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COMPLETE MONOTONICITY PROPERTIES OF FUNCTIONS INVOLVING q–GAMMA AND q–DIGAMMA FUNCTIONS

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Abstract. In this paper, the q-analogue of the Stirling formula (the Moak formula) for the q-gamma function is exploited to prove the complete monotonicity property of functions involving the q-gamma and the q-digamma functions. The monotonicity of these functions is used to establish sharp inequalities for the q-gamma and the q-polygamma functions and the q-Harmonic number.

1. Introduction

The q-analogue of the gamma function is defined as

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{k=0}^{\infty} \frac{1-q^{n+1}}{1-q^{n+x}}, \qquad 0 < q < 1, \tag{1.1}$$

and

$$\Gamma_q(x) = (q-1)^{1-x} q^{\frac{x(x-1)}{2}} \prod_{k=0}^{\infty} \frac{1-q^{-(n+1)}}{1-q^{-(n+x)}}, \qquad q > 1. \tag{1.2}$$

The logarithmic derivative $\psi_q(x)$ of the q-gamma function is known as the q-psi or the q-digamma function,

$$\psi_q(x) = \frac{d}{dx}(\log \Gamma_q(x)) = \frac{\Gamma_q'(x)}{\Gamma_q(x)} \tag{1.3}$$

which appeared in the work of Krattenthaler and Srivastava [9] when they studied the summations of basic hypergeometric series. Some of its properties are presented and proved in their work. They proved that $\psi_q(x)$ tends to the digamma function $\psi(x)$ when letting $q \to 1$. Some inequalities which involve the q-gamma and q-digamma functions have been proved in [2, 4–8, 11, 13]. For more details on the q-digamma function, see [12] where some properties and expansions associated with the q-digamma function are presented.

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From (1.1), for $0<q<1$ and for all real variable $x>0$, we get

$$\psi_q(x) = -\log(1-q) + \log q \sum_{k=1}^{\infty} \frac{q^{xk}}{1-q^k}, \tag{1.4}$$

and from (1.2), for $q>1$ and $x>0$, we obtain

$$\psi_q(x) = -\log(q-1) + \log q \left[x - \frac{1}{2} - \sum_{k=1}^{\infty} \frac{q^{-xk}}{1-q^{-k}} \right]. \tag{1.5}$$

From the previous definitions, for a positive x and $q \geqslant 1$, we get

$$\Gamma_q(x) = q^{\frac{(x-1)(x-2)}{2}} \Gamma_{q^{-1}}(x), \tag{1.6}$$

$$\psi_q(x) = \frac{2x-3}{2} \log q + \psi_{q^{-1}}(x). \tag{1.7}$$

An important fact for the gamma function in applied mathematics as well as in probability is the Stirling formula that gives a pretty accurate idea about the size of the gamma function. With the Euler-Maclaurin formula, Moak [10] obtained the following q-analogue of the Stirling formula (see also [11])

$$\log \Gamma_q(x) \sim \left(x - \frac{1}{2}\right) \log [x]_q + \frac{\mathrm{Li}_2(1-q^x)}{\log q} + \frac{1}{2} H(q-1) \log q + C_{\hat{q}}$$
$$+ \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left(\frac{\log \hat{q}}{\hat{q}^x - 1}\right)^{2k-1} \hat{q}^x P_{2k-3}(\hat{q}^x), \qquad x \to \infty \tag{1.8}$$

where $H(\cdot)$ denotes the Heaviside step function, B_k, $k = 1,2,\cdots$ are the Bernoulli numbers,

$$\hat{q} = \begin{cases} q & \text{if} \quad 0 < q \leqslant 1 \\ q^{-1} & \text{if} \quad q \geqslant 1, \end{cases}$$

$[x]_q = (1-q^x)/(1-q)$, $\mathrm{Li}_2(z)$ is the dilogarithm function defined for complex argument z as [1]

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t} dt, \qquad z \notin (1,\infty), \tag{1.9}$$

P_k is a polynomial of degree k satisfying

$$P_k(z) = (z - z^2) P'_{k-1}(z) + (kz+1) P_{k-1}(z), \qquad P_0 = P_{-1} = 1, \quad k = 1,2,\cdots \tag{1.10}$$

and

$$C_q = \frac{1}{2}\log(2\pi) + \frac{1}{2}\log\left(\frac{q-1}{\log q}\right) - \frac{1}{24}\log q + \log\left(\sum_{m=-\infty}^{\infty} \left(r^{m(6m+1)} - r^{(2m+1)(3m+1)}\right)\right) \tag{1.11}$$

where $r = \exp(4\pi^2/\log q)$. It is easy to see that

$$\lim_{q\to 1} C_q = C_1 = \frac{1}{2}\log(2\pi), \quad \lim_{q\to 1} \frac{\mathrm{Li}_2(1-q^x)}{\log q} = -x \quad \text{and} \quad P_k(1) = (k+1)! \tag{1.12}$$

and so (1.8) when letting $q \to 1$, tends to the ordinary Stirling formula [1]

$$\log\Gamma(x) \sim \left(x-\frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)}\frac{1}{x^{2k-1}}, \qquad x\to\infty. \tag{1.13}$$

A real-valued function f, defined on an interval I, is called completely monotonic, if f has derivatives of all orders and satisfies

$$(-1)^n f^{(n)}(x) \geqslant 0, \qquad n\in\mathbb{N};\ x\in I. \tag{1.14}$$

These functions have numerous applications in various branches, like, for instance, numerical analysis and probability theory.

This paper is devoted to prove the complete monotonicity properties of functions involving the q-gamma and the q-digamma functions by means of using the q-analogue of the Stirling formula (the Moak formula) for the q-gamma function. The monotonicity of these functions is exploited to establish sharp inequalities for the q-gamma and the q-polygamma functions and the q-Harmonic number. Our results are shown to be a generalization of the results obtained by Theorem 2.1 and Corollary 2.2 in the work of Batir [3].

2. The main results

We begin this section with the following lemma that we need to prove the main results.

LEMMA 2.1. *Suppose that $y\in\mathbb{R}$ such that $0<y<1$. Then, the function*

$$g(y) = \frac{\log(1-y)+\log(\log^2 y)-\log 3-\log(2\log y+\log^2 y+2(1-y))}{\log y} \tag{2.1}$$

is increasing from $(0,1)$ onto $(0,\frac{1}{4})$.

Proof. On differentiating $g(y)$ yields $g'(y) = f(y)/(y\log^2 y)$ where

$$f(y) = 2-\log(1-y)-\log(\log^2 y)+\log 3+\log(2(1-y)+2\log y+\log^2 y) \\ -\frac{y\log y}{1-y}+\frac{2\log y(1-y+\log y)}{2(1-y)+2\log y+\log^2 y}.$$

Again, differentiating $f(y)$ gives

$$f'(y) = \frac{h(y)}{y\log y((1-y)^2((2((1-y)+2\log y+\log^2 y)^2}$$

where

$$h(y) = -8(1-y)^4 - 16(1-y)^3\log y - 4(4-y)(1-y)^2\log^2 y \\ -4(1+y-3y^2+y^3)\log^3 y - 2y(3-y^2)\log^4 y - 4y\log^5 y - y\log^6 y. \quad (2.2)$$

Using the Cauchy product rule and binomial coefficients would yield

$$(1-y)^4 = y^4\left(e^{\log(1/y)}-1\right)^4 = 4y^4\sum_{n=4}^{\infty}\frac{\log^n(1/y)}{n!}(4^{n-1}-3^n+3\times 2^{n-1}-1). \quad (2.3)$$

Similarly, we can deduce that

$$(1-y)^3\log y = -y^4\sum_{n=4}^{\infty}\frac{\log^n(1/y)}{(n-1)!}(4^{n-1}-3^n+3\times 2^{n-1}-1), \quad (2.4)$$

$$(4-y)(1-y)^2\log^2 y = y^4\sum_{n=4}^{\infty}\frac{\log^n(1/y)}{(n-2)!}(4^{n-1}-3^n+3\times 2^{n-1}-1), \quad (2.5)$$

$$y^s\log^r y = y^4(-1)^r\sum_{n=r}^{\infty}\frac{\log^n(1/y)}{(n-r)!}(4-s)^{n-r}, \qquad s,r\in\mathbb{N}_0. \quad (2.6)$$

Substituting into (2.2) yields

$$h(y) = y^4\sum_{n=6}^{\infty}\frac{\log^n(1/y)}{n!}\alpha(n)$$

where

$$\alpha(n) = -4(n^2-5n+8)(4^{n-1}-3^n+3\times 2^{n-1}-1) \\ +4n(n-1)(n-2)(4^{n-3}+3^{n-3}-3\times 2^{n-3}+1) \\ -n(n-1)(n-2)(n-3)(2\times 3^{n-3}-2-4(n-4)\times 3^{n-5}+(n-4)(n-5)\times 3^{n-6})$$

which can be read as

$$\alpha(n) = 4^{n-2}\alpha_1(n) - 3\times 2^{n-1}\alpha_2(n) - 3^{n-6}\alpha_3(n) + \alpha_4(n) \quad (2.7)$$

where

$$\alpha_1(n) = n^3-19n^2+82n-128,$$
$$\alpha_2(n) = n^3+n^2-18n+32,$$
$$\alpha_3(n) = n^6-27n^5+259n^4-1077n^3-1124n^2+13632n-23328,$$
$$\alpha_4(n) = 2n^4-8n^3+14n^2-24n-32.$$

Now, let us suppose that

$$\beta_1(n) = 2^{n-5}\alpha_1(n) - 3\alpha_2(n), \quad (2.8)$$

$$\beta_2(n) = 81\left(\frac{4}{3}\right)^{n-3}\alpha_1(n) - \alpha_3(n). \quad (2.9)$$

Differentiation gives

$$\beta_1^{(4)}(n) = 2^{n-5}\log 2[\alpha_1(n)\log^3 2 + 4\alpha_1'(n)\log^2 2 + 6\alpha_1''(n)\log 2 + 4\alpha_1'(n)].$$

It is easy to see that $\alpha_i(n); i = 1,2,3,4$ and their derivatives are greater than or equal zero for $n \geqslant 15$ and thus $\beta_1^{(4)}(n) > 0$ for $n \geqslant 15$ which yields that $\beta_1^{(3)}(n)$ is increasing for $n \geqslant 15$. Since $\beta_1^{(i)}(15) > 0; i = 0,1,2,3$, then $\beta_1(n) > 0$ for $n \geqslant 15$. Similarly, we can deduce that $\beta_2(n) > 0$ for $n \geqslant 15$. In view of (2.7), (2.8), (2.9) and the previous, we get

$$\alpha(n) = 2^{n-1}\beta_1(n) + 3^{n-6}\beta_2(n) + \alpha_4(n) > 0, \qquad n \geqslant 15$$

and $\alpha(6) = \alpha(7) = \alpha(8) = \alpha(9) = 0$, $\alpha(10) = 18480$, $\alpha(11) = 535920$, $\alpha(12) = 8524032$, $\alpha(13) = 98731776$, $\alpha(14) = 932611680$. These conclude that $\alpha(n) \geqslant 0$ for all $n \geqslant 6$ and thus $h(y) > 0, (0 < y < 1)$ which reveals that $f'(y) < 0, (0 < y < 1)$. This concludes that $f(y)$ is decreasing on $(0,1)$ and by virtue that $\lim_{y\to 1} f(y) = 0$, we get $f(y) > 0, (0 < y < 1)$. Therefore, $g'(y) > 0, (0 < y < 1)$ which yields that $g(y)$ is increasing on $(0,1)$. Using l'Hospital's rule would yield that $\lim_{y\to 0} g(y) = 0$ and $\lim_{y\to 1} g(y) = \frac{1}{4}$. This ends the proof. □

According to Moak formula for the q-gamma function (1.8), the function $\mu_q(x)$ defined for all real number $q > 0$ as

$$\mu_q(x) = \log\Gamma_q(x) - \left(x - \frac{1}{2}\right)\log[x]_q - \frac{\mathrm{Li}_2(1-q^x)}{\log q} - \frac{1}{2}H(q-1)\log q - C_{\hat{q}} \quad (2.10)$$

which will appear in our main result, has the limit

$$\lim_{x\to\infty} \mu_q(x) = 0, \qquad q \in \mathbb{R}^+. \quad (2.11)$$

THEOREM 2.2. *Suppose that x, q and a are real. Then, the function*

$$F_a(x;q) = \mu_q(x) - \frac{1}{2}(\log[x]_q - \psi_q(x)) + \frac{1}{12}H(q-1)\log q - \frac{1}{6}\frac{q^{x-a}\log q}{1-q^{x-a}} \quad (2.12)$$

is completely monotonic on (a,∞) for all $q > 0$ if and only if $a \geqslant g(\hat{q})$ where g is defined as in (2.1). Also, the function $-F_b(x;q)$ is completely monotonic on $(0,\infty)$ for all $q > 0$ if and only if $b \leqslant 0$.

Proof. Differentiation gives

$$F_a'(x;q) = \frac{d}{dx}F_a(x;q) = \psi_q(x) - \log[x]_q + \frac{1}{2}\psi_q'(x) - \frac{1}{6}\frac{q^{x-a}\log^2 q}{(1-q^{x-a})^2} \quad (2.13)$$

and

$$F_a''(x;q) = \psi_q'(x) + \frac{1}{2}\psi_q''(x) + \frac{q^x\log q}{1-q^x} - \frac{1}{6}\frac{q^{x-a}\log^3 q(1+q^{x-a})}{(1-q^{x-a})^3}. \quad (2.14)$$

When $0 < q < 1$, (1.4) gives

$$F_a''(x;q) = \frac{1}{12}\sum_{k=1}^{\infty}\frac{q^{xk}\log q}{1-q^k}f(a,y)$$

where

$$f(a,y) = 12\log y + 6\log^2 y + 12(1-y) - 2y^{-a}(1-y)\log^2 y, \qquad y = q^k;\ k\in\mathbb{N}.$$

It is easy to see that the function $a \mapsto f(a,y)$ is decreasing on $\mathbb{R}$ and it has the only one zero function depending on y at $a = g(y)$, $y = q^k$; $k \in \mathbb{N}$. The function $f(0,y)$ can be computed as

$$f(0,y) = 12\log y + 6\log^2 y + 12(1-y) - 2(1-y)\log^2 y = 4y\sum_{n=3}^{\infty}\frac{\log^n(1/y)}{n(n-2)(n-4)!} > 0$$

which concludes that the function $-F_b''(x;q)$ is completely monotonic on $(0,\infty)$ for $0 < q < 1$ if $b \leqslant 0$. In view of the result obtained by Lemma 2.1, we have to take $a \geqslant g(q)$ to ensure that $f(a,y) < 0$ for all $y = q^k$; $k \in \mathbb{N}$ which reveals that $F_a''(x;q)$ is completely monotonic on (a,∞) for $0 < q < 1$ if $a > g(q)$. When $q > 1$, inserting the derivatives of (1.7) into (2.14) would yield $F_a''(x;q) = F_a''(x;q^{-1})$ which proves that $F_a''(x;q)$ is completely monotonic on (a,∞) for $q > 1$ if $a > g(q^{-1})$ and the function $-F_b''(x;q)$ is completely monotonic on $(0,\infty)$ for $q > 1$ if $b \leqslant 0$. In view of the previous results we conclude that $F_a''(x;q)$ is completely monotonic on (a,∞) for $q > 0$ if $a > g(\hat{q})$ and the function $-F_b''(x;q)$ is completely monotonic on $(0,\infty)$ for $q > 0$ if $b \leqslant 0$.

From (1.4) and (1.5), one can easily show that

$$\lim_{x\to\infty}(\psi_q(x) - \log[x]_q) = \begin{cases} 0 & \text{when} \quad 0 < q < 1, \\ -\frac{1}{2}\log q & \text{when} \quad q > 1, \end{cases} \tag{2.15}$$

and also

$$\lim_{x\to\infty}\psi_q'(x) = \begin{cases} 0 & \text{when} \quad 0 < q < 1, \\ \log q & \text{when} \quad q > 1, \end{cases} \tag{2.16}$$

which yield by applying (2.12) that $\lim_{x\to\infty}F_a'(x;q) = 0$ for all $q > 0$. The monotonicity property of $F_a''(x;q)$ shows that $F_a'(x;q) < 0$ and increasing on (a,∞) for $q > 0$ if $a > g(\hat{q})$ and the function $F_b'(x;q) > 0$ and decreasing on $(0,\infty)$ for $q > 0$ if $b \leqslant 0$. The relations (2.11), (2.12), (2.15) and (2.16) give $\lim_{x\to\infty}F_a(x;q) = 0$ for all $q > 0$. Therefore, The monotonicity property of $F_a'(x;q)$ shows that $F_a(x;q) > 0$ and decreasing on (a,∞) for $q > 0$ if $a > g(\hat{q})$ and the function $F_b(x;q) < 0$ and increasing on $(0,\infty)$ for $q > 0$ if $b \leqslant 0$ which conclude that $F_a(x;q)$ is completely monotonic on (a,∞) for $q > 0$ if $a > g(\hat{q})$ and the function $-F_b(x;q)$ is completely monotonic on $(0,\infty)$ for $q > 0$ if $b \leqslant 0$.

Conversely, let the function $F_a(x;q)$ is completely monotonic on (a,∞) for $q>0$. From Moak formula (1.8), we get

$$\begin{aligned}\lim_{x\to\infty}\hat{q}^{-x}\mu_q(x) &= \lim_{x\to\infty}\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left(\frac{\log\hat{q}}{\hat{q}^x-1}\right)^{2k-1}P_{2k-3}(\hat{q}^x)\\ &= \frac{-1}{\log\hat{q}}\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}(\log\hat{q})^{2k}P_{2k-3}(0)\\ &= \frac{1}{1-\hat{q}}+\frac{1}{\log\hat{q}}-\frac{1}{2}. \qquad (2.17)\end{aligned}$$

Here, we used $P_k(0)=1,\ k\in\mathbb{N}_0$ and the generating function of Bernoulli number

$$\frac{t}{e^t-1}=1-\frac{1}{2}t+\sum_{k=1}^{\infty}B_{2k}\frac{t^{2k}}{(2k)!}.$$

Also from (1.4) and (1.5), for $q>0$, we have

$$\lim_{x\to\infty}\hat{q}^{-x}\left(\psi_q(x)-\log[x]_q+\frac{1}{2}H(q-1)\log q\right)=1+\frac{\log\hat{q}}{1-\hat{q}}. \qquad (2.18)$$

It is easy to prove that

$$\lim_{x\to\infty}\hat{q}^{-x}\left(H(q-1)\log q+\frac{q^{x-a}\log q}{1-q^{x-a}}\right)=\hat{q}^{-a}\log\hat{q}. \qquad (2.19)$$

The monotonicity property of $F_a(x;q)$ on (a,∞) gives $\hat{q}^{-x}F_a(x;q)\geqslant 0$ and thus by substituting (2.17), (2.18) and (2.19) into (2.12) after multiplying by $\hat{q}^{-x}$ would yield

$$\frac{1}{1-\hat{q}}+\frac{1}{\log\hat{q}}-\frac{1}{2}+\frac{1}{2}\left(1+\frac{\log\hat{q}}{1-\hat{q}}\right)-\frac{1}{6}\hat{q}^{-a}\log\hat{q}\geqslant 0$$

or equivalently

$$a\geqslant\frac{\log(1-\hat{q})+\log(\log^2\hat{q})-\log 3-\log(2\log\hat{q}+\log^2\hat{q}+2(1-\hat{q}))}{\log\hat{q}}=g(\hat{q}).$$

Now suppose that $-F_b(x;q)$ (with $b>0$) is completely monotonic for $q>0$ on $(0,\infty)$. This means that $F_b(x;q)$ is negative on $(0,\infty)$. But, this contradicts

$$\lim_{x\to 0}F_b(x;q)=\infty.$$

This ends the proof. □

From the above theorem, the double inequalities $F_b(x;q)<0<F_a(x;q)$ hold for all $q\in\mathbb{R}^+;b\leqslant 0$, $a\geqslant g(\hat{q})$, $x>0$ for left side and $x>a$ for the right side which make us in a position to provide the following sharp inequalities for the q-gamma function.

COROLLARY 2.3. *Let α and β be real numbers. Then the inequalities*

$$\sqrt{2\pi}S_{\hat{q}}[x]_q^x q^{\frac{5}{12}H(q-1)} \exp\left(\frac{Li_2(1-q^x)}{\log q} - \frac{1}{2}\psi_q(x) + \frac{1}{6}\frac{q^{x-\alpha}\log q}{1-q^{x-\alpha}}\right) < \Gamma_q(x)$$
$$< \sqrt{2\pi}S_{\hat{q}}[x]_q^x q^{\frac{5}{12}H(q-1)} \exp\left(\frac{Li_2(1-q^x)}{\log q} - \frac{1}{2}\psi_q(x) + \frac{1}{6}\frac{q^{x-\beta}\log q}{1-q^{x-\beta}}\right) \qquad (2.20)$$

hold for all $x > \alpha$ for the left side and $x > 0$ for the right side with the best possible constants $\alpha = g(\hat{q})$ and $\beta = 0$ for all $q > 0$, where

$$S_q = q^{\frac{-1}{24}}\sqrt{\frac{q-1}{\log q}}\sum_{m=-\infty}^{\infty}\left(r^{m(6m+1)} - r^{(2m+1)(3m+1)}\right). \qquad (2.21)$$

A q-analogue of Harmonic number is defined by [14] as

$$H_{n,q} = \sum_{k=1}^{n}\frac{q^k}{1-q^k}, \qquad n \in \mathbb{N} \qquad (2.22)$$

which can be related to $\psi_q(n+1)$ for a positive integer n by

$$\psi_q(n+1) = \frac{\log q}{1-q}\gamma_q - \log q H_{n,q}, \qquad n \in \mathbb{N} \qquad (2.23)$$

where $\gamma_q = \frac{1-q}{\log q}\psi_q(1)$ is the q-analogue of the Euler-Mascheroni constant [12].
Also, in [12], the recursive formula

$$\psi_q(x+1) = \psi_q(x) - \frac{q^x\log q}{1-q^x}, \qquad x > 0 \qquad (2.24)$$

derived for $0 < q < 1$ and by (1.7), it can be extended for $q \geqslant 1$. Moak [10] proved the identity

$$\frac{d^n}{dx^n}\left[\frac{q^x\log q}{1-q^x}\right] = \left(\frac{\log q}{1-q^x}\right)^{n+1} q^x P_{n-1}(q^x), \qquad n \in \mathbb{N}_0. \qquad (2.25)$$

From the above theorem, the following double inequalities

$$F_a'(x;q) < 0 < F_b'(x;q), \qquad x > a$$
$$(-1)^n F_a^{(n+1)}(x;q) < 0 < (-1)^n F_b^{(n+1)}(x;q), \qquad x > a,\ n \in \mathbb{N}$$

are valid for all $a \geqslant g(\hat{q})$ and $b \leqslant 0$, which make us with the above identities in a position to provide the following sharp inequalities for the q-digamma, the q-polygamma functions and the q-Harmonic number.

COROLLARY 2.4. *Let α and β be real numbers. For all $x > \alpha$ we have*

$$\log[x]_q - \frac{1}{2}\psi_q'(x) + \frac{1}{6}\frac{q^{x-\beta}\log^2 q}{(1-q^{x-\beta})^2} < \psi_q(x) < \log[x]_q - \frac{1}{2}\psi_q'(x) + \frac{1}{6}\frac{q^{x-\alpha}\log^2 q}{(1-q^{x-\alpha})^2} \tag{2.26}$$

and for all positive integer numbers n, we have

$$\begin{aligned}
&\frac{1}{2}(-1)^{n+1}\psi_q^{(n+1)}(x) - (-1)^n\left(\frac{\log q}{1-q^x}\right)^n q^x P_{n-2}(q^x) \\
&+\frac{1}{6}(-1)^n\left(\frac{\log q}{1-q^{x-\beta}}\right)^{n+2} q^{x-\beta}P_n(q^{x-\beta}) \\
<&(-1)^n\psi_q^{(n)}(x) < \frac{1}{2}(-1)^{n+1}\psi_q^{(n+1)}(x) - (-1)^n\left(\frac{\log q}{1-q^x}\right)^n q^x P_{n-2}(q^x) \\
&+\frac{1}{6}(-1)^n\left(\frac{\log q}{1-q^{x-\alpha}}\right)^{n+2} q^{x-\alpha}P_n(q^{x-\alpha})
\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}
&\gamma_q + \frac{q-1}{\log q}\log[n]_q + \frac{1-q}{2\log q}\psi_q'(n) - \frac{\log q}{6(1-q)}\frac{q^{n-\alpha}}{[n-\alpha]_q^2} + \frac{q^n}{[n]_q} < (1-q)H_{n,q} \\
&< \gamma_q + \frac{q-1}{\log q}\log[n]_q + \frac{1-q}{2\log q}\psi_q'(n) - \frac{\log q}{6(1-q)}\frac{q^{n-\beta}}{[n-\beta]_q^2} + \frac{q^n}{[n]_q}
\end{aligned} \tag{2.28}$$

with the best possible constants $\alpha = g(\hat{q})$ and $\beta = 0$ for all $q > 0$.

REMARK 2.5. We concentrate in our work on the cases of $0 < q < 1$ and $q > 1$, because the case of $q = 1$ has been studied by Batir [3]. Due to approach $\Gamma_q(x)$ and $\psi_q(x)$ to $\Gamma(x)$ and $\psi(x)$, respectively, when letting $q \to 1$, our results (2.12) and (2.20) tend to the formulas of Theorem 2.1 and Corollary 2.2 in [3]. Also the inequalities (2.26), (2.27) and (2.28) in Corollary 2.4 when letting $q \to 1$, will approach to

$$\log x - \frac{1}{2}\psi'(x) + \frac{1}{6}\frac{1}{(x-\beta)^2} < \psi(x) < \log x - \frac{1}{2}\psi'(x) + \frac{1}{6}\frac{1}{(x-\alpha)^2} \tag{2.29}$$

and for all positive integer numbers n, we have

$$\begin{aligned}
&\frac{1}{2}(-1)^{n+1}\psi^{(n+1)}(x) - \frac{(n-1)!}{x^n} + \frac{1}{6}\frac{(n+1)!}{(x-\beta)^{n+2}} < (-1)^n\psi^{(n)}(x) \\
&< \frac{1}{2}(-1)^{n+1}\psi^{(n+1)}(x) - \frac{(n-1)!}{x^n} + \frac{1}{6}\frac{(n+1)!}{(x-\alpha)^{n+2}}
\end{aligned} \tag{2.30}$$

and

$$\gamma + \log n - \frac{1}{2}\psi'(n) + \frac{1}{6}\frac{1}{(n-\alpha)^2} + \frac{1}{n} < H_n < \gamma + \log n - \frac{1}{2}\psi'(n) + \frac{1}{6}\frac{1}{(n-\beta)^2} + \frac{1}{n} \tag{2.31}$$

with the best possible constants $\alpha = \frac{1}{4}$ and $\beta = 0$, where H_n is the Harmonic number defined as

$$H_n = \sum_{k=1}^{n} \frac{1}{k}, \qquad n \in \mathbb{N}.$$

REMARK 2.6. It is worth mentioning that, Salem [11] proved that the inequalities

$$\sqrt{2\pi} S_{\hat{q}}[x]_q^x q^{\frac{1}{2}\lambda H(q-1)} \exp\left(\frac{\mathrm{Li}_2(1-q^x)}{\log q} - \frac{1}{2}\psi_q(x+\alpha)\right) < \Gamma_q(x)$$
$$< \sqrt{2\pi} S_{\hat{q}}[x]_q^x q^{\frac{1}{2}\lambda H(q-1)} \exp\left(\frac{\mathrm{Li}_2(1-q^x)}{\log q} - \frac{1}{2}\psi_q(x+\beta)\right) \qquad (2.32)$$

hold for all $x > 0$ with the best possible constants $\alpha = g(\hat{q})$ and $\beta = 0$ when $q > 0$ and $\lambda = \alpha + \frac{1}{2}$ or $\alpha = \frac{1}{2}$ and $\beta = 0$ when $q > 1$ and $\lambda = 1$ where $g(q)$ is increasing function depending on q and satisfies the sharp inequality $0 \leqslant g(\hat{q}) \leqslant 1/3$.

Since $q^x \log q/(6(1-q^x)) < 0$ for all $x > 0$ and $q > 0$, then we get

$$\exp\left(\frac{\mathrm{Li}_2(1-q^x)}{\log q} - \frac{1}{2}\psi_q(x) + \frac{1}{6}\frac{q^x \log q}{1-q^{x-\beta}}\right) < \exp\left(\frac{\mathrm{Li}_2(1-q^x)}{\log q} - \frac{1}{2}\psi_q(x)\right)$$

for all $x > 0$ and $q > 0$ and so the right hand side of (2.20) is sharper than of (2.32) when $0 < q < 1$. Also, when $q > 1$ and $\lambda = 1$, we note that $q^{5/12} < q^{1/2}$ and so the right hand side of (2.20) is sharper than of (2.32) when $q > 1$. In conclusion, we see that the inequality (2.20) is a refinement of the inequality (2.32).

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