r \leqslant \Omega_N \int_G |f_\alpha(x)|^p dx, \tag{3}$$

for all $\alpha = 0, 1, \ldots, m_N - 1$. Since the left hand side of (3) is independent of α, multiplying both the sides of it by $(1/\lambda_{\alpha+1})$ and taking summation over α, we get

$$(B(h))^r \theta_{m_N} \leqslant \Omega_N \int_G \left(\sum_{\alpha=0}^{m_N-1} \frac{|f_\alpha(x)|^p}{\lambda_{\alpha+1}}\right) dx,$$

where $\theta_\nu = \sum_{j=1}^{\nu}(1/\lambda_j) = \sum_{j=0}^{\nu-1}(1/\lambda_{j+1})$, for all $\nu \in \mathbb{N}$; and hence

$$B(h) \leqslant \left(\frac{\Omega_N}{\theta_{m_N}}\right)^{1/r} \left\{\int_G \left(\sum_{\alpha=0}^{m_N-1} \frac{|f_\alpha(x)|^p}{\lambda_{\alpha+1}}\right) dx\right\}^{1/r}.$$

Integrating both sides of this inequality over G_N with respect to h we get

$$\int_{G_N} B(h) dh \leqslant \left(\frac{\Omega_N}{\theta_{m_N}}\right)^{1/r} \int_{G_N} \left\{\int_G \sum_{\alpha=0}^{m_N-1} \frac{|f_\alpha(x)|^p}{\lambda_{\alpha+1}} dx\right\}^{1/r} dh. \tag{4}$$

Now, for any $h \in G_N$ and any $x \in G$ the points $x + z_\alpha^{(N)} + h$ and $x + z_\alpha^{(N)}$ lie in the coset $x + z_\alpha^{(N)} + G_N$ of G_N in G and hence

$$|f_\alpha(x)| = |f(x + z_\alpha^{(N)} + h) - f(x + z_\alpha^{(N)})| \leqslant \operatorname{osc}(f, x + z_\alpha^{(N)} + G_N). \tag{5}$$

Since $f \in \Lambda\mathrm{GBF}^{(p)}$, for any $h \in G_N$ and $x \in G$, in view of (5), we have

$$\sum_{\alpha=0}^{m_N-1} \frac{|f_\alpha(x)|^p}{\lambda_{\alpha+1}} \leqslant \sum_{\alpha=0}^{m_N-1} \frac{\left(\operatorname{osc}(f; x + z_\alpha^{(N)} + G_N)\right)^p}{\lambda_{\alpha+1}} \leqslant (\Lambda GF_p(f;G))^p, \tag{6}$$

because for any $x \in G$, the finite sequence of cosets $\{x+z_\alpha^{(N)}+G_N : \alpha = 0,1,\ldots,m_N-1\}$ is a permutation of the sequence $\{z_\alpha^{(N)}+G_N : \alpha = 0,1,\ldots,m_N-1\}$. Further, from (2),

$$\int_{G_N} B(h)dh \geqslant \sum_{k=M}^{\infty} |\hat{f}(k)|^2 \int_{G_N} |\chi_k(h)-1|^2 dh = \left(\frac{2}{m_N}\right) \sum_{k=M}^{\infty} |\hat{f}(k)|^2, \tag{7}$$

in view of Lemma 1, because $k \geqslant M$ implies $k \geqslant m_N$. Using (6) and (7) in (4) we get

$$R_M \equiv \sum_{k=M}^{\infty} |\hat{f}(k)|^2 = O\left[\left(\frac{\Omega_N}{\theta_{m_N}}\right)^{1/r}\right]. \tag{8}$$

Applying Lemma 2 with $u_k = |\hat{f}(k)|^2$ and $F(u) = u^{\beta/2}$ we get

$$\sum_{k=1}^{\infty} |\hat{f}(k)|^\beta = \sum_{k=1}^{\infty} F(u_k) \leqslant 2\sum_{k=1}^{\infty} F\left(\frac{1}{k}\sum_{j=k}^{\infty} |\hat{f}(j)|^2\right) = 2\sum_{k=1}^{\infty} F\left(\frac{R_k}{k}\right). \tag{9}$$

Thus in view of (8) we get

$$\begin{aligned}
\sum_{k=1}^{\infty} |\hat{f}(k)|^\beta &= O(1)\sum_{k=1}^{\infty}\left(\frac{R_k}{k}\right)^{\beta/2} \\
&= O(1)\sum_{n=0}^{\infty}\sum_{k=m_n}^{m_{n+1}-1}\left(\frac{R_k}{k}\right)^{\beta/2} \\
&= O(1)\sum_{n=0}^{\infty}\sum_{k=m_n}^{m_{n+1}-1}\left[\frac{(\Omega_n)^{1/r}}{m_n(\theta_{m_n})^{1/r}}\right]^{\beta/2} \\
&= O(1)\sum_{n=0}^{\infty}\left[\frac{(\Omega_n)^{1/r}}{m_n(\theta_{m_n})^{1/r}}\right]^{\beta/2}(m_{n+1}-m_n) \\
&= O(1)\sum_{n=0}^{\infty}\left[\frac{(m_n)^{2/\beta-1}\left(\omega^{(p+(2-p)s)}(f,n)\right)^{2-p/r}}{\left(\sum_{j=1}^{m_n}\frac{1}{\lambda_j}\right)^{1/r}}\right]^{\beta/2} < \infty,
\end{aligned}$$

because G is bounded and by the assumption of theorem. Thus Theorem 1 is proved for $r>1$.

For the case $r=1$, $s=\infty$, simply note that

$$|f_\alpha(x)|^2 = |f_\alpha(x)|^{2-p}|f_\alpha(x)|^p \leqslant (\omega_N(f))^{2-p}|f_\alpha(x)|^p,$$

because

$$|f_\alpha(x)| = |f(x+z_\alpha^{(N)}+h) - f(x+z_\alpha^{(N)})| \leqslant \omega_N(f)$$

since $h \in G_N$; and proceed as above. $\square$

Proof of Corollary 2. If we put $r=1$, $s=\infty$ and $\Lambda=\{1\}$ in Corollary 1, then Condition (1) becomes

$$\sum_{n=0}^{\infty}\frac{(m_n)^{1/2}(\omega_n(f))^{1-p/2}}{(m_n)^{1/2}}<\infty,$$

which is same as $\sum_{n=0}^{\infty}(\omega_n(f))^{1-p/2}<\infty$. Now, if $f\in\operatorname{Lip}\alpha$ $(\alpha>0)$, then $\omega_n(f)=O(m_n^{-\alpha})$ so that

$$\sum_{n=0}^{\infty}(\omega_n(f))^{1-p/2}=O(1)\sum_{n=0}^{\infty}(m_n)^{-\alpha(1-p/2)}<\infty,$$

because $\alpha(1-p/2)>0$. Further, in this case, $\Lambda GBF^{(p)}=GBF^{(p)}$. Therefore, Corollary 2 follows from Corollary 1. □

Proof of Corollary 3. If we put $r=1$, $s=\infty$ and $p=1$ in Corollary 1, then Condition (1) becomes

$$\sum_{n=0}^{\infty}\frac{(m_n)^{1/2}(\omega_n(f))^{1/2}}{\left(\sum_{j=1}^{m_n}\frac{1}{\lambda_j}\right)^{1/2}}<\infty.$$

Since $\{\lambda_j\}$ is non-decreasing we have $\sum_{j=1}^{m_n}(1/\lambda_j)\geqslant m_n/\lambda_{m_n}$. Now, if $\lambda_{m_n}=O(m_n^{\gamma})$, with $0\leqslant\gamma<\alpha$, and $f\in\operatorname{Lip}\alpha$ $(\alpha>0)$ so that $\omega_n(f)=O(m_n^{-\alpha})$, then

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(m_n)^{1/2}(\omega_n(f))^{1/2}}{\left(\sum_{j=1}^{m_n}\frac{1}{\lambda_j}\right)^{1/2}}&=O(1)\sum_{n=0}^{\infty}(m_n)^{1/2}(\omega_n(f))^{1/2}\left(\frac{\lambda_{m_n}}{m_n}\right)^{1/2}\\&=O(1)\sum_{n=0}^{\infty}\left(m_n^{-\alpha}\right)^{1/2}(m_n^{\gamma})^{1/2}\\&=O(1)\sum_{n=0}^{\infty}m_n^{-\frac{1}{2}(\alpha-\gamma)}<\infty.\end{aligned}$$

Further, in this case, $\Lambda GBF^{(p)}=\Lambda GBF^{(1)}=\Lambda GBF$. Therefore, Corollary 3 follows from Corollary 1. □

Proof of Theorem 2. Since ϕ is convex on $[0,\infty)$ and $\phi(0)=0$, for any $0<\alpha<1$ and $x>0$ we have

$$\phi(\alpha x)=\phi(\alpha\cdot x+(1-\alpha)\cdot 0)\leqslant\alpha\phi(x)+(1-\alpha)\phi(0)=\alpha\phi(x). \tag{10}$$

Further, as $\phi(2x)\leqslant d\phi(x)$, for all $x\geqslant 0$, we get

$$\phi(ax)\leqslant d^{\log_2 a+1}\phi(x),\ \text{for all } x\geqslant 0 \text{ and for all } a\geqslant 1. \tag{11}$$

For, using induction on n we get

$$\phi(2^n x) \leqslant d^n \phi(x), \text{ for all } x \geqslant 0 \text{ and for all } n \in \mathbb{N}.$$

Next, if $a \geqslant 1$ is any real number, choosing $n \in \mathbb{N}$ such that $2^{n-1} \leqslant a < 2^n$ we get $0 < \frac{a}{2^n} < 1$. Therefore for all $x \geqslant 0$ we have

$$\phi(ax) = \phi\left(\frac{a}{2^n} \cdot 2^n x\right) \leqslant \frac{a}{2^n}\phi(2^n x) \leqslant \frac{a}{2^n} d^n \phi(x) < d^n \phi(x) \leqslant d^{\log_2 a+1}\phi(x).$$

Now, as $f \in \phi\Lambda GBF$, for any $x \in G$ we have

$$\begin{aligned}
|f(x)| &\leqslant |f(0)| + |f(x) - f(0)| \\
&\leqslant |f(0)| + \operatorname{osc}\left(f; z_0^{(0)} + G_0\right) \\
&= |f(0)| + \phi^{-1}\left[\lambda_1 \cdot \frac{\phi\left\{\operatorname{osc}\left(f; z_0^{(0)} + G_0\right)\right\}}{\lambda_1}\right] \\
&\leqslant |f(0)| + \phi^{-1}\left[\lambda_1 \cdot \Lambda GF_\phi(f;G)\right].
\end{aligned}$$

Therefore f is bounded on G and hence $f \in L^2(G)$. For $r > 1$, proceeding as in the proof of Theorem 1 we get (3). Since multiplying f by a positive constant alters $\omega^{(p)}(f,n)$ by the same constant, and ϕ is Δ_2, we may assume that $|f(x)| \leqslant \frac{1}{2}$ for all x. But then from (3) we get

$$(B(h))^r \leqslant \Omega_N \int_G |f_\alpha(x)|dx \qquad (\alpha = 0, 1, \ldots, m_N - 1).$$

Since $\Omega_N \geqslant 0$, if $\Omega_N < 1$ then from (10) we get

$$\phi((B(h))^r) \leqslant \phi\left(\Omega_N \int_G |f_\alpha(x)|dx\right) \leqslant \Omega_N \phi\left(\int_G |f_\alpha(x)|dx\right).$$

Further when $\Omega_N \geqslant 1$, in view of (11), we have

$$\begin{aligned}
\phi((B(h))^r) &\leqslant \phi\left(\Omega_N \int_G |f_\alpha(x)|dx\right) \\
&\leqslant d^{\log_2 \Omega_N+1}\phi\left(\int_G |f_\alpha(x)|dx\right) \\
&= d(\Omega_N)^{\log_2 d}\phi\left(\int_G |f_\alpha(x)|dx\right) \\
&= d(\Omega_N)^{\log_2 d-1}\Omega_N \phi\left(\int_G |f_\alpha(x)|dx\right) \\
&\leqslant d\Omega_N \phi\left(\int_G |f_\alpha(x)|dx\right),
\end{aligned}$$

because of the fact that $(\Omega_N)^{\log_2 d-1} \leqslant 1$, as $|f(x)| \leqslant \frac{1}{2}$, for all x and $\log_2 d - 1 \geqslant 0$. Since $d \geqslant 2$, in either case,

$$\phi\left((B(h))^r\right) \leqslant d\Omega_N \phi\left(\int_G |f_\alpha(x)|dx\right) \leqslant d\Omega_N \int_G \phi(|f_\alpha(x)|)dx,$$

in view of the Jensen's inequality. Now multiplying both the sides of this inequality by $(1/\lambda_{\alpha+1})$ and taking summation over $\alpha = 0, 1, \ldots, m_N - 1$ we get

$$\phi\left((B(h))^r\right) \leqslant d\left(\frac{\Omega_N}{\theta_{m_N}}\right)\int_G \left(\sum_{\alpha=0}^{m_N-1} \frac{\phi(|f_\alpha(x)|)}{\lambda_{\alpha+1}}\right) dx. \tag{12}$$

Since $f \in \phi\Lambda GBF$ and ϕ is increasing, for all $h \in G_N$ and $x \in G$ we have

$$\sum_{\alpha=0}^{m_N-1} \frac{\phi(|f_\alpha(x)|)}{\lambda_{\alpha+1}} \leqslant \sum_{\alpha=0}^{m_N-1} \frac{\phi\left(\operatorname{osc}(f; x + z_\alpha^{(N)} + G_N)\right)}{\lambda_{\alpha+1}} \leqslant \Lambda GF_\phi(f;G). \tag{13}$$

Using (13) in (12) we get $\phi\left((B(h))^r\right) \leqslant C\left(\frac{\Omega_N}{\theta_{m_N}}\right)$, where C is a constant such that $C \geqslant 1$. Thus

$$(B(h))^r \leqslant \phi^{-1}\left\{C\left(\frac{\Omega_N}{\theta_{m_N}}\right)\right\} \leqslant C\phi^{-1}\left(\frac{\Omega_N}{\theta_{m_N}}\right)$$

and therefore

$$B(h) = O\left[\left\{\phi^{-1}\left(\frac{\Omega_N}{\theta_{m_N}}\right)\right\}^{1/r}\right].$$

Integrating both sides of this inequality over G_N with respect to h, in view of (7) we get

$$R_M \equiv \sum_{k=M}^{\infty} |\hat{f}(k)|^2 \leqslant \left(\frac{m_N}{2}\right)\int_{G_N} B(h)dh = O\left[\left\{\phi^{-1}\left(\frac{\Omega_N}{\theta_{m_N}}\right)\right\}^{1/r}\right].$$

Thus in view of (9) we get

$$\begin{aligned}
\sum_{k=1}^{\infty} |\hat{f}(k)|^\beta &= O(1)\sum_{k=1}^{\infty}\left(\frac{R_k}{k}\right)^{\beta/2} \\
&= O(1)\sum_{n=0}^{\infty}\sum_{k=m_n}^{m_{n+1}-1}\left(\frac{R_k}{k}\right)^{\beta/2} \\
&= O(1)\sum_{n=0}^{\infty}\sum_{k=m_n}^{m_{n+1}-1}\left[\frac{1}{m_n}\left\{\phi^{-1}\left(\frac{\Omega_n}{\theta_{m_n}}\right)\right\}^{1/r}\right]^{\beta/2} \\
&= O(1)\sum_{n=0}^{\infty}\left[\frac{1}{m_n}\left\{\phi^{-1}\left(\frac{\Omega_n}{\theta_{m_n}}\right)\right\}^{1/r}\right]^{\beta/2}(m_{n+1} - m_n) \\
&= O(1)\sum_{n=0}^{\infty}\left[(m_n)^{2/\beta-1}\left\{\phi^{-1}\left(\frac{\Omega_n}{\theta_{m_n}}\right)\right\}^{1/r}\right]^{\beta/2} < \infty,
\end{aligned}$$

since G is bounded and in view of the assumption of the theorem. This completes the proof of Theorem 2 for $r > 1$. For the case $r = 1$, $s = \infty$, the proof is similar as that of Theorem 1. □

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