Mathematical
Inequalities
& Applications
Volume 17, Number 2 (2014), 779–799 doi:10.7153/mia-17-57

GENERALIZATIONS OF STEFFENSEN'S INEQUALITY VIA WEIGHTED MONTGOMERY IDENTITY

A. AGLIĆ ALJINOVIĆ, J. PEČARIĆ AND A. PERUŠIĆ

(*Communicated by I. Franjić*)

Abstract. Some new generalizations of Steffensen's inequality are obtained by means of weighted Montgomery identity and estimations between difference of two weighted integral means. Further, functionals associated to these new generalizations are observed and used to generate n-exponentially and exponentially convex functions as well as to obtain new Stolarsky type means related to these functionals.

1. Introduction

The well-known Steffensen's inequality states (see [10])

THEOREM 1. *Let $f,g:[a,b]\to\mathbb{R}$ be integrable mappings on $[a,b]$ such that f is nonincreasing and $0\leqslant g(t)\leqslant 1$ for $t\in[a,b]$. Then*

$$\int_{b-\lambda}^{b} f(t)\,dt \leqslant \int_a^b f(t)\,g(t)\,dt \leqslant \int_a^{a+\lambda} f(t)\,dt \tag{1.1}$$

where $\lambda=\int_a^b g(t)\,dt$.

J. F. Steffensen proved this inequality in 1918 and since then it was generalized in numerous ways. Extensive overview of these generalizations can be found in [5] or [9].

In [3] P. Cerone proved the following generalization of the Steffensen's inequality:

THEOREM 2. *Let $f,g:[a,b]\to\mathbb{R}$ be integrable mappings on $[a,b]$ and let f be nonincreasing. Further, let $0\leqslant g(t)\leqslant 1$ and $\lambda=\int_a^b g(t)\,dt=d_i-c_i$, where $[c_i,d_i]\subset[a,b]$ for $i=1,2$ and $d_1\leqslant d_2$. Then the result*

$$\int_{c_2}^{d_2} f(t)\,dt - r(c_2,d_2) \leqslant \int_a^b f(t)\,g(t)\,dt \leqslant \int_{c_1}^{d_1} f(t)\,dt + R(c_1,d_1) \tag{1.2}$$

holds where,

$$r(c_2,d_2)=\int_{d_2}^{b}(f(c_2)-f(t))\,g(t)\,dt\geqslant 0$$

Mathematics subject classification (2010): 26D15, 26A48.
Keywords and phrases: Steffensen's inequality, Montgomery identity, n-exponentially convex function.

and

$$R(c_1,d_1) = \int_a^{c_1} (f(t) - f(d_1))\, g(t)\, dt \geqslant 0.$$

It is easy to check that if one takes $c_1 = a$ and $d_2 = b$ and thus $d_1 = a + \lambda$ and $c_2 = b - \lambda$, (1.2) reduces to (1.1).

REMARK 1. Identity (1.2) can also be proved differently and more simple than in [3] by using the very Steffensen's inequality. Indeed, in order to prove the right-hand side inequality in (1.2) we observe

$$\begin{aligned}
&\int_a^b f(t)\, g(t)\, dt - \int_{c_1}^{d_1} f(t)\, dt \\
&= \int_a^b (f(t) - f(d_1))\, g(t)\, dt + f(d_1) \int_a^b g(t)\, dt - \int_{c_1}^{d_1} f(t)\, dt \\
&= \int_a^b (f(t) - f(d_1))\, g(t)\, dt + f(d_1)\,\lambda - \int_{c_1}^{c_1+\lambda} f(t)\, dt \\
&= \int_a^b (f(t) - f(d_1))\, g(t)\, dt - \int_{c_1}^{c_1+\lambda} (f(t) - f(d_1))\, dt
\end{aligned}$$

and apply the right-hand side of Steffensen's inequality for nonincreasing function $f(t) - f(d_1)$ on the interval $[c_1, b]$

$$\int_{c_1}^b (f(t) - f(d_1))\, g(t)\, dt \leqslant \int_{c_1}^{c_1+\lambda_1} (f(t) - f(d_1))\, dt.$$

Here we have $\lambda_1 = \int_{c_1}^b g(t)\, dt$ and thus obviously $\lambda_1 \leqslant \lambda$ which leads us to

$$\int_{c_1}^{c_1+\lambda_1} (f(t) - f(d_1))\, dt \leqslant \int_{c_1}^{c_1+\lambda} (f(t) - f(d_1))\, dt.$$

Finally

$$\begin{aligned}
&\int_a^b (f(t) - f(d_1))\, g(t)\, dt - \int_{c_1}^{c_1+\lambda} (f(t) - f(d_1))\, dt \\
&\leqslant \int_a^b (f(t) - f(d_1))\, g(t)\, dt - \int_{c_1}^b (f(t) - f(d_1))\, g(t)\, dt = R(c_1, d_1)
\end{aligned}$$

and the proof is complete. In a similar manner, the left-hand side inequality in (1.2) can be proved.

Let $w : [a,b] \to \mathbb{R}$ be a weight function, i.e. an integrable function such that $\int_a^b w(t)\, dt \neq 0$ and $W(x) = \int_a^x w(t)\, dt$, $x \in [a,b]$. Let also $f : [a,b] \to \mathbb{R}$ be a continuous function of bounded variation. Then the *weighted Montgomery identity* given by Pečarić in [6], states

$$f(x) - \frac{1}{\int_a^b w(t)\, dt} \int_a^b f(t)\, w(t)\, dt = \int_a^b P_w(x,t)\, df(t) \tag{1.3}$$

where $P_w(x,t)$ is the weighted Peano kernel, defined by

$$P_w(x,t) = \begin{cases} \frac{W(t)}{W(b)}, & a \leqslant t \leqslant x, \\ \frac{W(t)}{W(b)} - 1, & x < t \leqslant b. \end{cases} \tag{1.4}$$

Assumptions $W(t) = 0$ for $t \leqslant a$ and $W(t) = \int_a^b w(t)\,dt$ for $t \geqslant b$ allow us to subtract two weighted Montgomery identities, one for the interval $[a,b]$ and the other for $[c,d]$. In such a way the next result is obtained in [1].

THEOREM 3. *Let $f : [a,b] \cup [c,d] \to \mathbb{R}$ be a continuous function of bounded variation on $[a,b] \cup [c,d]$, $w : [a,b] \to \mathbb{R}$ and $u : [c,d] \to \mathbb{R}$ some weight functions, such that $\int_a^b w(t)\,dt \neq 0$, $\int_c^d u(t)\,dt \neq 0$ and*

$$W(x) = \begin{cases} 0, & x < a, \\ \int_a^x w(t)\,dt, & a \leqslant x \leqslant b, \\ \int_a^b w(t)\,dt, & x > b, \end{cases} \qquad U(x) = \begin{cases} 0, & x < c, \\ \int_c^x u(t)\,dt, & c \leqslant x \leqslant d, \\ \int_c^d u(t)\,dt, & x > d, \end{cases}$$

and $[a,b] \cap [c,d] \neq \emptyset$. Then, for both cases $[c,d] \subseteq [a,b]$ and $[a,b] \cap [c,d] = [c,b]$ (and also for $[a,b] \subseteq [c,d]$ and $[a,b] \cap [c,d] = [a,d]$) the next formula is valid

$$\frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) f(t)\,dt - \frac{1}{\int_c^d u(t)\,dt} \int_c^d u(t) f(t)\,dt = \int_{\min\{a,c\}}^{\max\{b,d\}} K(t)\,df(t) \tag{1.5}$$

where

$$K(t) = P_u(x,t) - P_w(x,t), \; t \in [\min\{a,c\}, \max\{b,d\}]$$

and $P_u(x,t)$, $P_w(x,t)$ are given by

$$P_w(x,t) = \begin{cases} \frac{W(t)}{W(b)}, & a \leqslant t \leqslant x, \\ \frac{W(t)}{W(b)} - 1, & x < t \leqslant b, \end{cases} \qquad P_u(x,t) = \begin{cases} \frac{U(t)}{U(d)}, & c \leqslant t \leqslant x, \\ \frac{U(t)}{U(d)} - 1, & x < t \leqslant d, \end{cases}$$

and thus

$$K(t) = \begin{cases} -\frac{W(t)}{W(b)}, & t \in [a,c\rangle, \\ -\frac{W(t)}{W(b)} + \frac{U(t)}{U(d)}, & t \in [c,d], \quad \text{if } [c,d] \subseteq [a,b], \\ 1 - \frac{W(t)}{W(b)}, & t \in \langle d,b], \end{cases} \tag{1.6}$$

$$K(t) = \begin{cases} -\frac{W(t)}{W(b)}, & t \in [a,c\rangle, \\ -\frac{W(t)}{W(b)} + \frac{U(t)}{U(d)}, & t \in [c,b\rangle, \quad \text{if } [a,b] \cap [c,d] = [c,b]. \\ \frac{U(t)}{U(d)} - 1, & t \in [b,d]. \end{cases} \tag{1.7}$$

This identity enables us to estimate the difference between two weighted integral means, each having its own weight, on two different intersecting intervals $[a,b]$ and $[c,d]$ for both possible cases, when one interval is a subset of the other $[c,d] \subseteq [a,b]$ and for overlapping intervals $[a,b] \cap [c,d] = [c,b]$. The other two possible cases, when $[a,b] \cap [c,d] \neq \emptyset$ we simply get by replacement $a \leftrightarrow c$, $b \leftrightarrow d$.

The special case of this identity for normalized weight function was obtained in [2].

The aim of this paper is to generalize Steffensen's inequality by using the weighted Montgomery identity and Theorem 3. In such a way, new generalizations of Steffensen's inequality for a monotonic function are obtained in Section 2, as well as the generalization of Cerone's Theorem 2 from the Introduction. In Section 3, estimates of the difference of the left-hand and right-hand sides of the obtained inequalities are given. In Section 4, three functionals associated to these new generalizations are considered and used to generate n-exponentially and exponentially convex functions. In Section 5, new Stolarsky type means related to these functionals are obtained.

2. Steffensen's inequality via estimates of the difference between two weighted integral means

THEOREM 4. *Let $f : [a,b] \cup [c,d] \to \mathbb{R}$ be a continuous function of bounded variation on $[a,b] \cup [c,d]$, $w : [a,b] \to \mathbb{R}$ and $u : [c,d] \to \mathbb{R}$ some weight functions, such that $\int_a^b w(t)\,dt \neq 0$, $\int_c^d u(t)\,dt \neq 0$ and $[a,b] \cap [c,d] \neq \emptyset$. Then*

$$\frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) f(t)\,dt \leqslant \frac{1}{\int_c^d u(t)\,dt} \int_c^d u(t) f(t)\,dt \tag{2.1}$$

holds for every nonincreasing function f if and only if $[c,d] \subseteq [a,b]$ and

$$\frac{W(x)}{W(b)} \leqslant 0 \text{ for } x \in [a,c\rangle,\ \frac{W(x)}{W(b)} \leqslant \frac{U(x)}{U(d)} \text{ for } x \in [c,d],\ \frac{W(x)}{W(b)} \leqslant 1 \text{ for } x \in \langle d,b],\tag{2.2}$$

or $[a,b] \cap [c,d] = [c,b]$ and

$$\frac{W(x)}{W(b)} \leqslant 0 \text{ for } x \in [a,c\rangle,\ \frac{W(x)}{W(b)} \leqslant \frac{U(x)}{U(d)} \text{ for } x \in [c,b\rangle,\ 1 \leqslant \frac{U(x)}{U(d)} \text{ for } x \in [b,d].\tag{2.3}$$

Proof. If $[c,d] \subseteq [a,b]$, we apply (2.1) for

$$f(t) = \begin{cases} 1, & t \leqslant x, \\ 0, & t > x, \end{cases}$$

with $x \in [a,c\rangle$, $x \in [c,d]$, $x \in \langle d,b]$, respectively, and inequalities in (2.2) follow. Similarly, if $[a,b] \cap [c,d] = [c,b]$, we apply (2.1) for f with $x \in [a,c\rangle$, $x \in [c,b\rangle$, $x \in [b,d]$, respectively, and inequalities (2.3) follow. Conversely, utilizing (1.5) for every nonincreasing function f, in both cases $[c,d] \subseteq [a,b]$ and $[a,b] \cap [c,d] = [c,b]$ we have $K(t) \geqslant 0$, $t \in [\min\{a,c\}, \max\{b,d\}]$ and thus $\int_{\min\{a,c\}}^{\max\{b,d\}} K(t)\,df(t) \leqslant 0$. □

REMARK 2. If f is a nondecreasing function, inequality (2.1) is reversed.

THEOREM 5. *Let* $f:[a,b]\cup[a,a+\lambda]\to\mathbb{R}$ *be a continuous function of bounded variation on* $[a,b]\cup[a,a+\lambda]$ *and let* $w:[a,b]\to\mathbb{R}$ *and* $u:[a,a+\lambda]\to\mathbb{R}$ *be some weight functions such that* $\int_a^b w(t)\,dt=\int_a^{a+\lambda}u(t)\,dt$. *Then*

$$\int_a^{a+\lambda} u(t)f(t)\,dt \leqslant \int_a^b w(t)f(t)\,dt \tag{2.4}$$

holds for every nonincreasing function f *if and only if* $0<\lambda\leqslant b-a$ *and*

$$\int_a^x u(t)\,dt \leqslant \int_a^x w(t)\,dt \text{ for } x\in[a,a+\lambda] \text{ and } \int_x^b w(t)\,dt \leqslant 0 \text{ for } x\in\langle a+\lambda,b]; \tag{2.5}$$

or $\lambda>b-a$ *and*

$$\int_a^x u(t)\,dt \leqslant \int_a^x w(t)\,dt \text{ for } x\in[a,b] \text{ and } \int_x^{a+\lambda} u(t)\,dt \geqslant 0 \text{ for } x\in\langle b,a+\lambda]. \tag{2.6}$$

Proof. If $0<\lambda\leqslant b-a$, we apply (2.4) for

$$f(t)=\begin{cases}1, & t\leqslant x,\\ 0, & t>x,\end{cases}$$

with $x\in[a,a+\lambda]$, $x\in\langle a+\lambda,b]$, respectively, and inequalities in (2.5) follow. Similarly, if $\lambda>b-a$, we apply (2.4) for f with $x\in[a,b]$, $x\in\langle b,a+\lambda]$, and inequalities in (2.6) follow. Conversely, from Theorem 3 applied with $[c,d]=[a,a+\lambda]$ we have

$$\int_a^b w(t)f(t)\,dt-\int_a^{a+\lambda}u(t)f(t)\,dt=\alpha\int_a^{\max\{b,a+\lambda\}}K(t)\,df(t)$$

where $\alpha=\int_a^b w(t)\,dt=\int_a^{a+\lambda}u(t)\,dt$, that is, $\alpha=W(b)=U(a+\lambda)$. First, we consider the case $0<\lambda\leqslant b-a$. We have $\max\{b,a+\lambda\}=b$. By utilizing (1.6) we obtain

$$\alpha K(t)=\begin{cases}U(t)-W(t), & t\in[a,a+\lambda],\\ \alpha-W(t), & t\in\langle a+\lambda,b].\end{cases}$$

Since f is nonincreasing, if $\int_a^x u(t)\,dt\leqslant\int_a^x w(t)\,dt$ for $x\in[a,a+\lambda]$ and $\alpha-\int_a^x w(t)\,dt$ $=\int_x^b w(t)\,dt\leqslant 0$ for $x\in\langle a+\lambda,b]$, we have $\alpha K(t)\leqslant 0$ and therefore $\int_a^b \alpha K(t)\,df(t)\geqslant 0$.

In case $\lambda>b-a$, we have $\max\{b,a+\lambda\}=a+\lambda$, and by utilizing (1.7)

$$\alpha K(t)=\begin{cases}U(t)-W(t), & t\in[a,b],\\ U(t)-\alpha, & t\in\langle b,a+\lambda].\end{cases}$$

Again, since f is nonincreasing, if $\int_a^x u(t)\,dt\leqslant\int_a^x w(t)\,dt$ for $x\in[a,b]$ and $\int_a^x u(t)\,dt-\alpha=-\int_x^{a+\lambda}u(t)\,dt\leqslant 0$ for $x\in\langle b,a+\lambda]$, we have $\alpha K(t)\leqslant 0$ and therefore $\int_a^b \alpha K(t)\,df(t)\geqslant 0$. □

COROLLARY 1. *Let* $f:[a,b]\cup[a,a+\lambda]\to\mathbb{R}$ *and* $g:[a,b]\to\mathbb{R}$ *be integrable functions, where* $\lambda=\int_a^b g(t)\,dt$*. Then*

$$\int_a^{a+\lambda} f(t)\,dt \leqslant \int_a^b f(t)\,g(t)\,dt \tag{2.7}$$

holds for every nonincreasing function f *if and only if* $0<\lambda\leqslant b-a$ *and*

$$x-a\leqslant\int_a^x g(t)\,dt \text{ for } x\in[a,a+\lambda] \text{ and } \int_x^b g(t)\,dt\leqslant 0 \text{ for } x\in\langle a+\lambda,b];$$

or $\lambda>b-a$ *and*

$$x-a\leqslant\int_a^x g(t)\,dt \ \text{ for } x\in[a,b].$$

Proof. Proof follows directly by applying Theorem 5 with weight functions $w(t)=g(t)$ for $t\in[a,b]$ and $u(t)=1$ for $t\in[a,a+\lambda]$. □

THEOREM 6. *Let* $f:[a,b]\cup[b-\lambda,b]\to\mathbb{R}$ *be a continuous function of bounded variation on* $[a,b]\cup[b-\lambda,b]$ *and let* $w:[a,b]\to\mathbb{R}$ *and* $u:[b-\lambda,b]\to\mathbb{R}$ *be some weight functions, such that* $\int_a^b w(t)\,dt=\int_{b-\lambda}^b u(t)\,dt$*. Then*

$$\int_a^b w(t)\,f(t)\,dt\leqslant\int_{b-\lambda}^b u(t)\,f(t)\,dt \tag{2.8}$$

holds for every nonincreasing function f *if and only if* $0<\lambda\leqslant b-a$ *and*

$$\int_a^x w(t)\,dt\leqslant 0 \text{ for } x\in[a,b-\lambda] \text{ and } \int_{b-\lambda}^x u(t)\,dt\geqslant\int_a^x w(t)\,dt \ \text{ for } x\in\langle b-\lambda,b]; \tag{2.9}$$

or $\lambda>b-a$ *and*

$$\int_{b-\lambda}^x u(t)\,dt\geqslant 0 \text{ for } x\in[b-\lambda,a] \text{ and } \int_{b-\lambda}^x u(t)\,dt\geqslant\int_a^x w(t)\,dt \text{ for } x\in\langle a,b]. \tag{2.10}$$

Proof. If $0<\lambda\leqslant b-a$ we apply (2.8) for

$$f(t)=\begin{cases}1, & t\leqslant x,\\ 0, & t>x.\end{cases}$$

with $x\in[a,b-\lambda]$, $x\in\langle b-\lambda,b]$, and inequalities in (2.9) follow. Similarly, if $\lambda>b-a$, we apply (2.8) for f with $x\in[b-\lambda,a]$, $x\in\langle a,b]$, and (2.10) follow. Conversely from Theorem 3 applied with $[c,d]=[b-\lambda,b]$ we have

$$\int_a^b w(t)\,f(t)\,dt-\int_{b-\lambda}^b u(t)\,f(t)\,dt=\alpha\int_{\min\{a,b-\lambda\}}^b K(t)\,df(t)$$

where $\alpha = \int_a^b w(t)\,dt = \int_{b-\lambda}^b u(t)\,dt$ and in case $0 < \lambda \leqslant b-a$

$$\alpha K(t) = \begin{cases} -W(t), & t \in [a, b-\lambda], \\ U(t) - W(t), & t \in \langle b-\lambda, b], \end{cases}$$

while in case $\lambda > b-a$

$$\alpha K(t) = \begin{cases} U(t), & t \in [b-\lambda, a], \\ U(t) - W(t), & t \in \langle a, b]. \end{cases}$$

The rest of the proof can be obtained by proceeding in the similar way as in the proof of Theorem 5. □

COROLLARY 2. *Let $f : [a,b] \cup [b-\lambda, b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be integrable functions, where $\lambda = \int_a^b g(t)\,dt$. Then*

$$\int_a^b f(t)\,g(t)\,dt \leqslant \int_{b-\lambda}^b f(t)\,dt \tag{2.11}$$

holds for every nonincreasing function f if and only if $0 < \lambda \leqslant b-a$ and

$$\int_a^x g(t)\,dt \leqslant 0 \text{ for } x \in [a, b-\lambda] \text{ and } b - x \leqslant \int_x^b g(t)\,dt \text{ for } x \in \langle b-\lambda, b];$$

or $\lambda > b-a$ and

$$b - x \leqslant \int_x^b g(t)\,dt \text{ for } x \in [a,b].$$

Proof. Proof follows directly by applying Theorem 6 with weight functions $w(t) = g(t)$ for $t \in [a,b]$ and $u(t) = 1$ for $t \in [b-\lambda, b]$. □

REMARK 3. Corollaries 1 and 2 were previously obtained by Pečarić in [7] (see also monograph [9]).

REMARK 4. If f is a nondecreasing function, inequalities (2.4), (2.7), (2.8) and (2.11) are reversed.

Finally, we give a generalization of Theorem 2 from the Introduction.

THEOREM 7. *Let $f : [a,b] \to \mathbb{R}$ be nonincreasing function. Also, let $w : [a,b] \to [0,\infty\rangle$ and $u_i : [c_i, d_i] \to [0,\infty\rangle$, $i = 1,2$, be some weight functions, such that $\int_a^b w(t)\,dt = \int_{c_i}^{d_i} u_i(t)\,dt \neq 0$ and $0 \leqslant w(t) \leqslant u_i(t)$, $t \in [c_i, d_i]$, where $[c_i, d_i] \subset [a,b]$ for $i = 1,2$ and $c_1 \leqslant c_2$. Then*

$$\int_{c_2}^{d_2} u_2(t)\,f(t)\,dt - r(c_2, d_2) \leqslant \int_a^b w(t)\,f(t)\,dt \leqslant \int_{c_1}^{d_1} u_1(t)\,f(t)\,dt + R(c_1, d_1) \tag{2.12}$$

holds, where

$$r(c_2,d_2)=\int_{d_2}^{b}(f(c_2)-f(t))\,w(t)\,dt\geqslant 0$$

and

$$R(c_1,d_1)=\int_{a}^{c_1}(f(t)-f(d_1))\,w(t)\,dt\geqslant 0.$$

Proof. First we prove the right-hand side of the double inequality. We denote $\lambda=\int_a^b w(t)\,dt=\int_{c_i}^{d_i}u_i(t)\,dt\neq 0$, $i=1,2$. Multiplying (1.5) with λ and utilizing (1.6) we have

$$\begin{aligned}&\int_a^b w(t)f(t)\,dt-\int_{c_1}^{d_1}u_1(t)f(t)\,dt\\&=-\int_a^{c_1}W(t)\,df(t)+\int_{c_1}^{d_1}(-W(t)+U_1(t))\,df(t)+\int_{d_1}^{b}(\lambda-W(t))\,df(t).\end{aligned}$$

Since $\lambda-W(t)=\int_t^b w(t)\geqslant 0$ and f is nonincreasing, we have $\int_{d_1}^{b}(\lambda-W(t))\,df(t)\leqslant 0$. Thus, interchanging the order of the integration leads us to

$$\begin{aligned}&\int_a^b w(t)f(t)\,dt-\int_{c_1}^{d_1}u_1(t)f(t)\,dt\\&\leqslant-\int_a^{c_1}W(t)\,df(t)+\int_{c_1}^{d_1}(-W(t)+U_1(t))\,df(t)\\&=-\int_a^{d_1}W(t)\,df(t)+\int_{c_1}^{d_1}U_1(t)\,df(t)\\&=-\int_a^{d_1}\left(\int_a^t w(s)\,ds\right)df(t)+\int_{c_1}^{d_1}\left(\int_{c_1}^{t}u_1(s)\,ds\right)df(t)\\&=-\int_a^{d_1}\left(\int_s^{d_1}df(t)\right)w(s)\,ds+\int_{c_1}^{d_1}\left(\int_s^{d_1}df(t)\right)u(s)\,ds\\&=-\int_a^{d_1}(f(d_1)-f(s))\,w(s)\,ds+\int_{c_1}^{d_1}(f(d_1)-f(s))\,u(s)\,ds\\&=\int_a^{c_1}(f(s)-f(d_1))\,w(s)\,ds+\int_{c_1}^{d_1}(f(d_1)-f(s))(u(s)-w(s))\,ds\\&\leqslant\int_a^{c_1}(f(s)-f(d_1))\,w(s)\,ds=R(c_1,d_1).\end{aligned}$$

The last inequality holds since $f(d_1)\leqslant f(s)$ and $u(s)\geqslant w(s)$ for $s\in[c_1,d_1]$. The left-hand side inequality can be proved in a similar manner. □

REMARK 5. If we take $u_i(x)=1$, $x\in[c_i,d_i]$, for $i=1,2$ the previous theorem reduces to Cerone's Theorem 2 from the Introduction.

3. Estimates of the difference of the left-hand and right-hand side of Steffensen's inequality

THEOREM 8. *Let* $f : [a,b] \cup [a,a+\lambda] \to \mathbb{R}$ *be a continuous and nonincreasing function on* $[a,b] \cup [a,a+\lambda]$ *and let* $w : [a,b] \to \mathbb{R}$ *and* $u : [a,a+\lambda] \to \mathbb{R}$ *be some weight functions, such that* $\int_a^b w(t)\,dt \neq 0$, $\int_a^{a+\lambda} u(t)\,dt \neq 0$. *Let also* $W(x) = \int_a^x w(t)\,dt$, $x \in [a,b]$ *and* $U(x) = \int_a^x u(t)\,dt$, $x \in [a,a+\lambda]$. *Then, if* $a+\lambda \leqslant b$, *it holds that*

$$\left| \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) f(t)\,dt - \frac{1}{\int_a^{a+\lambda} u(t)\,dt} \int_a^{a+\lambda} u(t) f(t)\,dt \right| \tag{3.1}$$
$$\leqslant (f(a) - f(b)) \cdot \max \left\{ \max_{t\in[a,a+\lambda]} \left| \frac{U(t)}{U(a+\lambda)} - \frac{W(t)}{W(b)} \right|, \max_{t\in[a+\lambda,b]} \left| 1 - \frac{W(t)}{W(b)} \right| \right\}$$

and if $\lambda \geqslant b-a$

$$\left| \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) f(t)\,dt - \frac{1}{\int_a^{a+\lambda} u(t)\,dt} \int_a^{a+\lambda} u(t) f(t)\,dt \right| \tag{3.2}$$
$$\leqslant (f(a) - f(a+\lambda)) \cdot \max \left\{ \max_{t\in[a,b]} \left| \frac{U(t)}{U(a+\lambda)} - \frac{W(t)}{W(b)} \right|, \max_{t\in[b,a+\lambda]} \left| \frac{U(t)}{U(a+\lambda)} - 1 \right| \right\}.$$

Both inequalities are sharp.

Proof. By applying Theorem 3 with $[c,d] = [a,a+\lambda]$ we obtain

$$\frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) f(t)\,dt - \frac{1}{\int_a^{a+\lambda} u(t)\,dt} \int_a^{a+\lambda} u(t) f(t)\,dt$$
$$= \int_a^{\max\{b,a+\lambda\}} K(t)\,df(t).$$

Since $K(t)$ is continuous on $[a,b]$ and f is of bounded variation on $[a,b] \cup [a,a+\lambda]$, in the case $a+\lambda \leqslant b$ we have

$$\left| \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) f(t)\,dt - \frac{1}{\int_a^{a+\lambda} u(t)\,dt} \int_a^{a+\lambda} u(t) f(t)\,dt \right|$$
$$= \left| \int_a^b K(t)\,df(t) \right| \leqslant \bigvee_a^b (f) \cdot \sup_{t\in[a,b]} |K(t)|$$
$$= (f(a) - f(b)) \cdot \sup_{t\in[a,b]} |K(t)|$$

where $\bigvee_a^b (f)$ is the total variation of function f and $K(t)$ is given by

$$K(t) = \begin{cases} \frac{U(t)}{U(a+\lambda)} - \frac{W(t)}{W(b)}, & t \in [a,a+\lambda], \\ 1 - \frac{W(t)}{W(b)}, & t \in \langle a+\lambda, b]. \end{cases}$$

Thus (3.1) follows. In order to prove the sharpness of (3.1) consider the nonincreasing function f defined by

$$f(t) = \begin{cases} 1, \ t \in [a, a+\lambda], \\ 0, \ t \in \langle a+\lambda, b], \end{cases}$$

and weight functions $w(t) = 1$, $t \in [a,b]$, $u(t) = 1$, $t \in [a, a+\lambda]$. It is easy to check that then equality in (3.1) holds. In case $a + \lambda \geqslant b$ we have

$$K(t) = \begin{cases} \frac{U(t)}{U(a+\lambda)} - \frac{W(t)}{W(b)}, & t \in [a,b], \\ \frac{U(t)}{U(a+\lambda)} - 1, & t \in \langle b, a+\lambda], \end{cases}$$

and the proof of (3.2) can be obtained in the similar manner. In this case, to prove the sharpness of (3.2), we consider

$$f(t) = \begin{cases} 1, & t \in [a,b], \\ 0, & t \in \langle b, a+\lambda], \end{cases}$$

and weight functions $w(t) = 1$, $t \in [a,b]$, $u(t) = 1$, $t \in [a, a+\lambda]$. This completes the proof. □

COROLLARY 3. *Suppose that all the assumptions of the previous theorem hold. Additionally, assume that $g : [a,b] \to \mathbb{R}$ is an integrable function. Let $G(x) = \int_a^x g(t)\,dt$, $x \in [a,b]$ and $\lambda = G(b)$. Then, if $a + \lambda \leqslant b$, it holds that*

$$\left| \int_a^b g(t) f(t)\,dt - \int_a^{a+\lambda} f(t)\,dt \right| \\ \leqslant (f(a) - f(b)) \max \left\{ \max_{t \in [a, a+\lambda]} |t - a - G(t)|, \max_{t \in [a+\lambda, b]} |\lambda - G(t)| \right\}$$

and if $\lambda \geqslant b - a$

$$\left| \int_a^b g(t) f(t)\,dt - \int_a^{a+\lambda} f(t)\,dt \right| \\ \leqslant (f(a) - f(a+\lambda)) \max \left\{ \max_{t \in [a,b]} |t - a - G(t)|, \max_{t \in [b, a+\lambda]} |t - a - \lambda| \right\}.$$

Both inequalities are sharp.

Proof. Proof follows by applying Theorem 8 with weight functions $w(t) = g(t)$ for $t \in [a,b]$ and $u(t) = 1$ for $t \in [a, a+\lambda]$. □

THEOREM 9. *Let $f : [a,b] \cup [b-\lambda, b] \to \mathbb{R}$ be a continuous and nonincreasing function on $[a,b] \cup [b-\lambda, b]$ and let $w : [a,b] \to \mathbb{R}$ and $u : [b-\lambda, b] \to \mathbb{R}$ be some weight functions, such that $\int_a^b w(t)\,dt \neq 0$, $\int_{b-\lambda}^b u(t)\,dt \neq 0$. Let also $W(x) =$*

$\int_a^x w(t)\,dt$, $x \in [a,b]$ and $U(x) = \int_{b-\lambda}^x u(t)\,dt$, $x \in [b-\lambda,b]$. Then, if $a+\lambda \leqslant b$, it holds that

$$\left| \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t)f(t)\,dt - \frac{1}{\int_{b-\lambda}^b u(t)\,dt} \int_{b-\lambda}^b u(t)f(t)\,dt \right| \tag{3.3}$$
$$\leqslant (f(a)-f(b)) \cdot \max\left\{ \max_{t\in[a,b-\lambda]} \left|\frac{W(t)}{W(b)}\right|, \max_{t\in[b-\lambda,b]} \left|\frac{U(t)}{U(b)} - \frac{W(t)}{W(b)}\right| \right\}$$

and if $\lambda \geqslant b-a$

$$\left| \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t)f(t)\,dt - \frac{1}{\int_{b-\lambda}^b u(t)\,dt} \int_{b-\lambda}^b u(t)f(t)\,dt \right| \tag{3.4}$$
$$\leqslant (f(b-\lambda)-f(b)) \cdot \max\left\{ \max_{t\in[b-\lambda,a]} \left|\frac{U(t)}{U(b)}\right|, \max_{t\in[a,b]} \left|\frac{U(t)}{U(b)} - \frac{W(t)}{W(b)}\right| \right\}.$$

Both inequalities are sharp.

Proof. By applying Theorem 3 with $[c,d] = [b-\lambda,b]$ we obtain

$$\frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t)f(t)\,dt - \frac{1}{\int_{b-\lambda}^b u(t)\,dt} \int_{b-\lambda}^b u(t)f(t)\,dt$$
$$= \int_{\min\{a,b-\lambda\}}^b K(t)\,df(t)$$

where if $a+\lambda \leqslant b$

$$K(t) = \begin{cases} -\frac{W(t)}{W(b)}, & t \in [a,b-\lambda], \\ \frac{U(t)}{U(b)} - \frac{W(t)}{W(b)}, & t \in \langle b-\lambda,b], \end{cases}$$

and if $a+\lambda \geqslant b$

$$K(t) = \begin{cases} \frac{U(t)}{U(b)}, & t \in [b-\lambda,a], \\ \frac{U(t)}{U(b)} - \frac{W(t)}{W(b)}, & t \in \langle a,b]. \end{cases}$$

The rest of the proof of (3.3) and (3.4) is similar to the proof of Theorem 9. In order to prove the sharpness of (3.3) consider the nonincreasing function f defined by

$$f(t) = \begin{cases} 1, & t \in [a,b-\lambda], \\ 0, & t \in \langle b-\lambda,b], \end{cases}$$

weight functions $w(t) = 1$, $t \in [a,b]$, $u(t) = 1$, $t \in [b-\lambda,b]$, and to prove the sharpness of (3.4) consider

$$f(t) = \begin{cases} 1, & t \in [b-\lambda,a], \\ 0, & t \in \langle a,b], \end{cases}$$

weight functions $w(t) = 1$, $t \in [a,b]$, $u(t) = 1$, $t \in [b-\lambda,b]$. This completes the proof. $\square$

COROLLARY 4. *Suppose that all the assumptions of the previous theorem hold. Additionally, assume that* $g:[a,b]\to\mathbb{R}$ *is an integrable function. Let* $G(x)=\int_a^x g(t)\,dt$, $x\in[a,b]$ *and* $\lambda=G(b)$. *Then, if* $a+\lambda\leqslant b$, *it holds that*

$$\left|\int_a^b g(t)f(t)\,dt-\int_{b-\lambda}^b f(t)\,dt\right|$$
$$\leqslant (f(a)-f(b))\max\left\{\max_{t\in[a,b-\lambda]}|-G(t)|,\max_{t\in[b-\lambda,b]}|t-b+\lambda-G(t)|\right\}$$

and if $\lambda\geqslant b-a$

$$\left|\int_a^b g(t)f(t)\,dt-\int_{b-\lambda}^b f(t)\,dt\right|$$
$$\leqslant (f(b-\lambda)-f(b))\max\left\{\max_{t\in[b-\lambda,a]}|t-b+\lambda|,\max_{t\in[a,b]}|t-b+\lambda-G(t)|\right\}.$$

Both inequalities are sharp.

Proof. Proof follows by applying Theorem 9 with weight functions $w(t)=g(t)$ for $t\in[a,b]$ and $u(t)=1$ for $t\in[b-\lambda,b]$. □

REMARK 6. By using Theorem 3 applied for $[c,d]=[a,a+\lambda]$ and $[c,d]=[b-\lambda,b]$, with an additional assumption that f is derivable and that $|f'|^p$ is an integrable function, analogous inequalities for L_p spaces (as in [1]) could be obtained.

4. Applications via n-exponential convexity and exponential covexity

We begin this section by giving some definitions and notions which are used frequently in the results. For more details see e.g. [4] and [8].

DEFINITION 1. A function $\psi\colon I\to\mathbb{R}$ is n-*exponentially convex in the Jensen sense* on I if

$$\sum_{i,j=1}^{n}\xi_i\xi_j\psi\left(\frac{x_i+x_j}{2}\right)\geqslant 0 \tag{4.1}$$

holds for all choices $\xi_i\in\mathbb{R}$ and $x_i\in I$, $i=1,\dots,n$.

A function $\psi\colon I\to\mathbb{R}$ is n-*exponentially convex* if it is n-exponentially convex in the Jensen sense and continuous on I.

REMARK 7. n-exponentially convex function in the Jensen sense is $k-$exponentially convex in the Jensen sense for every $k\in\mathbb{N}$, $k\leqslant n$.

DEFINITION 2. A function $\psi\colon I\to\mathbb{R}$ is *exponentially convex in the Jensen sense on* I if it is n-exponentially convex in the Jensen sense for all $n\in\mathbb{N}$.

A function $\psi\colon I\to\mathbb{R}$ is *exponentially convex* if it is exponentially convex in the Jensen sense and continuous.

REMARK 8. In [4] it is showed that $\psi : I \to \mathbb{R}$ is a log-convex in the Jensen sense if and only if

$$\alpha^2\psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2\psi(y) \geqslant 0,$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense. A positive function is log-convex if and only if it is 2-exponentially convex.

PROPOSITION 1. If f is a convex function on I and if $x_1 \leqslant y_1$, $x_2 \leqslant y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leqslant \frac{f(y_2) - f(y_1)}{y_2 - y_1}.$$

If the function f is concave, the inequality is reversed.

DEFINITION 3. Let f be a real-valued function defined on $[a,b]$. *n-th order divided difference* of f at distinct points $x_0, x_1, \ldots, x_n$ in $[a,b]$ is defined recursively by

$$[x_j; f] = f(x_j), \quad j = 0, \ldots, n,$$

and

$$[x_0, x_1, \ldots, x_n; f] = \frac{[x_1, \ldots, x_n; f] - [x_0, \ldots, x_{n-1}; f]}{x_n - x_0}.$$

REMARK 9. The value $[x_0, x_1, \ldots, x_n; f]$ is independent of the order of the points $x_0, \ldots, x_n$. Previous definition can be extended to include the case in which some or all of the points coincide by assuming that $x_0 \leqslant \ldots \leqslant x_n$ and letting

$$[\underbrace{x, \ldots, x}_{j+1 \text{ times}}; f] = \frac{f^{(j)}(x)}{j!},$$

provided that $f^{(j)}$ exists.

Next, we define the following functionals, under the assumptions of Theorem 4

$$L_1(f) = \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) f(t)\,dt - \frac{1}{\int_c^d u(t)\,dt} \int_c^d u(t) f(t)\,dt, \tag{4.2}$$

under the assumptions of Theorem 5

$$L_2(f) = \int_a^{a+\lambda} u(t) f(t)\,dt - \int_a^b w(t) f(t)\,dt \tag{4.3}$$

and under the assumptions of Theorem 6

$$L_3(f) = \int_a^b w(t) f(t)\,dt - \int_{b-\lambda}^b u(t) f(t)\,dt. \tag{4.4}$$

REMARK 10. Under the assumptions of Theorems 4, 5 and 6, respectively, it holds $L_k(f) \geqslant 0$, $k = 1,2,3$ for all nondecreasing functions f.

Now, we state and prove the Lagrange type mean value theorem for defined functionals. We denote $I_1 = [a,b] \cup [c,d]$, $I_2 = [a,b] \cup [a, a+\lambda]$ and $I_3 = [a,b] \cup [b-\lambda, b]$.

THEOREM 10. *Suppose that w and u are weight functions as in Theorem 4, 5 and 6 respectively and $[a,b] \cap [c,d] \neq \emptyset$. Additionally, assume that $f \in C^1(I_k)$ for $k = 1,2,3$. If (2.2) and (2.3) hold in case $k = 1$ or (2.5) and (2.6) in case $k = 2$ or (2.9) and (2.10) in case $k = 3$ then there exists $\xi_k \in I_k$ such that*

$$L_k(f) = f'(\xi_k) L_k(id) \tag{4.5}$$

where $L_k(f)$, $k = 1,2,3$ are defined by (4.2), (4.3) and (4.4) respectively.

Proof. First we consider $k = 1$. Since f' is continuous on $[a,b] \cup [c,d]$, there exists

$$m = \min_{x \in [a,b] \cup [c,d]} f'(x) \text{ and } M = \max_{x \in [a,b] \cup [c,d]} f'(x).$$

Let us consider functions $F_1, F_2 : I \to \mathbb{R}$ defined by

$$F_1(x) = Mx - f(x) \quad \text{ and } \quad F_2(x) = f(x) - mx.$$

Since $F_1'(x) = M - f'(x) \geqslant 0$ and $F_2'(x) = f'(x) - m \geqslant 0$, functions F_1 and F_2 are nondecreasing and thus of bounded variation on $[a,b] \cup [c,d]$. From Theorem 4 for a nondecreasing function F_1, we have

$$\begin{aligned}
0 &\leqslant \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) F_1(t) dt - \frac{1}{\int_c^d u(t)\,dt} \int_c^d u(t) F_1(t) dt \\
&= \frac{1}{\int_a^b w(t)\,dt} \left(M \int_a^b t w(t)\,dt - \int_a^b f(t) w(t)\,dt \right) \\
&\quad - \frac{1}{\int_c^d u(t)\,dt} \left(M \int_c^d t u(t)\,dt - \int_c^d f(t) u(t) dt \right)
\end{aligned}$$

i.e.

$$L_1(f) \leqslant M \left[\frac{1}{\int_a^b w(t)\,dt} \int_a^b t w(t)\,dt - \frac{1}{\int_c^d u(t)\,dt} \int_c^d t u(t)\,dt \right].$$

Similarly, for nondecreasing function F_2 from Theorem 4 we have

$$L_1(f) \geqslant m \left[\frac{1}{\int_a^b w(t)\,dt} \int_a^b t w(t)\,dt - \frac{1}{\int_c^d u(t)\,dt} \int_c^d t u(t)\,dt \right],$$

that is

$$m\left[\frac{1}{\int_a^b w(t)\,dt}\int_a^b tw(t)\,dt-\frac{1}{\int_c^d u(t)\,dt}\int_c^d tu(t)\,dt\right]\leqslant L_1(f)$$
$$\leqslant M\left[\frac{1}{\int_a^b w(t)\,dt}\int_a^b tw(t)\,dt-\frac{1}{\int_c^d u(t)\,dt}\int_c^d tu(t)\,dt\right].$$

If $\frac{1}{\int_a^b w(t)dt}\int_a^b tw(t)\,dt=\frac{1}{\int_c^d u(t)dt}\int_c^d tu(t)\,dt$, (4.5) holds for all $\xi_1\in[a,b]\cup[c,d]$. Otherwise,

$$m\leqslant\frac{L_1(f)}{\frac{1}{\int_a^b w(t)dt}\int_a^b tw(t)\,dt-\frac{1}{\int_c^d u(t)dt}\int_c^d tu(t)\,dt}\leqslant M.$$

Since $f'(x)$ is continuous on $[a,b]\cup[c,d]$ there exists $\xi_1\in[a,b]\cup[c,d]$ such that (4.5) holds and the proof is complete. In case $k=2$ and $k=3$ the proof can be obtained similarly by utilizing Theorems 5 and 6. □

Next is the Cauchy type mean value theorem for functionals $L_k(f)$, $k=1,2,3$.

THEOREM 11. *Let $f,\widehat{f}:I_k\to\mathbb{R}$, $k=1,2,3$ such that $f,\widehat{f}\in C^1(I_k)$. If (2.2) and (2.3) hold in case $k=1$ or (2.5) and (2.6) in case $k=2$ or (2.9) and (2.10) in case $k=3$ then there exists $\xi_k\in I_k$ such that*

$$\frac{L_k(f)}{L_k\left(\widehat{f}\right)}=\frac{f'(\xi_k)}{\widehat{f}'(\xi_k)}\tag{4.6}$$

holds for $k=1,2,3$, provided that the denominators differ from zero.

Proof. First we consider $k=1$. Define $\Phi_1(t)=f(t)L_1\left(\widehat{f}\right)-\widehat{f}(t)L_1(f)$. Note that $\Phi_1(t)\in C^1(I_1)$. By Theorem 10 there exists $\xi_1\in[a,b]\cup[c,d]$ such that

$$L_1(\Phi_1)=\Phi_1'(\xi_1)\;L_1(id).$$

From $L_1(\Phi_1)=0$ it follows that $\Phi_1'(\xi_1)=f'(\xi_1)L_1\left(\widehat{f}\right)-\widehat{f}'(\xi_1)L_1(f)=0$ which implies (4.6). In case $k=2$ and $k=3$ the proof can be obtained similarly by utilizing Theorems 5 and 6. □

Now, we will use an idea from [4] to generate n-exponentially and exponentially convex functions applying defined functionals. In the sequel J will be an interval in $\mathbb{R}$.

THEOREM 12. *Let $\Omega=\{f_s:s\in J\}$, where J is an interval in $\mathbb{R}$, be a family of continuous functions defined on I_k, $k=1,2,3$ in $\mathbb{R}$, such that the function $s\mapsto[x_0,x_1;f_s]$ is n-exponentially convex in the Jensen sense on J for mutually different points $x_0,x_1\in I_k,k=1,2,3$. Let L_k, $k=1,2,3$ be linear functionals defined by (4.2),* *(4.3), (4.4). Then $s \mapsto L_k(f_s)$ is n-exponentially convex function in the Jensen sense on J.*

If the function $s \mapsto L_k(f_s)$ is continuous on J, then it is $n-$exponentially convex on J.

Proof. For $\xi_j \in \mathbb{R}$, $s_j \in J$, $j = 1, \ldots, n$ and $s_{ij} = \frac{s_i + s_j}{2}$ we define the function

$$g(x) = \sum_{i,j=1}^{n} \xi_i \xi_j f_{s_{ij}}(x).$$

Since $s \mapsto [x_0, x_1; f_s]$ is n-exponentially convex in the Jensen sense, we have

$$[x_0, x_1; g] = \sum_{i,j=1}^{n} \xi_i \xi_j \left[x_0, x_1; f_{s_{ij}}\right] \geqslant 0,$$

which implies that g is a nondecreasing function on I_k. Therefore, from Remark 10 we have $L_k(g) \geqslant 0$, and thus

$$\sum_{i,j=1}^{n} \xi_i \xi_j L_k(f_{s_{ij}}) \geqslant 0$$

holds. Hence, we can conclude that the function $s \mapsto L_k(f_s)$ is $n-$exponentially convex on J in the Jensen sense.

If the function $s \mapsto L_k(f_s)$ is also continuous on J, then $s \mapsto L_k(f_s)$ is n- exponentially convex by definition. □

The following corollary is a simple consequence of the previous theorem.

COROLLARY 5. *Let $\Omega = \{f_s : s \in J\}$ be a family of continuous functions defined on $I_k, k = 1,2,3$ in $\mathbb{R}$ such that the function $s \mapsto [x_0, x_1; f_s]$ is exponentially convex in the Jensen sense on J for mutually different points $x_0, x_1 \in I_k$, $k = 1,2,3$. Let L_k, $k = 1,2,3$ be linear functionals defined by (4.2), (4.3), (4.4). Then $s \mapsto L_k(f_s)$ is exponentially convex function in the Jensen sense on J.*

If the function $s \mapsto L_k(f_s)$ is continuous on J, then it is exponentially convex on J.

Now, we will prove a corollary of Theorem 12 which will be used in the next section for obtaining new Stolarsky type means.

COROLLARY 6. *Let $\Omega = \{f_s : s \in J\}$ be a family of continuous functions defined on I_k, $k = 1,2,3$ in $\mathbb{R}$, such that the function $s \mapsto [x_0, x_1; f_s]$ is $2-$exponentially convex in the Jensen sense on J for mutually different points $x_0, x_1 \in I_k$, $k = 1,2,3$. Let $L_k, k = 1,2,3$ be linear functionals defined by (4.2), (4.3), (4.4). Then the following statements hold:*

(i) If the function $s \mapsto L_k(f_s)$ is continuous on J, then it is 2-exponentially convex function on J. If $s \mapsto L_k(f_s)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true:

$$[L_k(f_s)]^{t-r} \leqslant [L_k(f_r)]^{t-s} [L_k(f_t)]^{s-r} \tag{4.7}$$

for every choice $r,s,t \in J$, such that $r < s < t$.

(ii) If the function $s \mapsto L_k(f_s)$ is strictly positive and differentiable on J, then for every $s,q,u,v \in J$ such that $s \leqslant u$ and $q \leqslant v$, we have

$$M_{s,q}(L_k,\Omega) \leqslant M_{u,v}(L_k,\Omega), \tag{4.8}$$

where

$$M_{s,q}(L_k,\Omega) = \begin{cases} \left(\frac{L_k(f_s)}{L_k(f_q)}\right)^{\frac{1}{s-q}}, & s \neq q; \\ \exp\left(\frac{\frac{d}{ds}L_k(f_s)}{L_k(f_s)}\right), & s = q \end{cases} \tag{4.9}$$

for $f_s, f_q \in \Omega$.

Proof. The first statement (i) is a consequence of Theorem 12 and Remark 8.

Further, since $s \mapsto L_k(f_s)$ is positive and continuous, by (i) we have that $s \mapsto L_k(f_s)$ is log-convex on J, that is, $s \mapsto \log L_k(f_s)$ is convex on J. Applying Proposition 1 we get

$$\frac{\log L_k(f_s) - \log L_k(f_q)}{s-q} \leqslant \frac{\log L_k(f_u) - \log L_k(f_v)}{u-v}$$

for $s \leqslant u$, $q \leqslant v$, $s \neq q$, $u \neq v$. Hence, we conclude that

$$M_{s,q}(L_k,\Omega) \leqslant M_{u,v}(L_k,\Omega).$$

Cases $s = q$ and $u = v$ follow as limit cases. □

REMARK 11. Results from Theorem 12, Corollaries 5 and 6 still hold when $x_0 = x_1 \in I_k$. This follows from Remark 9.

5. Stolarsky type means

In this section we will apply general results obtained in previous section to several families of functions which fulfil conditions of obtained general results. Using these families of functions, we will obtain new Stolarsky type means related to functionals defined by (4.2)-(4.4).

EXAMPLE 1. Consider a family of functions

$$\Omega_1 = \{f_s : (0,\infty) \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(x) = \begin{cases} \frac{x^s}{s}, & s \neq 0; \\ \log x, & s = 0. \end{cases}$$

Since $\frac{df_s}{dx}(x) = x^{s-1} = e^{(s-1)\log x} > 0$ for $x > 0$, then f_s is a nondecreasing function for $x > 0$ and $s \mapsto \frac{df_s}{dx}(x)$ is exponentially convex by definition. Analogously as in the proof of Theorem 12 we have that $s \mapsto [x_0, x_1; f_s]$ is exponentially convex (and so exponentially convex in the Jensen sense). Using Corollary 5 we conclude that $s \mapsto L_k(f_s)$, $k = 1,2,3$ are exponentially convex in the Jensen sense. It is easy to verify that these mappings are continuous (although mapping $s \mapsto f_s$ is not continuous for $s = 0$), so they are exponentially convex. In this case we assume that $I_k \subset \mathbb{R}^+$.

For this family of functions $M_{s,q}(L_k, \Omega_1)$ from (4.9) is equal to

$$M_{s,q}(L_k, \Omega_1) = \begin{cases} \left(\frac{L_k(f_s)}{L_k(f_q)}\right)^{\frac{1}{s-q}}, & s \neq q; \\ \exp\left(\frac{L_k(f_s f_0)}{L_k(f_s)} - \frac{1}{s}\right), & s = q \neq 0; \\ \exp\left(\frac{L_k(f_0^2)}{2L_k(f_0)}\right), & s = q = 0. \end{cases}$$

Applying Theorem 11 for functions $f_s, f_q \in \Omega_1$ there exists $\xi_k \in I_k$, $k = 1,2,3$ such that

$$\xi_k^{s-q} = \frac{L_k(f_s)}{L_k(f_q)}, \quad k = 1,2,3.$$

Since the function $\xi_k \mapsto \xi_k^{s-q}$ is invertible for $s \neq q$ we have

$$\min\{a, b-\lambda, c\} \leqslant \left(\frac{L_k(f_s)}{L_k(f_q)}\right)^{\frac{1}{s-q}} \leqslant \max\{a+\lambda, b, d\}, \quad k = 1,2,3$$

which together with the fact that $M_{s,q}(L_k, \Omega_1)$ is continuous, symmetric and monotonic shows that $M_{s,q}(L_k, \Omega_1)$ is mean.

EXAMPLE 2. Consider a family of functions

$$\Omega_2 = \{g_s \colon \mathbb{R} \to (0, \infty) : s \in \mathbb{R}\}$$

defined by

$$g_s(x) = \begin{cases} \frac{e^{sx}}{s}, & s \neq 0; \\ x, & s = 0. \end{cases}$$

Here, $\frac{dg_s}{dx}(x) = e^{sx} > 0$, which shows that g_s is a nondecreasing function on $\mathbb{R}$ for every $s \in \mathbb{R}$ and $s \mapsto \frac{dg_s}{dx}(x)$ is exponentially convex by definition. As in Example 1 we conclude that $s \mapsto L_k(g_s)$, $k = 1,2,3$ are exponentially convex.

For this family of functions from (4.9) we have

$$M_{s,q}(L_k,\Omega_2)=\begin{cases}\left(\frac{L_k(g_s)}{L_k(g_q)}\right)^{\frac{1}{s-q}}, & s\neq q;\\ \exp\left(\frac{L_k(g_0\cdot g_s)}{L_k(g_s)}-\frac{1}{s}\right), & s=q\neq 0;\\ \exp\left(\frac{L_k(g_0^2)}{2L_k(g_0)}\right), & s=q=0.\end{cases}$$

Applying Theorem 11 for functions $g_s,g_q\in\Omega_2$ there exists $\xi_k\in I_k$, $k=1,2,3$ such that

$$e^{\xi_k(s-q)}=\frac{L_k(g_s)}{L_k(g_q)},\quad k=1,2,3.$$

Therefore

$$S_{s,q}(L_k,\Omega_2)=\log M_{s,q}(L_k,\Omega_2)$$

satisfies $\min\{a,b-\lambda,c\}\leqslant S_{s,q}(L_k,\Omega_2)\leqslant\max\{a+\lambda,b,d\}$, so $S_{s,q}(L_k,\Omega_2)$ is a monotonic mean.

EXAMPLE 3. Consider a family of functions

$$\Omega_3=\{\phi_s\colon(0,\infty)\to(0,\infty):s\in(0,\infty)\}$$

defined by

$$\phi_s(x)=\begin{cases}\frac{-s^{-x}}{\log s}, & s\neq 1;\\ x, & s=1.\end{cases}$$

Since $\frac{d\phi_s}{dx}(x)=s^{-x}>0$ for $s,x\in(0,\infty)$, ϕ_s is a nondecreasing function for $x>0$. $\frac{d\phi_s}{dx}(x)=s^{-x}$ is Laplace transform of non-negative function, that is $s^{-x}=\frac{1}{\Gamma(x)}\int_0^\infty e^{-st}t^{x-1}dt$, so $s\mapsto\frac{d\phi_s}{dx}(x)$ is exponentially convex on $(0,\infty)$. As in Example 1 we conclude that $s\mapsto L_k(\phi_s)$, $k=1,2,3$ are exponentially convex. In this case we assume that $I_k\subset\mathbb{R}^+$.

For this family of functions, from (4.9) we have

$$M_{s,q}(L_k,\Omega_3)=\begin{cases}\left(\frac{L_k(\phi_s)}{L_k(\phi_q)}\right)^{\frac{1}{s-q}}, & s\neq q;\\ \exp\left(\frac{-L_k(\phi_1\cdot\phi_s)}{sL_k(\phi_s)}-\frac{1}{s\log s}\right), & s=q\neq 1;\\ \exp\left(\frac{-L_k(\phi_1^2)}{2L_k(\phi_1)}\right), & s=q=1.\end{cases}$$

Applying Theorem 11 for functions $\phi_s,\phi_q\in\Omega_3$ there exists $\xi_k\in I_k$, $k=1,2,3$ such that

$$\left(\frac{s}{q}\right)^{-\xi_k}=\frac{L_k(\phi_s)}{L_k(\phi_q)},\quad k=1,2,3$$

Therefore

$$S_{s,q}(L_k,\Omega_3)=-L(s,q)\log M_{s,q}(L_k,\Omega_3)$$

satisfies $\min\{a, b-\lambda, c\} \leqslant S_{s,q}(L_k, \Omega_3) \leqslant \max\{a+\lambda, b, d\}$, so $S_{s,q}(L_k, \Omega_3)$ is a monotonic mean. $L(s,q)$ is logarithmic mean defined by

$$L(s,q) = \begin{cases} \frac{s-q}{\log s - \log q}, & s \neq q \\ s, & s = q. \end{cases}$$

EXAMPLE 4. Consider a family of functions

$$\Omega_4 = \{\psi_s \colon (0,\infty) \to (0,\infty) : s \in (0,\infty)\}$$

defined by

$$\psi_s(x) = \frac{-e^{-x\sqrt{s}}}{\sqrt{s}}.$$

For every $s > 0$, ψ_s are nondecreasing functions for $x > 0$. Again we conclude, $s \mapsto \frac{d\psi_s}{dx}(x) = e^{-x\sqrt{s}}$ is Laplace transform of non-negative function, so it is exponentially convex on $(0,\infty)$. As in Example 1 we conclude that $s \mapsto L_k(\psi_s)$, $k = 1,2,3$ are exponentially convex. In this case we assume that $I_k \subset \mathbb{R}^+$. For this family of functions, from (4.9) we have

$$M_{s,q}(L_k, \Omega_4) = \begin{cases} \left(\frac{L_k(\psi_s)}{L_k(\psi_q)}\right)^{\frac{1}{s-q}}, & s \neq q; \\ \exp\left(\frac{-L_k(id \cdot \psi_s)}{2\sqrt{s}L_k(\psi_s)} - \frac{1}{2s}\right), & s = q. \end{cases}$$

Applying Theorem 11 for functions $\psi_s, \psi_q \in \Omega_4$ there exists $\xi_k \in I_k$, $k = 1,2,3$ such that

$$e^{-\xi_k(\sqrt{s}-\sqrt{q})} = \frac{L_k(\psi_s)}{L_k(\psi_q)}, \quad k = 1,2,3.$$

Therefore

$$S_{s,q}(L_k, \Omega_4) = -(\sqrt{s}+\sqrt{q}) \log M_{s,q}(L_k, \Omega_4)$$

satisfies $\min\{a, b-\lambda, c\} \leqslant S_{s,q}(L_k, \Omega_4) \leqslant \max\{a+\lambda, b, d\}$, so $S_{s,q}(L_k, \Omega_4)$ is a monotonic mean.

Acknowledgement. This research was supported by the Croatian Ministry of Science, Education, and Sports, under Research Grants 036-1170889-1054 (first author), 117-1170889-0888 (second author) and 114-0000000-3145 (third author).

REFERENCES

[1] A. AGLIĆ ALJINOVIĆ AND J. PEČARIĆ, *Error estimates for approximations of the Fourier transform of functions in L_p spaces*, Aust. J. Math. Anal. Appl. **1** (2) (2004), Article 10.

[2] A. AGLIĆ ALJINOVIĆ, J. PEČARIĆ AND I. PERIĆ, *Estimates of the difference between two weighted integral means via weighted Montgomery identity*, Math. Inequal. Appl. **7** (3) (2004), 315–336.

[3] P. CERONE, *On some generalisations of Steffensen's inequality and related results*, J. Inequal. Pure Appl. Math. (JIPAM) **2** (3) (2001), Article 28.

[4] J. JAKŠETIĆ, J. PEČARIĆ, *Exponential Convexity Method*, J. Convex Anal. **20** (1) (2013), 181–197.
[5] D. S. MITRINOVIĆ, J. E. PEČARIĆ AND A. M. FINK, *Classical and New Inequalities in Analysis, Mathematics and Its Applications (Eastern European Series)*, vol. 61, Kluwer Academic Publishers, Dordrecht, Boston and London, 1993.
[6] J. E. PEČARIĆ, *On the Čebyšev inequality*, Bul. Inst. Politehn. Timisoara **25** (39) (1980), 10–11.
[7] J. E. PEČARIĆ, *Inverse of Steffensen's inequality*, Glas. Mat. Ser. III **17** (37) (1982), 265–270.
[8] J. PEČARIĆ, J. PERIĆ, *Improvements of the Giaccardi and the Petrović inequality and related results*, An. Univ. Craiova Ser. Mat. Inform. **39** (1) (2012), 65–75.
[9] J. E. PEČARIĆ, F. PROSCHAN, Y. L. TONG, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, Inc, 1992.
[10] J. F. STEFFENSEN, *On certain inequalities between mean values, and their applications to actuarial problems*, Skand. Aktuarietidskr. (1918), 82–97.

(Received February 25, 2013)

A. Aglić Aljinović
Department of Applied Mathematics
Faculty of Electrical Engineering and Computing
University of Zagreb
Unska 3, 10 000 Zagreb, Croatia
e-mail: andrea.aglic@fer.hr

J. Pečarić
Faculty of Textile Technology
University of Zagreb
Prilaz baruna Filipovića 28a
10 000 Zagreb, Croatia
e-mail: pecaric@hazu.hr

A. Perušić
Faculty of Civil Engineering, University of Rijeka
Radmile Matejčić 3
51 000 Rijeka, Croatia
e-mail: anamarija.perusic@gradri.hr