Mathematical
Inequalities
& Applications
Volume 17, Number 3 (2014), 831–842 doi:10.7153/mia-17-61

WEAK FORMS OF SUPERCYCLITY AND A CLASS OF PARANORMAL OPERATORS

SUNGEUN JUNG, INSOOK KIM AND EUNGIL KO

(*Communicated by J.-C. Bourin*)

Abstract. In this paper, we give several properties of class A operators, an interesting subclass of paranormal operators. In particular, we consider the operators $T \in \mathscr{L}(\mathscr{H})$ such that $T-\lambda$ is a class A operator for every $\lambda \in \mathbb{C}$. We also provide some cases for class A operators to have a nontrivial invariant subspace. Finally, we prove that there are no N-supercyclic class A operators with trivial kernel for any positive integer N and that weakly supercyclic class A operators with trivial kernel must be normal.

1. Introduction

Let $\mathscr{H}$ be a complex separable Hilbert space and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. If $T \in \mathscr{L}(\mathscr{H})$, we write $\rho(T)$, $\sigma(T)$, $\sigma_p(T)$, $\sigma_e(T)$, and $r(T)$ for the resolvent set, the spectrum, the point spectrum, the essential spectrum, and the spectral radius of T, respectively.

An arbitrary operator $T \in \mathscr{L}(\mathscr{H})$ has a unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the appropriate partial isometry satisfying $\ker(U) = \ker(|T|) = \ker(T)$ and $\ker(U^*) = \ker(T^*)$. Associated with T is a related operator $|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ called the *Aluthge transform* of T, and denoted throughout this paper by $\widetilde{T}$. For an arbitrary operator $T \in \mathscr{L}(\mathscr{H})$, the sequence $\{\widetilde{T}^{(n)}\}$ of Aluthge iterates of T is defined by $\widetilde{T}^{(0)} = T$ and $\widetilde{T}^{(n+1)} = \widetilde{\widetilde{T}^{(n)}}$ for every nonnegative integer n (see [2] and [23]).

An operator $T \in \mathscr{L}(\mathscr{H})$ is said to be p-*hyponormal* if $(TT^*)^p \leqslant (T^*T)^p$, where $0 < p < \infty$. In particular, 1-hyponormal operators and $\frac{1}{2}$-hyponormal operators are called *hyponormal* operators and *semi-hyponormal* operators, respectively. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to be w-*hyponormal* if $|\widetilde{T}| \geqslant |T| \geqslant |(\widetilde{T})^*|$ (see [2]), and an operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *paranormal* if $\|Tx\|^2 \leqslant \|T^2x\|\|x\|$ for all $x \in \mathscr{H}$. We say that an operator $T \in \mathscr{L}(\mathscr{H})$ is called *totally paranormal* if $T-\lambda$ is paranormal for every $\lambda \in \mathbb{C}$. An operator $T \in \mathscr{L}(\mathscr{H})$ with $\|T\| = r(T)$ is called *normaloid*. Furuta-Ito-Yamazaki first introduced class A operators in [18]. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to be a *class A operator* (or *belong to class A*) if it satisfies the condition $|T^2| \geqslant |T|^2$.

Mathematics subject classification (2010): 47B20, 47A16.
Keywords and phrases: Class A operator, invariant subspace, N-supercyclic, weakly supercyclic.
This work was supported by Priority Research Centers Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2009-0093827).

Paper MIA-17-61

There are a lot of consequences concerning class A operators ([18], [20], [21], [22], [28], [33], etc). It is well-known from [18] that

$$\text{Hyponormal} \Rightarrow p\text{-Hyponormal } (0 < p \leqslant 1) \Rightarrow w\text{-Hyponormal}$$
$$\Rightarrow \text{Class } A \Rightarrow \text{Paranormal} \Rightarrow \text{Normaloid.}$$

For $T \in \mathcal{L}(\mathcal{H})$ and $x \in \mathcal{H}$, $\{T^n x\}_{n=0}^{\infty}$ is called the *orbit* of x under T, and is denoted by $O(x,T)$. When the linear span of the orbit $O(x,T)$ is norm dense in $\mathcal{H}$, x is called a *cyclic* vector for T and T is said to be a *cyclic* operator. If $O(x,T)$ is norm dense in $\mathcal{H}$, then x is called a *hypercyclic* vector for T. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *hypercyclic* if there is at least one hypercyclic vector for T. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *hypertransitive* if every nonzero vector in $\mathcal{H}$ is hypercyclic for T. Denote the set of all nonhypertransitive operators in $\mathcal{L}(\mathcal{H})$ by (NHT). *The hypertransitive operator problem* is the open question whether $(NHT) = \mathcal{L}(\mathcal{H})$. We note that $T \in (NHT)$ if and only if it has a nontrivial invariant closed set.

We say that an operator $T \in \mathcal{L}(\mathcal{H})$ is *supercyclic* if there exists a vector $x \in \mathcal{H}$ such that $\mathbb{C}O(x,T) := \{\lambda T^n x : \lambda \in \mathbb{C}, n = 0,1,2,\cdots\}$ is norm dense in $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly supercyclic* if there exists a vector $x \in \mathcal{H}$ such that $\mathbb{C}O(x,T)$ is weakly dense in $\mathcal{H}$. For a positive integer N, an operator $T \in \mathcal{L}(\mathcal{H})$ is called N*-supercyclic* if there exists an N-dimensional subspace $\mathcal{M}$ such that $\bigcup_{n=0}^{\infty} T^n(\mathcal{M})$ is norm dense in $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be N*-multicyclic* if there exist N vectors $x_1, x_2, \cdots, x_N$ in $\mathcal{H}$ such that the linear span of $\{P(T)x_m : m = 1,2,\cdots,N \text{ and } P \in Rat(\sigma(T))\}$ is norm dense in $\mathcal{H}$ where $Rat(\sigma(T))$ is the algebra of complex-valued rational functions with poles off $\sigma(T)$. It is evident that supercyclic operators are 1-supercyclic and the relations between the properties above are as follows:

$$\begin{array}{ccc}
\text{Hypercyclic} & \Rightarrow & \text{Weakly hypercyclic} \\
\Downarrow & & \Downarrow \\
\text{Supercyclic} & \Rightarrow & \text{Weakly supercyclic} \\
\Downarrow & & \Downarrow \\
N\text{-Supercyclic} & & \text{Cyclic} \\
 & & \Downarrow \\
 & & N\text{-Multicyclic}
\end{array}$$

It is known from [6] that any hyponormal operator is not N-supercyclic, while every weakly supercyclic hyponormal operator is a scalar multiple of a unitary operator.

In this paper, we give several properties of class A operators, an interesting subclass of paranormal operators. In particular, we consider the operators $T \in \mathcal{L}(\mathcal{H})$ such that $T - \lambda$ is a class A operator for every $\lambda \in \mathbb{C}$. We also provide some cases for class A operators to have a nontrivial invariant subspace. Finally, we prove that there are no N-supercyclic class A operators with trivial kernel for any positive integer N and that weakly supercyclic class A operators with trivial kernel must be normal.

2. Preliminaries

An operator $T \in \mathscr{L}(\mathscr{H})$ is called *left semi-Fredholm* if its range is closed and $\dim(\ker(T)) < \infty$, while T is called *right semi-Fredholm* if its range is closed and $\dim(\mathscr{H}/\mathrm{ran}(T)) < \infty$. When T is left semi-Fredholm or right semi-Fredholm, T is said to be *semi-Fredholm*. In this case, the *Fredholm index* of T is defined by $\mathrm{ind}(T) := \dim(\ker(T)) - \dim(\mathscr{H}/\mathrm{ran}(T))$. Note that $\mathrm{ind}(T)$ is an integer or $\pm\infty$. We define $\rho_{sF}(T) := \{\lambda \in \mathbb{C} : T-\lambda \text{ is not semi-Fredholm}\}$ and $\sigma_{sF}(T) := \mathbb{C} \setminus \rho_{sF}(T)$. We say that T is *Fredholm* if it is both left and right semi-Fredholm. In particular, an operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *Weyl* if it is Fredholm of index zero. *The Weyl spectrum* of T is given by $\sigma_w(T) := \{\lambda \in \mathbb{C} : T-\lambda \text{ is not Weyl}\}$.

An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have *the single-valued extension property* (or SVEP) if for every open subset G of $\mathbb{C}$ and any analytic function $f : G \to \mathscr{H}$ such that $(T-z)f(z) \equiv 0$ on G, it results $f(z) \equiv 0$ on G. For an operator $T \in \mathscr{L}(\mathscr{H})$ and $x \in \mathscr{H}$, we consider the set $\rho_T(x)$ of elements z_0 in $\mathbb{C}$ such that there exists an analytic function $f(z)$ defined in a neighborhood of z_0, with values in $\mathscr{H}$, which verifies $(T-z)f(z) \equiv x$. The *local spectrum* of T at x, with notation $\sigma_T(x)$, is given by the complement of $\rho_T(x)$, that is, $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$. Using local spectra, we define the *local spectral subspace* for T by $H_T(F) = \{x \in \mathscr{H} : \sigma_T(x) \subseteq F\}$ where F is a subset of $\mathbb{C}$. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have *Dunford's property* (C) if $H_T(F)$ is closed for each closed subset F of $\mathbb{C}$. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have *property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to \mathscr{H}$ of $\mathscr{H}$-valued analytic functions such that $(T-z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. It is well known from [25] that

$$\text{Property } (\beta) \Rightarrow \text{Dunford's property } (C) \Rightarrow \text{SVEP}.$$

It is shown in [22] that every class A operator has property (β).

3. Some properties

In this section, we give several properties of class A operators. We first examine some invariant properties of such operators. It is clear that every scalar multiple of a class A operator is also a class A operator, and the collection of all class A operators is closed under unitary equivalence relation. However, we remark that it is not translation-invariant, i.e., there is a class A operator $T \in \mathscr{L}(\mathscr{H})$ such that $T-\lambda$ does not belong to class A for some $\lambda \in \mathbb{C}$; indeed, setting $T := 4S^2 + S^{*2} + 2SS^* + 2$ where $S \in \mathscr{L}(\mathscr{H})$ is the unilateral shift on $\mathscr{H}$, we get that T is semi-hyponormal but $T-4$ is not paranormal from [8], and hence T is a class A operator but $T-4$ is not.

In the following theorem, we give an inequality for an operator $T \in \mathscr{L}(\mathscr{H})$ such that $T-\lambda$ is a class A operator for each $\lambda \in \mathbb{C}$.

THEOREM 3.1. Let $T \in \mathscr{L}(\mathscr{H})$. If $T-\lambda$ is a class A operator for each $\lambda \in \mathbb{C}$, then

$$\|Tx\| \geqslant |\langle T^2x, x\rangle|^{\frac{1}{2}} \tag{1}$$

for all $x \in \mathscr{H}$.

Proof. Since $T-\lambda$ is a class A operator for each $\lambda \in \mathbb{C}$, it follows from [17] that

$$(T^*-\overline{\lambda})^2(T-\lambda)^2 - 2r(T^*-\overline{\lambda})(T-\lambda) + r^2 \geqslant 0 \tag{2}$$

for all $r>0$ and $\lambda \in \mathbb{C}$. Set $\lambda = \rho e^{i\theta}$ where $0 \leqslant \theta < 2\pi$ and $\rho > 0$. Putting $r=\rho^2$ in (2), we have

$$\begin{aligned} 0 &\leqslant (T^*-\rho e^{-i\theta})^2(T-\rho e^{i\theta})^2 - 2\rho^2(T^*-\rho e^{-i\theta})(T-\rho e^{i\theta}) + \rho^4 \\ &= T^{*2}T^2 - 2\rho e^{-i\theta}T^*T^2 - 2\rho e^{i\theta}T^{*2}T + \rho^2(e^{2i\theta}T^{*2} + e^{-2i\theta}T^2 + 2T^*T). \end{aligned}$$

Dividing both sides by ρ^2 and then letting $\rho \to \infty$, we get that

$$e^{2i\theta}T^{*2} + e^{-2i\theta}T^2 + 2T^*T \geqslant 0$$

for every $0 \leqslant \theta < 2\pi$. Thus it holds that

$$2\|Tx\|^2 \geqslant -\langle e^{-2i\theta}T^2x,x\rangle - \overline{\langle e^{-2i\theta}T^2x,x\rangle} = -2Re\big(\langle e^{-2i\theta}T^2x,x\rangle\big)$$

for every $0 \leqslant \theta < 2\pi$ and $x \in \mathscr{H}$. Taking θ so that $Re\big(\langle e^{-2i\theta}T^2x,x\rangle\big) = -|\langle T^2x,x\rangle|$, we obtain that

$$\|Tx\|^2 \geqslant |\langle T^2x,x\rangle|$$

for all $x \in \mathscr{H}$. $\square$

REMARK. The converse of Theorem 3.1 does not hold. If $T \in \mathscr{L}(\mathscr{H})$ is nilpotent of order 2, then (1) is clearly satisfied for each $x \in \mathscr{H}$. However, since every class A operator is normaloid by [17], the only nilpotent class A operator is the zero operator, but $T \neq 0$. Hence T does not belong to class A.

EXAMPLE 3.2. Since every hyponormal operator is translation invariant, it is obvious that if $T \in \mathscr{L}(\mathscr{H})$ is hyponormal, then it satisfies inequalities (1) by Theorem 3.1. For example, consider a weighted shift W_α given by $W_\alpha e_n = \alpha_n e_{n+1}$ where $\{e_n\}_{n=1}^\infty$ is an orthonormal basis for $\mathscr{H}$ and $\{\alpha_n\}_{n=1}^\infty$ is a bounded sequence of $\mathbb{C}$. Since it is easy to see that W_α is a class A operator if and only if the weight sequence $\{\alpha_n\}$ is increasing, we get that W_α is a class A operator if and only if it is hyponormal. Hence every weighted shift W_α with increasing weight sequence $\{\alpha_n\}$ satisfies that $W_\alpha - \lambda$ is a class A operator for each $\lambda \in \mathbb{C}$ and so (1) holds for such a weighted shift.

From the next proposition, we consider a special case. Recall that an operator $T \in \mathscr{L}(\mathscr{H})$ is called *k-quasihyponormal* if $T^{*k}(T^*T-TT^*)T^k \geqslant 0$, where k is a positive integer. In particular, if $k=1$, then we say that T is *quasihyponormal*.

PROPOSITION 3.3. For $T \in \mathscr{L}(\mathscr{H})$, the following assertions hold.

(i) If T is k-quasihyponormal and $\operatorname{ran}(T^{k-1})$ is norm dense in $\mathscr{H}$, then T belongs to class A, where k is a positive integer. In particular, every quasihyponormal operator belongs to class A.

(ii) $T-\lambda$ is quasihyponormal for all $\lambda \in \mathbb{C}$ if and only if T is hyponormal.

Proof. (i) Since T is k-quasihyponormal, we have

$$0 \leqslant T^{*k}(T^*T - TT^*)T^k = T^{*k-1}(|T^2|^2 - |T|^4)T^{k-1}.$$

Let $x \in \mathscr{H}$ be given. Since $\overline{\operatorname{ran}(T^{k-1})} = \mathscr{H}$, there exists a sequence $\{x_n\}$ in $\mathscr{H}$ such that $\lim_{n\to\infty} T^{k-1}x_n = x$ in norm. Then it holds that

$$\begin{aligned} \langle (|T^2|^2 - |T|^4)x, x\rangle &= \lim_{n\to\infty} \langle (|T^2|^2 - |T|^4)T^{k-1}x_n, T^{k-1}x_n\rangle \\ &= \lim_{n\to\infty} \langle T^{*k-1}(|T^2|^2 - |T|^4)T^{k-1}x_n, x_n\rangle \geqslant 0 \end{aligned}$$

Thus $|T^2|^2 \geqslant |T|^4$, which implies the inequality $|T^2| \geqslant |T|^2$ by Löwner's inequality.

(ii) Suppose that $T - \lambda$ is quasihyponormal for all $\lambda \in \mathbb{C}$. Then we have

$$(T^* - \overline{\lambda})^2(T - \lambda)^2 - [(T^* - \overline{\lambda})(T - \lambda)]^2 \geqslant 0$$

for all $\lambda \in \mathbb{C}$. Expanding the left side of this inequality, we obtain that

$$TT^* - T^*T \leqslant \frac{1}{|\lambda|^2}(|T^2|^2 - |T|^4) - \frac{1}{\overline{\lambda}}T^{*2}T - \frac{1}{\lambda}T^*T^2 + \frac{1}{\overline{\lambda}}T^*TT^* + \frac{1}{\lambda}TT^*T$$

for all nonzero $\lambda \in \mathbb{C}$. Therefore we get that

$$\begin{aligned} \langle (TT^* - T^*T)x, x\rangle \leqslant\ & \frac{1}{|\lambda|^2}|\langle (|T^2|^2 - |T|^4)x, x\rangle| + \frac{1}{|\lambda|}|\langle T^{*2}Tx, x\rangle| \\ & + \frac{1}{|\lambda|}|\langle T^*T^2x, x\rangle| + \frac{1}{|\lambda|}|\langle T^*TT^*x, x\rangle| + \frac{1}{|\lambda|}|\langle TT^*Tx, x\rangle| \end{aligned}$$

for all $x \in \mathscr{H}$ and all nonzero $\lambda \in \mathbb{C}$. Letting $|\lambda| \to \infty$, we conclude that $TT^* \leqslant T^*T$, i.e., T is hyponormal. The converse statement is trivial. □

It is easy to show that the set of all class A operators in $\mathscr{L}(\mathscr{H})$ is norm closed.

PROPOSITION 3.4. *If $\{T_n\}_{n=1}^{\infty}$ is a sequence of class A operators in $\mathscr{L}(\mathscr{H})$ such that* $\lim_{n\to\infty} \|T_n - T\| = 0$ *for some $T \in \mathscr{L}(\mathscr{H})$, then T is a class A operator.*

COROLLARY 3.5. *Under the same hypotheses as in Proposition 3.4, we have*

$$\lim_{n\to\infty} r(T_n) = r(T).$$

Proof. By Proposition 3.4, T is a class A operator. Hence T is normaloid, i.e., $r(T) = \|T\|$. Since $|r(T_n) - r(T)| = \big|\|T_n\| - \|T\|\big| \leqslant \|T_n - T\|$, the equality $\lim_{n\to\infty} r(T_n) = r(T)$ holds. □

Recall that an operator $X \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ is said to be a *quasiaffinity* if it has trivial kernel and dense range. An operator $S \in \mathscr{L}(\mathscr{H})$ is said to be a *quasiaffine transform* of an operator $T \in \mathscr{L}(\mathscr{K})$ if there is a quasiaffinity $X \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ such that $XS =$

TX. Also, we say that $S \in \mathcal{L}(\mathcal{H})$ and $T \in \mathcal{L}(\mathcal{K})$ are *quasisimilar* if there are quasiaffinities $X \in \mathcal{L}(\mathcal{H},\mathcal{K})$ and $Y \in \mathcal{L}(\mathcal{K},\mathcal{H})$ such that $XS = TX$ and $SY = YT$.

REMARK. Let $T \in \mathcal{L}(\mathcal{H})$ be a class A operator. If (i) T^{*n} is a class A operator for some positive integer n or (ii) T is quasisimilar to a normal operator and $\ker(T) \subseteq \ker(T^*)$, then

$$\sigma_T(x) = \sigma_B(x) \cup \sigma_C(x) \cup [-\sigma_C(x)]$$

for every vector $x \in \mathcal{H}$, and

$$H_T(F) = H_B(F) \oplus H_C(F) \oplus H_C(-F)$$

for any closed subset F of $\mathbb{C}$, where B and C are normal and $-F := \{-\lambda : \lambda \in F\}$.

Indeed, if T^{*n} belongs to class A for some positive integer n, then T^n is a class A operator from [20], and so T^n is normal by [20]. Since T is paranormal, it is normal from [3]. If T is quasisimilar to a normal operator, then T^2 is also quasisimilar to a normal operator. Since T^2 is w-hyponormal from [20] and $\ker(T^2) = \ker(T) \subseteq \ker(T^*) \subseteq \ker(T^{2^*})$ by [21], T^2 is normal by [27]. Hence T is normal from [3]. Since T is normal in both the cases (i) and (ii), T^2 is also normal. Hence from [30] we get that

$$T = B \oplus \begin{pmatrix} C & D \\ 0 & -C \end{pmatrix}$$

where B and C are normal and D is an operator commuting with C. Since T is normal, D must be the zero operator, i.e., $T = B \oplus C \oplus (-C)$. Then we obtain from [25] that

$$\begin{cases} \sigma_T(x) = \sigma_B(x) \cup \sigma_C(x) \cup \sigma_{-C}(x) \\ H_T(F) = H_B(F) \oplus H_C(F) \oplus H_{-C}(F). \end{cases}$$

Since the equalities $\sigma_{-C}(x) = -\sigma_C(x)$ and $H_{-C}(F) = H_C(-F)$ hold from [25], we get our results.

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *quasitriangular* if there is a sequence $\{P_n\}$ of finite rank orthogonal projections on $\mathcal{H}$ converging strongly to the identity operator I on $\mathcal{H}$ such that $\lim_{n\to\infty} \|(I - P_n)TP_n\| = 0$. When both T and T^* are quasitriangular, we say that T is *biquasitriangular*. From the following theorem, we provide a sufficient condition for a class A operator to have a nontrivial invariant subspace.

THEOREM 3.6. *Let $T \in \mathcal{L}(\mathcal{H})$ be a class A operator that is not a scalar multiple of the identity operator. If $\sigma(T)$ is not the closure of the union of all singleton components of $\sigma(T)$, then T has a nontrivial invariant subspace.*

Proof. Suppose that T has no nontrivial invariant subspaces. Provided that there exists $\lambda \in \sigma(T) \setminus \sigma_e(T)$, then $T - \lambda$ is Fredholm but not invertible. Since $\ker(T - \lambda) = \ker(T^* - \overline{\lambda}) = \{0\}$, we get that $\operatorname{ran}(T - \lambda) = \ker(T^* - \overline{\lambda})^{\perp} = \mathcal{H}$, which is a contradiction. So, it holds that $\sigma(T) = \sigma_e(T)$. Since $\sigma_p(T^*) = \emptyset$, it is easy to show that T^* has the single-valued extension property. In addition, T has the single-valued extension property from [22]. Hence

$$\sigma(T) = \sigma_e(T) = \sigma_w(T) = \sigma_{sF}(T)$$

by [1, Corollary 3.53]. Then it is clear that $\text{ind}(T-\lambda)=0$ for all $\lambda\in\rho_{sF}(T)$. Furthermore, we obtain from [24, Theorem 2.3.21] that T is biquasitriangular. Since σ is continuous at T by [12] and T is biquasitriangular, it follows from [5, Thoerem 14.15] and [19, Theorem 6.15] that $\sigma(T)=\overline{\Gamma_0(T)}$ where $\Gamma_0(T)$ is the union of all singleton components of

$$\sigma(T)\cup\left[\text{Int}(\overline{\rho^0_{sF}(T)})\setminus\rho^0_{sF}(T)\right]$$

and $\rho^0_{sF}(T)$ stands for the set of all $\lambda\in\mathbb{C}$ such that $T-\lambda$ is semi-Fredholm with $\text{ind}(T-\lambda)=0$. Since $\rho^0_{sF}(T)=\mathbb{C}\setminus\sigma_w(T)=\rho(T)$, we have that $\Gamma_0(T)$ is the union of all singleton components of the set $\sigma(T)\cup\left[\text{Int}(\overline{\rho(T)})\setminus\rho(T)\right]$. Since

$$\sigma(T)\cup\left[\text{Int}(\overline{\rho(T)})\setminus\rho(T)\right]\subseteq\sigma(T)\cup(\partial\rho(T))=\sigma(T)\cup\partial\sigma(T)=\sigma(T),$$

$\Gamma_0(T)$ is the union of all singleton components of $\sigma(T)$. $\square$

COROLLARY 3.7. *Let $T\in\mathcal{L}(\mathcal{H})$ be a class A operator whose spectrum is a line segment or a circle. Then T has a nontrivial invariant subspace.*

Proof. Since a line segment or a circle is a connected set that is not singleton, $\sigma(T)$ is not the closure of the union of all singleton components of $\sigma(T)$. Hence T has a nontrivial invariant subspace from Theorem 3.6. $\square$

Next we have the following results from some applications of [32].

THEOREM 3.8. *Let $T\in\mathcal{L}(\mathcal{H})$ be a class A operator with $T\neq\lambda I$ for any $\lambda\in\mathbb{C}$. If there is a nonzero vector $x\in\mathcal{H}$ such that* (i) $\sigma_T(x)\subsetneqq\sigma(T)$ *or* (ii) $\|T^nx\|\leqslant Cr^n$ *for all positive integers n and some constants $C>0$ and $0<r<\|T\|$, then T has a nontrivial hyperinvariant subspace.*

Proof. (i) If there exists a nonzero vector $x\in\mathcal{H}$ such that $\sigma_T(x)\subsetneqq\sigma(T)$, set

$$\mathcal{M}:=H_T(\sigma_T(x)),\ \text{i.e.,}\ \mathcal{M}=\{y\in\mathcal{H}:\sigma_T(y)\subseteq\sigma_T(x)\}.$$

Since T has Dunford's property (C) by [22], $\mathcal{M}$ is a T-hyperinvariant subspace from [9] or [25]. Since $x\in\mathcal{M}$, we get that $\mathcal{M}\neq\{0\}$. Suppose that $\mathcal{M}=\mathcal{H}$. Since T has the single-valued extension property, it follows from [25] that

$$\sigma(T)=\bigcup\{\sigma_T(y):y\in\mathcal{H}\}\subseteq\sigma_T(x)\subsetneqq\sigma(T).$$

So we have a contradiction. Hence $\mathcal{M}$ is a nontrivial T-hyperinvariant subspace.

(ii) Assume that there is a nonzero vector $x\in\mathcal{H}$ such that

$$\|T^nx\|\leqslant Cr^n$$

for all positive integers n and some constants $C>0$ and $0<r<\|T\|$. Put $f(z):=-\sum_{n=0}^{\infty}z^{-(n+1)}T^nx$, which is analytic for $|z|>r$; in fact, if we set $\omega=z^{-1}$ for $|z|>r$, then $f(\omega)=-\sum_{n=0}^{\infty}\omega^{n+1}T^nx$ for $0<|\omega|<\frac{1}{r}$. Since the hypothesis implies that

$\limsup_{n\to\infty}\|T^n x\|^{\frac{1}{n}} \leqslant r$, the radius of convergence for the power series $\sum_{n=0}^{\infty}\omega^{n+1}T^n x$ is at least $\frac{1}{r}$. Setting $f(0) := 0$, we get that $f(\omega)$ is analytic for $|\omega| < \frac{1}{r}$, i.e., $f(z)$ is analytic for $|z| > r$. Since

$$(T-z)f(z) = -\sum_{n=0}^{\infty} z^{-(n+1)}T^{n+1}x + \sum_{n=0}^{\infty} z^{-n}T^n x = x$$

for all $z \in \mathbb{C}$ with $|z| > r$, we have $\rho_T(x) \supseteq \{z \in \mathbb{C} : |z| > r\}$, i.e.,

$$\sigma_T(x) \subseteq \{z \in \mathbb{C} : |z| \leqslant r\}.$$

Since $r < \|T\|$ and T is normaloid by [21], it holds that $\sigma_T(x) \subsetneqq \sigma(T)$. By (i), we conclude that T has a nontrivial hyperinvariant subspace. □

COROLLARY 3.9. *Let $T \in \mathcal{L}(\mathcal{H})$ be a class A operator. If T has a nonzero invariant subspace $\mathcal{M}$ such that $\sigma(T|_{\mathcal{M}}) \subsetneqq \sigma(T)$, then T has a nontrivial hyperinvariant subspace.*

Proof. For any nonzero $x \in \mathcal{M}$, we have

$$\sigma_T(x) \subseteq \sigma_{T|_{\mathcal{M}}}(x) \subseteq \sigma(T|_{\mathcal{M}}) \subsetneqq \sigma(T).$$

Hence T has a nontrivial hyperinvariant subspace by Theorem 3.8. □

4. Weak forms of supercyclicity

In this section we consider weak forms of supercyclicity for class A operators and provide several properties of such operators. The results in Theorem 4.2 are generalizations of F. Bayart and E. Matheron's results in [6]. For this we need the following lemma.

LEMMA 4.1. *Any p-hyponormal operator for $0 < p < 1$ is not N-supercyclic for any positive integer N.*

Proof. Suppose that $T \in \mathcal{L}(\mathcal{H})$ is p-hyponormal and N-supercyclic for some positive integer N. Let q be any positive integer and let $w_1, w_1, \cdots, w_q$ be pairwise distinct complex numbers on the unit circle that are of the form $w_j = e^{i2\pi r_j}$ for some rational numbers r_j. We set $S = w_1 T \bigoplus \cdots \bigoplus w_q T$. Then S is N-multicyclic by [6]. Since T is p-hyponormal, S is also p-hyponormal. Therefore, by [33] we have

$$\operatorname{tr}\left((|S|^{2p} - |S^*|^{2p})^{\frac{1}{p}}\right) \leqslant \frac{N}{\pi}\mu(\sigma(S))$$

where μ denotes the planar Lebesgue measure. We note that $\operatorname{tr}\left((|S|^{2p} - |S^*|^{2p})^{\frac{1}{p}}\right) = q\operatorname{tr}\left((|T|^{2p} - |T^*|^{2p})^{\frac{1}{p}}\right)$. Moreover since $\sigma(S) = \bigcup_{j=1}^{q}\sigma(w_j T) \subseteq \{z \in \mathbb{C} : |z| \leqslant \|T\|\}$, we get that

$$\begin{aligned}\operatorname{tr}\big((|T|^{2p}-|T^*|^{2p})^{\frac{1}{p}}\big) &= \Big(\frac{1}{q}\Big)\operatorname{tr}\big((|S|^{2p}-|S^*|^{2p})^{\frac{1}{p}}\big)\\ &\leqslant \Big(\frac{1}{q}\Big)\frac{N}{\pi}\mu(\sigma(S)) \leqslant \frac{N\|T\|^2}{q}\end{aligned}$$

for all positive integers q, and so $\operatorname{tr}\big((|T|^{2p}-|T^*|^{2p})^{\frac{1}{p}}\big)\leqslant 0$. Since T is p-hyponormal, $(|T|^{2p}-|T^*|^{2p})^{\frac{1}{p}}$ is a positive operator and hence $\operatorname{tr}\big((|T|^{2p}-|T^*|^{2p})^{\frac{1}{p}}\big)\geqslant 0$. So we have $\operatorname{tr}\big((|T|^{2p}-|T^*|^{2p})^{\frac{1}{p}}\big)=0$. Thus $(|T|^{2p}-|T^*|^{2p})^{\frac{1}{p}}=0$ and this implies that $|T|^2=|T^*|^2$, i.e., T is normal. But there is no normal operator that is N-supercyclic, and so we have a contradiction. Hence T is not N-supercyclic. $\square$

Using Lemma 4.1, we obtain the following theorem.

THEOREM 4.2. *If $T\in\mathscr{L}(\mathscr{H})$ is a class A operator with* $\ker(T)=\{0\}$, *then the following statements hold.*

(i) *If T is weakly supercyclic, then it is a scalar multiple of a unitary operator.*

(ii) *T is not N-supercyclic for any positive integer N.*

Proof. (i) Assume that T is weakly supercyclic. Then there is $x\in\mathscr{H}$ such that $\mathbb{C}O(x,T)$ is weakly dense in $\mathscr{H}$. If $S=T^2$, then S is w-hyponormal by [20]. Moreover, S is weakly supercyclic by an application of [4]. Since $\widetilde{S}|S|^{\frac{1}{2}}=|S|^{\frac{1}{2}}S$, it holds that

$$\mathbb{C}O(|S|^{\frac{1}{2}}x,\widetilde{S})=|S|^{\frac{1}{2}}\mathbb{C}O(x,S).$$

Similarly, we obtain that

$$\mathbb{C}O(|\widetilde{S}|^{\frac{1}{2}}|S|^{\frac{1}{2}}x,\widetilde{S}^{(2)})=|\widetilde{S}|^{\frac{1}{2}}\mathbb{C}O(|S|^{\frac{1}{2}}x,\widetilde{S})=|\widetilde{S}|^{\frac{1}{2}}|S|^{\frac{1}{2}}\mathbb{C}O(x,S).$$

Since $\mathbb{C}O(x,S)$ is weakly dense in $\mathscr{H}$ and $\ker(|\widetilde{S}|^{\frac{1}{2}})=\ker(|S|^{\frac{1}{2}})=\ker(S)=\{0\}$, it follows that $|\widetilde{S}|^{\frac{1}{2}}|S|^{\frac{1}{2}}\mathbb{C}O(x,S)$ is also weakly dense in $\mathscr{H}$. This means that $\widetilde{S}^{(2)}$ is weakly supercyclic. Since $\widetilde{S}^{(2)}$ is hyponormal from [2] and [20], it is a scalar multiple of a unitary operator by [6]. Since $\ker(S)=\{0\}$, $\widetilde{S}^{(2)}=\widetilde{S}=S$ by applying [2]. In particular, $S=T^2$ is normal. Hence T is normal from [3]. Therefore T is a scalar multiple of a unitary operator by [6].

(ii) If T is N-supercyclic, then there exists an N-dimensional subspace $\mathscr{M}$ such that $\bigcup_{n=0}^{\infty}T^n(\mathscr{M})$ is norm dense in $\mathscr{H}$. Let $\mathscr{N}=\operatorname{span}\{\mathscr{M},T(\mathscr{M})\}$. Then $\mathscr{N}$ is a subspace of $\mathscr{H}$ with $\dim\mathscr{N}\leqslant 2N$. We claim that T^2 is K-supercyclic where $K=\dim\mathscr{N}$. Indeed, for any $x\in\mathscr{M}$, x and Tx belong to $\mathscr{N}$. So we obtain that $(T^2)^n x$, $(T^2)^n(Tx)\in(T^2)^n(\mathscr{N})$ for all $n\geqslant 0$. Therefore, it holds that

$$\bigcup_{n=0}^{\infty}T^n(\mathscr{M})=\bigcup_{n=0}^{\infty}\{(T^2)^n(\mathscr{M})\cup(T^2)^n(T(\mathscr{M}))\}\subseteq\bigcup_{n=0}^{\infty}(T^2)^n(\mathscr{N})$$

and the inclusion implies that $\bigcup_{n=0}^{\infty}(T^2)^n(\mathscr{N})$ is norm dense in $\mathscr{H}$. Hence T^2 is K-supercyclic with K-dimensional subspace $\mathscr{N}$. Since T^2 is one-to-one, $|T^2|^{\frac{1}{2}}$ is one-to-one and hence $|T^2|^{\frac{1}{2}}(\mathscr{N})$ is a K-dimensional subspace of $\mathscr{H}$. Furthermore, $\bigcup_{n=0}^{\infty}(\widetilde{T^2})^n(|T^2|^{\frac{1}{2}}(\mathscr{N})) = |T^2|^{\frac{1}{2}}(\bigcup_{n=0}^{\infty}(T^2)^n(\mathscr{N}))$ is norm dense in $\mathscr{H}$. So $\widetilde{T^2}$ is K-supercyclic with K-dimensional subspace $|T^2|^{\frac{1}{2}}(\mathscr{N})$. But this contradicts to Lemma 4.1, since $\widetilde{T^2}$ is semi-hyponormal. Therefore T is not N-supercyclic. $\square$

COROLLARY 4.3. *Let $T \in \mathscr{L}(H)$ be a p-hyponormal operator for $0 < p < 1$. If T is weakly supercyclic, then T is a scalar multiple of a unitary operator.*

Proof. Since T is a p-hyponormal operator for $0 < p < 1$, it is a class A operator. If T is weakly supercyclic, we know that $\sigma_p(T^*) = \emptyset$ or $\sigma_p(T^*) = \{\lambda\}$ for some $\lambda \neq 0$ by a result of [29]. So T^* is one-to-one. Since T is p-hyponormal, we have $\ker(T) \subseteq \ker(T^*) = \{0\}$ and hence $\ker(T) = \{0\}$. Thus by Theorem 4.2, T is a scalar multiple of a unitary operator. $\square$

From applications of Theorem 4.2, we obtain hypertransitivity for the product of a class A operator and an algebraic operator which are commuting.

COROLLARY 4.4. If $R = TA$ is an operator in $\mathscr{L}(\mathscr{H})$ where T is a class A operator, A is algebraic, and $TA = AT$, then R is nonhypertransitive.

Proof. If $\ker(R) \neq \{0\}$, then R is clearly nonhypertransitive. Suppose that $\ker(R) = \{0\}$. Let $x \in \mathscr{H}$ be any nonzero vector. If A is algebraic of order k, then A^n can be written as a linear combination of $\{I, A, A^2, \cdots, A^{k-1}\}$ for each positive integer n. Set $\mathscr{M} := \text{span}\{x, Ax, \cdots, A^{k-1}x\}$. Then $\mathscr{M}$ is a subspace of $\mathscr{H}$ with $\dim \mathscr{M} \leqslant k$, and we obtain that

$$\mathbb{C}O(x,R) = \{\lambda T^n A^n x : \lambda \in \mathbb{C}, n = 0, 1, 2, \cdots\} \subseteq \bigcup_{n=0}^{\infty} T^n \mathscr{M}.$$

Since T is a class A operator with $\ker(T) = \{0\}$, Theorem 4.2 ensures that T is not N-supercyclic for any positive integer N. Hence we get that $\bigcup_{n=0}^{\infty} T^n \mathscr{M}$ is not norm dense in $\mathscr{H}$, which implies that $\mathbb{C}O(x,R)$ is not norm dense in $\mathscr{H}$, that is, R is not supercyclic. Thus R is nonhypertransitive. $\square$

COROLLARY 4.5. *Let $T \in \mathscr{L}(\mathscr{H})$ be a class A operator with* $\ker(T) = \{0\}$. *Then given $\rho > 0$, there exists $\varepsilon > 0$ such that neither* $\text{span}\{\ker(T-\lambda) : \rho < |\lambda| < \rho + \varepsilon\}$ *nor* $\text{span}\{\ker(T-\lambda) : \rho - \varepsilon < |\lambda| < \rho\}$ *is dense in $\mathscr{H}$.*

Proof. The proof follows from Theorem 4.2 and [15]. $\square$

We denote the direct sum of n copies of $T \in \mathscr{L}(\mathscr{H})$ by $T^{(n)}$, where n is a cardinal number with $1 \leqslant n \leqslant \aleph_0$. For two operators T and S in $\mathscr{L}(\mathscr{H})$, we say that T is *ampliation quasisimilar* to S if there exist cardinal numbers n_1 and n_2 with $1 \leqslant n_1, n_2 \leqslant \aleph_0$ such that $T^{(n_1)}$ is quasisimilar to $S^{(n_2)}$.

COROLLARY 4.6. *Let $T \in \mathscr{L}(\mathscr{H})$ be a class A operator. If T is weakly or N-supercyclic for some positive integer N, then T has a nontrivial hyperinvariant subspace. Moreover, if $S \in \mathscr{L}(\mathscr{H})$ is ampliation quasisimilar to T, then S has a nontrivial hyperinvariant subspace.*

Proof. If T is weakly supercyclic and $\ker(T) \neq \{0\}$, then $\ker(T)$ is a nontrivial T-hyperinvariant subspace. Otherwise, T is normal from Theorem 4.2. Therefore T has a nontrivial hyperinvariant subspace by [31]. If T is N-supercyclic, then $\ker(T) \neq \{0\}$ by Theorem 4.2. So the result follows.

Suppose that $S^{(n_1)}$ is quasisimilar to $T^{(n_2)}$ for some cardinal numbers n_1 and n_2 with $1 \leqslant n_1, n_2 \leqslant \aleph_0$. Since T has a nontrivial hyperinvariant subspace, we obtain from [16] that there exists a nontrivial hyperinvariant subspace for $T^{(n_2)}$. Then $S^{(n_1)}$ has a nontrivial hyperinvariant subspace from [31], and so does S by [16]. □

Acknowledgement. The authors would like to express their cordial thanks to the referee for his kind suggestions and valuable comments.

REFERENCES

[1] P. AIENA, *Fredholm and local spectral theory with applications to multipliers*, Kluwer Academic Publishers, 2004.

[2] A. ALUTHGE AND D. WANG, *w-hyponormal operators*, Int. Eq. Op. Th. **36** (2000), 1–10.

[3] T. ANDO, *Operators with a norm condition*, Acta Sci. Math. **33** (1972), 169–178.

[4] S. ANSARI, *Hypercyclic and cyclic vectors*, J. Func. Analysis **128** (1995), 374–383.

[5] C. APOSTOL, L. A. FIALKOW, D. A. HERRERO, AND D. VOICULESCU, *Approximation of Hilbert space operators*, Vol. II, Research Notes in Mathematics **102**, Pitman, Boston, 1984.

[6] F. BAYART AND E. MATHERON, *Hyponormal opertors, weighted shifts and weak forms of supercyclicity*, Proc. Edinb. Math. Soc. **49** (2006), 1–15.

[7] P. S. BOURDON, *Orbits of hyponormal operators*, Mich. Math. J. **44** (1997), 345–353.

[8] M. CHO AND J. I. LEE, *p-Hyponormality is not translation-invariant*, Proc. Amer. Math. Soc. **131** (2003), 3109–3111.

[9] I. COLOJOARA AND C. FOIAS, *Theory of generalized spectral operators*, Gordon and Breach, New York, 1968.

[10] S. V. DJORDJEVIĆ AND B. P. DUGGAL, *Weyl's theorems and continuity of spectra in the class of p-hyponormal operators*, Studia Math. **143**(2000), 23–32.

[11] S. V. DJORDJEVIĆ AND D. S. DJORDJEVIĆ, *Weyl's theorem: continuity of the spectrum and quasi-hyponormal operators*, Acta Sci. Math. (Szeged) **64** (1998), 259–269.

[12] B. P. DUGGAL, *Spectral continuity of k-th roots of hyponormal operators*, Oper. Matrices **1**(2007), 209–215.

[13] K. DYKEMA AND H. SCHULTZ, *Brown measure and iterates of the Aluthge transform for some operators arising from measurable actions*, Trans. Amer. Math. Soc. **361** (2009), 6583–6593.

[14] N. S. FELDMAN, *N-supercyclic operators*, Studia Math. **151** (2002), 141–159.

[15] N. S. FELDMAN, V. G. MILLER, AND T. L. MILLER, *Hypercyclic and supercyclic cohyponormal operators*, Acta Sci. Math. (Szeged) **68** (2002), 965–990.

[16] C. FOIAS AND C. PEARCY, *On hyperinvariant subspace problem*, J. Funct. Anal. **219** (2005), 134–142.

[17] T. FURUTA, *Invitation to linear operators*, Taylor and Francis, 2001.

[18] T. FURUTA, M. ITO, AND T. YAMAZAKI, *A subclass of paranormal operators including class of log-hyponormal and several related classes*, Scientiae Mathematicae, **1** (1998), 389–403.

[19] D. A. HERRERO, *Approximation of Hilbert space operators*, Vol. 1, Second edition, Pitman Research Notes in Math. Series **224**, 1989.

[20] M. ITO AND T. YAMAZAKI, *Relations between two inequalities* $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{p}{p+r}} \geqslant B^r$ *and* $A^p \geqslant (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$ *and their applications*, Int. Eq. Op. Th. **44** (2002), 442–450.
[21] I. H. JEON AND B. P. DUGGAL, *On operators with an absolute value conditions*, J. Kor. Math. Soc. **41** (2004), 617–627.
[22] S. JUNG, E. KO, AND M. LEE, *On class A operators*, Studia Math. **198** (2010), 249–260.
[23] I. B. JUNG, E. KO, AND C. PEARCY, *Aluthge transforms of operators*, Int. Eq. Op. Th. **37** (2000), 437–448.
[24] R. LANGE AND S. WANG, *New approaches in spectral decomposition*, Contemp. Math. **128**, Amer. Math. Soc., 1992.
[25] K. B. LAURSEN AND M. M. NEUMANN, *Introduction to local spectral theory*, London Math. Soc. Monograghs New Series. Claredon Press, Oxford, 2000.
[26] J. D. NEWBURGH, *The variation of spectra*, Duke Math. J. **8** (1951), 165–176.
[27] M. O. OTIENO, *On quasi-similarity and w-hyponormal operators*, Opuscula Math. **27**(2007), 73–81.
[28] S. M. PATEL, M. CHŌ, K. TANAHASHI, AND A. UCHIYAMA, *Putnam's inequality for class A operators and an operator transform by Chō and Yamazaki*, Scientiae Mathematicae Japonicae **67** (2008), 393–402.
[29] A. PERIS, *Multi-hypercyclic operators are hypercyclic*, Math. Z. **236**(2001), 779–786.
[30] H. RADJAVI AND P. ROSENTHAL, *On roots of normal operators*, J. Math. Anal. Appl. **34**(1971), 653–664.
[31] H. RADJAVI AND P. ROSENTHAL, *Invariant subspaces*, Springer-Verlag, 1973.
[32] J. G. STAMPFLI, *A local spectral theory for operators. V: spectral subspaces for hyponormal operators*, Trans. Amer. Math. Soc. **217** (1976), 285–296.
[33] A. UCHIYAMA, *Berger-Shaw's theorem for p-hyponormal operators*, Int. Eq. Op. Th. **33** (1999), 221–230.

(Received August 13, 2012)

Sungeun Jung
Institute of Mathematical Sciences
Ewha Womans University
Seoul, 120–750
Korea
e-mail: ssung105@ewhain.net

Insook Kim
Division of General Mathematics
University of Seoul
Seoul, 130-743
Korea
e-mail: ikim@uos.ac.kr; ikim99@hanmail.net

Eungil Ko
Department of Mathematics
Ewha Womans University
Seoul, 120-750
Korea
e-mail: eiko@ewha.ac.kr