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LOCATION OF THE ZEROS OF TRINOMIALS AND QUADRINOMIALS

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Abstract. In this paper, we prove certain results concerning the location of the zeros of quadrinomials, which in particular considerably improves a result due to Landau. We also present a very simple proof of a known result for trinomials, which provides a refinement of another result of Landau.

1. Introduction

Quite a few results giving bound for all the zeros of a polynomial $P(z) = \sum_{j=0}^{n} a_j z^j$ were expressed (see [7,8]) as functions of all the coefficients. It seems natural to ask whether, there exist some bounds for the k zeros of smallest modulus, $k < n$, which would be independent of certain coefficients a_j. Landau first, raised this question in connection with his study of the Picard's Theorem. In [5] and [6] Landau proved that every trinomial

$$a_0 + a_1 z + a_n z^n, \quad a_1 a_n \neq 0, \quad n \geqslant 2,$$

has at least one zero in the circle

$$|z| \leqslant 2|a_0/a_1| \tag{1}$$

and every quadrinomial

$$a_0 + a_1 z + a_m z^m + a_n z^n, \quad a_1 a_m a_n \neq 0, \quad 0 \leqslant m < n,$$

has at least one zero in the circle

$$|z| \leqslant \frac{17}{3} \left| \frac{a_0}{a_1} \right|. \tag{2}$$

For every $n \geqslant 2$, as a refinement of (1), the trinomial

$$a_0 + a_1 z + a_n z^n, \quad a_1 a_n \neq 0,$$

is known [2] to have a zero in both the regions

$$\left| z + \frac{a_0}{a_1} \right| \leqslant \left| \frac{a_0}{a_1} \right| \quad \text{and} \quad \left| z + \frac{a_0}{a_1} \right| \geqslant \left| \frac{a_0}{a_1} \right|. \tag{3}$$

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Joyal, Labelle and Rahman [4] gave an alternative proof of this fact by using Gauss-Lucas theorem. In literature, there exist several results about zero distribution of trinomials equations, for example see [1] and [3].

Here, in this paper, we present certain results for quadrinomials and also give a very simple proof of (3), independent of Gauss-Lucas theorem, which is perhaps the simplest one can think of. We start by proving the following result which considerably improves (2) due to Landau.

THEOREM 1. *At least one zero of the quadrinomial*

$$a_0 + a_1 z + a_m z^m + a_n z^n, \quad a_1 a_m a_n \neq 0, \quad 2 \leqslant m < n,$$

lie in the circle

$$|z| \leqslant \frac{2n}{n-1}\left|\frac{a_0}{a_1}\right| \leqslant 3\left|\frac{a_0}{a_1}\right|. \tag{4}$$

Applying this result to the polynomial $z^n P(1/z)$ where $P(z) = a_0 + a_p z^p + a_{n-1} z^{n-1} + z^n$, we get the following:

COROLLARY 1. *At least one zero of the quadrinomial*

$$a_0 + a_p z^p + a_{n-1} z^{n-1} + z^n, \quad a_0 a_p a_{n-1} \neq 0, \quad 1 \leqslant p \leqslant n-2,$$

lie in the circle

$$|z| \geqslant \frac{n-1}{2n}\left|a_{n-1}\right|. \tag{5}$$

Next we prove the following result for quadrinomials.

THEOREM 2. *For every $n \geqslant 3$, the quadrinomial*

$$a_0 + a_1 z + a_2 z^2 + a_n z^n, \quad a_2 a_n \neq 0,$$

has at least one zero in both the regions,

$$|z| \leqslant \left\{\frac{n}{n-2}\left|\frac{a_0}{a_2}\right|\right\}^{1/2} \tag{6}$$

and

$$\left|z + \frac{a_1}{2a_2}\right| \geqslant \left|\frac{a_1}{2a_2}\right|. \tag{7}$$

For the proofs of the theorems, we shall make use of the following Lemma.

LEMMA. *Let $P(z) = a_0 + \cdots + a_p z^p + \cdots + a_n z^n$, $a_p a_n \neq 0$, $1 \leqslant p \leqslant n$, be a polynomial of degree n. Then at least one zero of $P(z)$ lies in each of the n circles*

$$|z| \leqslant \left\{C(n,p)\left|\frac{a_0}{a_p}\right|\right\}^{1/p}, \quad p = 1, 2, \cdots, n, \tag{8}$$

where

$$C(n,p) = \frac{n!}{p!(n-p)!}.$$

Proof. If $a_0 = 0$, then $z = 0$ is a zero of $P(z)$ and the Lemma follows in this case. So we suppose that $a_0 \neq 0$. We take

$$Q(z) = z^n P(1/z) = a_0 z^n + \cdots + a_p z^{n-p} + \cdots + a_n.$$

Let $z_1, z_2, \cdots, z_n$ be the zeros of $Q(z)$ such that

$$|z_1| \leqslant |z_2| \leqslant \cdots \leqslant |z_n|. \tag{9}$$

Then

$$a_0 z^n + \cdots + a_p z^{n-p} + \cdots + a_n = Q(z) = a_0 \prod_{j=1}^{n} (z - z_j). \tag{10}$$

Equating the coefficients of the like powers of z on the two sides of (10), we get

$$|a_p| = |a_0| \left| \{ z_1 z_2 \cdots z_p + z_2 z_3 \cdots z_{p+1} + \cdots \} \right|,$$

where the number of terms inside the brackets is $C(n,p) = C(n,n-p)$. Therefore, by (9), we have

$$|a_p| \leqslant |a_0| C(n,p) |z_n|^p, \quad p = 1, 2, \cdots n,$$

which gives,

$$|z_n| \geqslant \left\{ \frac{1}{C(n,p)} \left| \frac{a_p}{a_0} \right| \right\}^{1/p}, \quad p = 1, 2, \cdots, n.$$

This shows that the polynomial $Q(z)$ has at least one zero in

$$|z| \geqslant \left\{ \frac{1}{C(n,p)} \left| \frac{a_p}{a_0} \right| \right\}^{1/p}, \quad p = 1, 2, \cdots, n.$$

Since $P(z) = z^n Q(1/z)$, it follows that $P(z)$ has at least one zero in each of the n circles defined by (8) and the proof of the Lemma is complete. □

2. The proofs

Proof of Theorem 1. If $a_0 = 0$, the assertion is obviously true, so we assume that $a_0 \neq 0$. We write

$$S(z) = a_0 + a_1 z + a_m z^m + a_n z^n.$$

If possible, suppose all the zeros of $S(z)$ lie in

$$|z| > \frac{2n}{n-1} \left| \frac{a_0}{a_1} \right|.$$

Then all the zeros of

$$T(z) = z^n S(1/z) = a_n + a_m z^{n-m} + a_1 z^{n-1} + a_0 z^n$$

lie in

$$|z| < \frac{n-1}{2n}|a_1/a_0|.$$

By Gauss-Lucas theorem, all the zeros of the derived polynomial

$$T'(z) = (n-m)a_m z^{n-m-1} + (n-1)a_1 z^{n-2} + na_0 z^{n-1}$$

also lie in

$$|z| < \frac{n-1}{2n}|a_1/a_0|.$$

This shows that all the zeros of the trinomial

$$z^{n-1}T'(1/z) = (n-m)a_m z^m + (n-1)a_1 z + na_0$$

lie in

$$|z| > \frac{2n}{n-1}|a_0/a_1|.$$

But this is a contradiction, because by (1), the trinomial $z^{n-1}T'(1/z)$ has at least one zero in

$$|z| \leqslant \frac{2n}{n-1}|a_0/a_1|.$$

Thus the quadrinomial $S(z)$ has at least one zero in the circle defined by (4) and the proof of Theorem 1 is complete. □

Proof of Theorem 2. We write

$$F(z) = a_0 + a_1 z + a_2 z^2 + a_n z^n.$$

The case $a_0 = 0$ is trivial, so to prove (6), we suppose that all the zeros of $F(z)$ lie in

$$|z| > \left\{ \frac{n}{n-2}\left|\frac{a_0}{a_2}\right| \right\}^{1/2}.$$

Then all the zeros of quadrinomial

$$G(z) = z^n F(1/z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + a_n$$

lie in

$$|z| < \left\{ \frac{n-2}{n}\left|\frac{a_2}{a_0}\right| \right\}^{1/2}.$$

By the Gauss-Lucas theorem all the zeros of the derived polynomial

$$G'(z) = na_0 z^{n-1} + (n-1)a_1 z^{n-2} + (n-2)a_2 z^{n-3}$$

also lie in

$$|z| < \left\{ \frac{n-2}{n} \left| \frac{a_2}{a_0} \right| \right\}^{1/2}.$$

This implies that all the zeros of quadratic

$$H(z) = na_0 z^2 + (n-1)a_1 z + (n-2)a_2$$

lie in

$$|z| < \left\{ \frac{n-2}{n} \left| \frac{a_2}{a_0} \right| \right\}^{1/2}.$$

Equivalently, all the zeros of the quadratic

$$z^2 H(1/z) = (n-2)a_2 z^2 + (n-1)a_1 z + na_0, \quad a_2 \neq 0,$$

lie in

$$|z| > \left\{ \frac{n}{n-2} \left| \frac{a_0}{a_2} \right| \right\}^{1/2}.$$

But by the Lemma above (with $p = n = 2$), it follows that the quadratic $z^2 H(1/z)$ has at least one in

$$|z| \leqslant \left\{ \frac{n}{n-2} \left| \frac{a_0}{a_2} \right| \right\}^{1/2}.$$

Hence we arrived at a contradiction and therefore $F(z)$ must have at least one zero in the circle defined by (6).

Now to prove (7), suppose that all the zeros of $F(z)$ lie in

$$\left| z + \frac{a_1}{2a_2} \right| < \left| \frac{a_1}{2a_2} \right|.$$

Then by the Gauss-Lucas theorem all the zeros of

$$F'(z) = a_1 + 2a_2 z + na_n z^{n-1}$$

lie in

$$\left| z + \frac{a_1}{2a_2} \right| < \left| \frac{a_1}{2a_2} \right|.$$

But by (3), the trinomial $F'(z)$ has at least one zero in the region

$$\left| z + \frac{a_1}{2a_2} \right| \geqslant \left| \frac{a_1}{2a_2} \right|.$$

Thus we again get a contradiction and therefore, $F(z)$ must have at least one zero in the region defined by (7). This completes the proof of Theorem 2. □

Proof of (3). We write

$$R(z) = a_0 + a_1 z + a_n z^n.$$

The case $a_0 = 0$ is trivial. Hence suppose $a_0 \neq 0$. In order to prove that $R(z)$ has a zero in both the regions defined by (3), we show that $F(z) = R\left(\frac{z-a_0}{a_1}\right)$ has a zero in both the regions $|z| \leqslant |a_0|$ and $|z| \geqslant |a_0|$. Now,

$$\begin{aligned} F(z) &= R\left(\frac{z-a_0}{a_1}\right) \\ &= z + a_n\left(\frac{z-a_0}{a_1}\right)^n \\ &= z + \frac{a_n}{a_1^n}\left(z^n - \cdots + (-1)^n a_0^n\right). \end{aligned}$$

Applying the Lemma above with $p = n$, it follows that at least one zero of $F(z) = R\left(\frac{z-a_0}{a_1}\right)$ lie in $|z| \leqslant |a_0|$.

Now to show that $F(z) = R\left(\frac{z-a_0}{a_1}\right)$ has a zero in $|z| \geqslant |a_0|$, we show that $z^n F(1/z)$ has a zero in $|z| \leqslant (1/|a_0|)$. Since

$$z^n F(1/z) = z^{n-1} + \frac{a_n}{a_1^n}(1 - \cdots + (-1)^n a_0^n z^n),$$

an application of the Lemma above with $p = n$ shows that the polynomial $z^n F(1/z)$ has a zero in $|z| \leqslant (1/|a_0|)$. Replacing z by $a_1 z + a_0$ in $F(z)$ and noting that $F(a_1 z + a_0) \equiv R(z)$, it follows that $R(z)$ has at least one zero in both the regions

$$|a_1 z + a_0| \leqslant |a_0| \quad \text{and} \quad |a_1 z + a_0| \geqslant |a_0|,$$

which readily implies (3). This completes the proof of (3). □

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