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LYAPUNOV–TYPE INEQUALITIES FOR TWO CLASSES OF DIRICHLET QUASILINEAR SYSTEMS

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Abstract. In this paper, we establish several new Lyapunov-type inequalities for two classes of Dirichlet quasilinear systems, which almost generalize and extend all related existing results in the literature. As an application, we also obtain sharp lower bounds for the eigenvalues of corresponding systems.

1. Introduction

In this paper, we state and prove new generalized Lyapunov-type inequalities for the following systems

$$\left.\begin{aligned} -\left((r_1(x)\phi_{p_1}(u_1')\right)' &= f_1(x)\phi_{\alpha_1}(u_1)\,|u_2|^{\alpha_2} \\ -\left((r_2(x)\phi_{p_2}(u_2')\right)' &= f_2(x)\phi_{\beta_2}(u_2)\,|u_1|^{\beta_1} \end{aligned}\right\}, \tag{1.1}$$

where $\phi_\gamma(u) = |u|^{\gamma-2}u,\ \gamma > 1,\ r_k, f_k \in C([a,b],\mathbb{R}),\ r_k(x) > 0$ for $k = 1,2$ and $x \in \mathbb{R}$, $(u_1(x), u_2(x))$ is a real nontrivial solution of the system (1.1) such that

$$u_k(a) = 0 = u_k(b) \tag{1.2}$$

for $k = 1,2,\ a,b \in \mathbb{R}$ with $a < b$ are consecutive zeros, u_k for $k = 1,2$ are not identically zero on $[a,b]$, $1 < p_k < \infty$ and $\alpha_2 \geqslant 0,\ \beta_1 \geqslant 0$ satisfy

$$\frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} = 1 \quad \text{and} \quad \frac{\beta_1}{p_1} + \frac{\beta_2}{p_2} = 1. \tag{1.3}$$

We also consider the following system

$$-\left(r_k(x)\,\phi_{p_k}(u_k')\right)' = f_k(x)\phi_{\alpha_k}(u_k)\prod_{i=1(\neq k)}^{n}|u_i|^{\alpha_i}, \tag{1.4}$$

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where $n \in \mathbb{N}$, $\phi_\gamma(u) = |u|^{\gamma-2} u$, $\gamma > 1$, $r_k, f_k \in C([a,b],\mathbb{R})$, $r_k(x) > 0$ for $k = 1,2,...,n$ and $x \in \mathbb{R}$, $(u_1(x), u_2(x), \ ..., u_n(x))$ is a real nontrivial solution of the system (1.4) such that

$$u_k(a) = 0 = u_k(b) \tag{1.5}$$

for $k = 1,2,...,n$, $a,b \in \mathbb{R}$ with $a < b$ are consecutive zeros, u_k for $k = 1,2,...,n$ are not identically zero on $[a,b]$, $1 < p_k < \infty$ and the nonnegative parameters α_k satisfy

$$\sum_{k=1}^{n} \frac{\alpha_k}{p_k} = 1. \tag{1.6}$$

As an application, we have also investigated in the lower bounds on the eigenvalues of the following problem. As usual, it is easier to find upper bounds for eigenvalues than lower bounds. In fact, they can be obtained by using elementary inequalities. Finding the estimated lower bounds is based on giving a suitable Lyapunov inequality for the corresponding systems.

Let λ_k for $k = 1,2,...,n$ be generalized eigenvalues of the problem (1.4)–(1.6) and $r(x)$ be a positive function for $x \in \mathbb{R}$. Then, the problem (1.4)–(1.6) with $f_k(x) = \lambda_k \alpha_k r(x) > 0$ for $k = 1,2,...,n$ and $x \in \mathbb{R}$ reduces to the following problem

$$-\left(r_k(x)\,\phi_{p_k}\left(u_k'\right)\right)' = \lambda_k \alpha_k r(x) \phi_{\alpha_k}(u_k) \prod_{i=1(\neq k)}^{n} |u_i|^{\alpha_i}, \tag{1.7}$$

$$u_k(a) = 0 = u_k(b), \ k = 1,2,...,n, \tag{1.8}$$

$$\sum_{k=1}^{n} \frac{\alpha_k}{p_k} = 1. \tag{1.9}$$

Before we proceed with the description of the main problem, we discuss a few hints concerning the literature on the results obtained in the problem (1.4)–(1.6) and the special cases of problem (1.7)–(1.9).

Firstly, we give the following results for second order differential equations.

If $n = 1$, $p_1 = 2$ and $r_1(x) = 1$, then the problem (1.4)–(1.6) reduces to the following problem

$$-u_1'' = f_1(x)\, u_1, \tag{1.10}$$

$$u_1(a) = 0 = u_1(b). \tag{1.11}$$

Lyapunov [2] proved the following remarkable result:

THEOREM A. *If $f_1 \in C([a,b],[0,\infty))$ and u_1 is a nontrivial solution on $[a,b]$ for the problem (1.10)–(1.11), then the so-called Lyapunov inequality*

$$\int_a^b f_1(s)\, ds \geqslant \frac{4}{b-a} \tag{1.12}$$

holds.

The Lyapunov inequality and many of its generalizations have proven to be useful tools in oscillation theory, disconjugacy, eigenvalue problems, and numerous other applications for the theories of differential and difference equations. A thorough literature review of continuous and discrete Lyapunov inequalities and their applications can be found in the survey paper by Cheng [3] and the references quoted therein. For some of the most recent works on Lyapunov-type inequalities, the reader is referred to [1–42].

Since then, there have been several results to generalize the above linear equation in many directions. Before stating many efforts, it is worth to the mention following works.

By using Green's function, Hartman [17] obtained the generalized inequality as follows:

THEOREM B. *If* $f_1 \in C([a,b],[0,\infty))$ *and* u_1 *is a nontrivial solution on* $[a,b]$ *for the problem* (1.10)–(1.11)*, then the inequality*

$$\int_a^b f_1(s)\frac{(s-a)(b-s)}{b-a}ds \geqslant 1 \tag{1.13}$$

holds.

We know that the inequality

$$4AB \leqslant (A+B)^2 \tag{1.14}$$

holds where A and B are positive numbers. If we take $A = x-a > 0$ and $B = b-x > 0$ for $x \in (a,b)$ in the function $M(x) := (x-a)(b-x)$, then we obtain the following inequality

$$(x-a)(b-x) \leqslant \left(\frac{b-a}{2}\right)^2, \tag{1.15}$$

i.e.

$$\max_{a<x<b} M(x) = M\left(\frac{a+b}{2}\right) = \left(\frac{b-a}{2}\right)^2. \tag{1.16}$$

Thus, condition (1.13) is a generalization of condition (1.12).

If $n = 1$ and $r_1(x) = 1$, then the problem (1.4)–(1.6) reduces to the following problem

$$-(\phi_{p_1}(u_1'))' = f_1(x)\phi_{p_1}(u_1) \tag{1.17}$$

$$u_1(a) = 0 = u_1(b). \tag{1.18}$$

Pinasco [28] extended the Lyapunov inequality from the linear equation to the half-linear equation as follows:

THEOREM C. *If* $f_1 \in C([a,b],\mathbb{R})$ *be a bounded positive function and* u_1 *is a nontrivial solution on* $[a,b]$ *for the problem* (1.17)–(1.18)*, then the inequality*

$$\int_a^b f_1(s)\,ds \geqslant 2\left(\frac{2}{b-a}\right)^{p_1-1} \tag{1.19}$$

holds.

Sim and Lee [31] obtained the generalized inequality (1.19) as follows:

THEOREM D. *If* $f_1 \in C([a,b],[0,\infty))$ *and* u_1 *is a nontrivial solution on* $[a,b]$ *for the problem* (1.17)–(1.18)*, then the inequality*

$$\int_a^b f_1(s)\left[\frac{2(s-a)(b-s)}{b-a}\right]^{p_1-1} ds \geqslant 2 \tag{1.20}$$

holds.

Note that when $p_1 = 2$ in the problem (1.17)–(1.18), the condition (1.20) coincides with the condition (1.13). But Hartman's argument does not work here, due to the lack of Green's function for p_1-Laplacian. It is easy to see that, by using the inequality (1.14), condition (1.20) is a generalization of condition (1.19).

It is clear that the problem (1.1)–(1.3) with the condition

$$(\alpha_2 = 0 \text{ and } \alpha_1 = p_1) \text{ or } (\beta_1 = 0 \text{ and } \beta_2 = p_2), \tag{1.21}$$

or the problem (1.4)–(1.6) for $n = 1$ reduces to the following type problem

$$-\left(r_1(x)\phi_{p_1}\left(u_1'\right)\right)' = f_1(x)\phi_{p_1}(u_1) \tag{1.22}$$

$$u_1(a) = 0 = u_1(b). \tag{1.23}$$

Moreover, when $\alpha_k = p_k$ for $k = 1,2,\ldots,n$, and for $i \neq k$, $\alpha_i = 0$ for $i = 1, 2,\ldots,n$, we obtain a single equation from system (1.4).

Now, throughout the paper for the sake of brevity, we denote

$$f_k^+(x) = \max\{0, f_k(x)\} \text{ is the nonnegative part of } f_k(x), \tag{1.24}$$

$$D_k(x) = \frac{[\xi_k(x)\eta_k(x)]^{p_k-1}}{\xi_k^{p_k-1}(x) + \eta_k^{p_k-1}(x)}, \quad E_k(x) = 2^{p_k-2}\left(\frac{\xi_k(x)\eta_k(x)}{\xi_k(x)+\eta_k(x)}\right)^{p_k-1} \tag{1.25}$$

and

$$F_k = 2^{-p_k}\left(\xi_k(x) + \eta_k(x)\right)^{p_k-1} = 2^{-p_k}\left(\int_a^b r_k^{1/(1-p_k)}(s)ds\right)^{p_k-1}, \tag{1.26}$$

where

$$\xi_k(x) = \int_a^x r_k^{1/(1-p_k)}(s)ds \quad \text{and} \quad \eta_k(x) = \int_x^b r_k^{1/(1-p_k)}(s)ds \tag{1.27}$$

for $k = 1,2,\ldots,n$.

Wang [42] obtained the following inequality:

THEOREM E. *If* $f_1 \in C([a,b],\mathbb{R})$ *and* u_1 *is a nontrivial solution on* $[a,b]$ *for the problem* (1.22)–(1.23)*, then the inequality*

$$\int_a^b f_1^+(s)E_1(s)\,ds > 2^{p_1-2}C(p_1) \tag{1.28}$$

holds, where

$$C(p_1) = \begin{cases} 2^{p_1-2}, & 1 < p_1 < 2 \\ 1, & p_1 \geqslant 2. \end{cases} \tag{1.29}$$

Secondly, we give the following results for systems (1.1) and (1.4).

Napoli and Pinasco [21] were interested in the problem of finding Lyapunov-type inequality for the system (1.4) with $n=2$ and $r_k(x)=1$ for $k=1,2$ and obtained a generalization of the inequalities (1.12) and (1.19) as follows:

THEOREM F. *If $f_k \in C([a,b],[0,\infty))$ for $k=1,2$ and $(u_1(x),u_2(x))$ is a nontrivial solution on $[a,b]$ for the problem (1.4)–(1.6) with $n=2$ and $r_k(x)=1$ for $k=1,2$, then the inequality*

$$\prod_{k=1}^{2}\left(\int_a^b f_k(s)ds\right)^{\alpha_k/p_k} \geqslant 2^{\alpha_1+\alpha_2}(b-a)^{1-(\alpha_1+\alpha_2)} \tag{1.30}$$

holds.

By using the inequality (1.30), Napoli and Pinasco [21] have also obtained the lower bounds on the eigenvalues of system (1.7) with $n=2$ and $r_k(x)=1$ for $k=1,2$ as follows:

THEOREM G. *There exist a function $k_1(\lambda_1)$ such that $\lambda_2 \geqslant k_1(\lambda_1)$ for every generalized eigenvalue (λ_1,λ_2) of the problem (1.7)–(1.9) with $n=2$ and $r_k(x)=1$ for $k=1,2$, where*

$$k_1(\lambda_1)=\frac{1}{\alpha_2}\left\{2^{\alpha_1+\alpha_2}(b-a)^{1-(\alpha_1+\alpha_2)}\left[(\lambda_1\alpha_1)^{\alpha_1/p_1}\int_a^b r(s)\,ds\right]^{-1}\right\}^{p_2/\alpha_2}. \tag{1.31}$$

Çakmak and Tiryaki [7] obtained the following inequality for system (1.1).

THEOREM H. *If $f_k \in C([a,b],\mathbb{R})$ for $k=1,2$ and $(u_1(x),u_2(x))$ is a nontrivial solution on $[a,b]$ for the problem (1.1)–(1.3), then the inequality*

$$\begin{aligned}&\left(\int_a^b r_1^{1/(1-p_1)}(s)ds\right)^{\frac{\beta_1(p_1-1)}{p_1}}\left(\int_a^b r_2^{1/(1-p_2)}(s)ds\right)^{\frac{\alpha_2(p_2-1)}{p_2}}\\ &\times\left(\int_a^b f_1^+(s)\,ds\right)^{\frac{\beta_1}{p_1}}\left(\int_a^b f_2^+(s)\,ds\right)^{\frac{\alpha_2}{p_2}} > 2^{\alpha_2+\beta_1}\end{aligned} \tag{1.32}$$

holds.

Çakmak and Tiryaki [8] extended and generalized the system (1.4) with $n=2$ and $r_k(x)=1$ for $k=1,2$ to the system (1.4) with $r_k(x)=1$ for $k=1,2,\ldots,n$ as follows:

THEOREM I. *If $f_k \in C([a,b],\mathbb{R})$ for $k=1,2,\ldots,n$ and $(u_1(x),u_2(x),\ldots,u_n(x))$ is a nontrivial solution on $[a,b]$ for the problem (1.4)–(1.6) with $r_k(x)=1$ for $k=1,2,\ldots,n$, then the inequality*

$$\prod_{k=1}^{n}\left(\int_a^b f_k^+(s)ds\right)^{\alpha_k/p_k} \geqslant \prod_{k=1}^{n}\left[(c_k-a)^{1-p_k}+(b-c_k)^{1-p_k}\right]^{\alpha_k/p_k} \tag{1.33}$$

holds, where $|u_k(c_k)| = \max_{a<x<b} |u_k(x)|$ *for* $k = 1,2,...,n$.

By using the inequality (1.33), Çakmak and Tiryaki [8] have also obtained the lower bounds on the eigenvalues of system (1.7) with $r_k(x) = 1$ for $k = 1,2,...,n$ as follows:

THEOREM J. *There exist a function* $k_2(\lambda_1,\lambda_2,...,\lambda_{n-1})$ *such that* $\lambda_n \geqslant k_2(\lambda_1,\lambda_2,...,\lambda_{n-1})$ *for every generalized eigenvalue* $(\lambda_1,\lambda_2,...,\lambda_n)$ *of the problem* (1.7)–(1.9) *with* $r_k(x) = 1$ *for* $k = 1,2,...,n$, *where* $|u_k(c_k)| = \max_{a<x<b} |u_k(x)|$ *for* $k = 1,2,...,n$ *and*

$$k_2(\lambda_1,\lambda_2,...,\lambda_{n-1}) = \frac{1}{\alpha_n}\left\{\prod_{k=1}^{n}\left[(c_k-a)^{1-p_k}+(b-c_k)^{1-p_k}\right]^{\frac{\alpha_k}{p_k}}\left[\prod_{k=1}^{n-1}(\lambda_k\alpha_k)^{\frac{\alpha_k}{p_k}}\int_a^b r(s)ds\right]^{-1}\right\}^{\frac{p_n}{\alpha_n}}. \quad (1.34)$$

If we use inequality (1.14), by choosing $A = x-a > 0$ and $B = b-x > 0$ for $x \in (a,b)$ in the function $m(x) := \frac{1}{x-a} + \frac{1}{b-x}$, then we have the following inequality

$$\frac{1}{x-a} + \frac{1}{b-x} \geqslant \frac{4}{b-a} \quad (1.35)$$

for $x \in (a,b)$, i.e.

$$\min_{a<x<b} m(x) = m\left(\frac{a+b}{2}\right) = \frac{4}{b-a}.$$

If we take $n = 2$ in the problem (1.7)–(1.9), since the inequality (1.35) holds, we observe that Theorem J improves and generalizes Theorem G. Similarly, by using the inequality (1.35), Theorem H also generalizes Theorem F.

More recently, by using the ideas of Çakmak and Tiryaki [7, 8] with a slight modification, Tang and He [32] obtained the following inequalities for systems (1.1) and (1.4).

THEOREM K. *If* $f_k \in C([a,b],\mathbb{R})$ *for* $k = 1,2$ *and* $(u_1(x), u_2(x))$ *is a nontrivial solution on* $[a,b]$ *for the problem* (1.1)–(1.3), *then the inequality*

$$\left(\int_a^b f_1^+(s)D_1(s)ds\right)^{\frac{\alpha_1\beta_1}{p_1^2}}\left(\int_a^b f_2^+(s)D_1(s)ds\right)^{\frac{\beta_1\alpha_2}{p_1p_2}}$$

$$\times\left(\int_a^b f_1^+(s)D_2(s)ds\right)^{\frac{\beta_1\alpha_2}{p_1p_2}}\left(\int_a^b f_2^+(s)D_2(s)ds\right)^{\frac{\alpha_2\beta_2}{p_2^2}} > 1 \quad (1.36)$$

holds.

THEOREM L. *If* $f_k \in C([a,b],\mathbb{R})$ *for* $k = 1,2,...,n$ *and* $(u_1(x), u_2(x), ..., u_n(x))$ *is a nontrivial solution on* $[a,b]$ *for the problem* (1.4)–(1.6)*, then the inequality*

$$\prod_{k=1}^{n} \prod_{i=1}^{n} \left(\int_a^b f_i^+(s) D_k(s)\, ds \right)^{\frac{\alpha_k \alpha_i}{p_k p_i}} > 1 \tag{1.37}$$

holds.

Note that Theorem K with the condition (1.21) (or Theorem 2.2 given in Tang and He [32]) or Theorem L with $n = 1$ gives a better result than Theorem E given by Wang [42] in both cases, i.e. $1 < p_1 < 2$ and $p_1 > 2$, from the inequality (2.16) in [42].

By using similar technique to Theorem 1.4 of Napoli and Pinasco [21], Tang and He [32] also obtained the following result, which gives lower bounds for the n-th eigenvalue of λ_n, from Theorem L for system (1.7) with $r_k(x) = 1$ for $k = 1, 2, ..., n$.

THEOREM M. *There exists a function* $k_3(\lambda_1, \lambda_2, ..., \lambda_{n-1})$ *such that* $\lambda_n > k_3(\lambda_1, \lambda_2, ..., \lambda_{n-1})$ *for every generalized eigenvalue* $(\lambda_1, \lambda_2, ..., \lambda_n)$ *of the problem* (1.7)–(1.9) *with* $r_k(x) = 1$ *for* $k = 1, 2, ..., n$*, where*

$$k_3(\lambda_1, \lambda_2, ..., \lambda_{n-1}) = \frac{1}{\alpha_n} \left[\prod_{i=1}^{n-1} (\lambda_i \alpha_i)^{\alpha_i/p_i} \prod_{k=1}^{n} \left(\int_a^b \frac{[(s-a)(b-s)]^{p_k-1}}{(s-a)^{p_k-1} + (b-s)^{p_k-1}} r(s)\, ds \right)^{\frac{\alpha_k}{p_k}} \right]^{-\frac{p_n}{\alpha_n}}. \tag{1.38}$$

In this paper, our motivation comes from the recent papers of Çakmak and Tiryaki [7, 8], Sim and Lee [31] and Tang and He [32]. We state and prove several new generalized Lyapunov-type inequalities for the problems (1.1)–(1.3) and (1.4)–(1.6). In fact, we almost generalize and extend all related existing results in the literature.

In [7] and [8], the authors derive a Lyapunov-type inequality which relates both points a and b in $I = [x_0, \infty) \subset \mathbb{R}$ at which all components of the solution have consecutive zeros and any point in (a,b) where all components of the solution are maximized. But here, we derive Lyapunov-type inequalities for the problem (1.1)–(1.3) or (1.4)–(1.6), where all components of the solution have only consecutive zeros at the points $a, b \in \mathbb{R}$ with $a < b$ in I. Namely, we do not required that all of the components of the solution are maximized at any point in (a,b).

Since our attention is restricted to the Lyapunov-type inequalities for the quasilinear systems of differential equations, we shall assume the existence of the nontrivial solution of the system (1.1) or (1.4). For readers who contributed to the existence of the solution of these type systems, we refer to the paper by Afrouzi and Heidarkhani [43].

This paper is organized as follows. In sections 2 and 3, we shall present new Lyapunov-type inequalities for systems (1.1) and (1.4), respectively. In section 4, we present an application of this type inequality obtained for the problem (1.4)–(1.6).

Now, we present some inequalities on $D_k(x)$, $E_k(x)$ and F_k for $k = 1,2,...,n$ which are useful in the comparison of our main results. We know that since the function $h(x) = x^{p_k-1}$ is concave for $x > 0$ and $1 < p_k < 2$, Jensen's inequality $h\left(\frac{\omega+\nu}{2}\right) \geqslant$

$\frac{1}{2}[h(\omega)+h(v)]$ with $\omega=\dfrac{1}{\xi_k(x)}$ and $v=\dfrac{1}{\eta_k(x)}$ implies

$$D_k(x) \geqslant E_k(x) \tag{1.39}$$

for $1<p_k<2,\ k=1,2,...,n$. If $p_k>2$ for $k=1,2,...,n$, then the function $h(x)=x^{p_k-1}$ is convex for $x>0$. Thus, the inequality (1.39) is reversed, i.e.

$$D_k(x) \leqslant E_k(x) \tag{1.40}$$

for $p_k>2,\ k=1,2,...,n$. In addition, since the function $l(x)=x^{1-p_k}$ is convex for $x>0$ and $p_k>1$, Jensen's inequality $l\left(\frac{\omega+v}{2}\right)\leqslant\frac{1}{2}[l(\omega)+l(v)]$ with $\omega=\xi_k(x)$ and $v=\eta_k(x)$ implies

$$D_k(x) \leqslant F_k \tag{1.41}$$

for $k=1,2,...,n$. By using inequality (1.14) with $A=\xi_k(x)>0$ and $B=\eta_k(x)>0$ for $k=1,2,...,n$ in $E_k(x)$, we obtain the following inequality

$$E_k(x) \leqslant F_k \tag{1.42}$$

for $k=1,2,...,n$.

2. Lyapunov-type inequalities for system (1.1)

For system (1.1), one of the main results of this section is the following theorem.

THEOREM 2.1. *If $f_k\in C([a,b],\mathbb{R})$ for $k=1,2$ and $(u_1(x),u_2(x))$ is a nontrivial solution on $[a,b]$ for the problem (1.1)–(1.3), then the inequality*

$$1<\left(\int_a^b f_1^+(s)\,D_1^{\alpha_1/p_1}(s)\,D_2^{\alpha_2/p_2}(s)\,ds\right)^{\beta_1/p_1}\left(\int_a^b f_2^+(s)\,D_1^{\beta_1/p_1}(s)\,D_2^{\beta_2/p_2}(s)\,ds\right)^{\alpha_2/p_2} \tag{2.1}$$

holds.

Proof. Let $u_k(a)=0=u_k(b)$ for $k=1,2$ where $a,b\in\mathbb{R}$ with $a<b$ are consecutive zeros, and u_k for $k=1,2$ are not identically zero on $[a,b]$. Multiplying the first equation of system (1.1) by u_1 and the second equation of system (1.1) by u_2, integrating from a to b and taking into account that $u_k(a)=0=u_k(b)$ for $k=1,2$, we get

$$\int_a^b r_1(s)\left|u_1'(s)\right|^{p_1}ds=\int_a^b f_1(s)\,|u_1(s)|^{\alpha_1}\,|u_2(s)|^{\alpha_2}\,ds \tag{2.2}$$

and

$$\int_a^b r_2(s)\left|u_2'(s)\right|^{p_2}ds=\int_a^b f_2(s)\,|u_1(s)|^{\beta_1}\,|u_2(s)|^{\beta_2}\,ds. \tag{2.3}$$

By using $u_k(a)=0$ and Hölder's inequality, we get

$$|u_k(x)|\leqslant\int_a^x\left|u_k'(s)\right|ds\leqslant\left(\int_a^x r_k^{1/(1-p_k)}(s)\,ds\right)^{(p_k-1)/p_k}\left(\int_a^x r_k(s)\left|u_k'(s)\right|^{p_k}ds\right)^{1/p_k}$$

for $k = 1,2$ and $x \in [a,b]$. Thus, we get

$$|u_k(x)|^{p_k} \xi_k^{1-p_k}(x) \leqslant \int_a^x r_k(s) \left|u_k'(s)\right|^{p_k} ds \tag{2.4}$$

for $k = 1,2$. Similarly, by using $u_k(b) = 0$ and Hölder's inequality, we get

$$|u_k(x)|^{p_k} \eta_k^{1-p_k}(x) \leqslant \int_x^b r_k(s) \left|u_k'(s)\right|^{p_k} ds \tag{2.5}$$

for $k = 1,2$ and $x \in [a,b]$. Adding (2.4) and (2.5), we have

$$|u_k(x)|^{p_k} \leqslant D_k(x) \int_a^b r_k(s) \left|u_k'(s)\right|^{p_k} ds \tag{2.6}$$

for $k = 1,2$ and $x \in [a,b]$. After that by using similar technique to the proof of Theorem 2.1 given by Tang and He [32], it can be showed that the equality case in (2.6) does not hold. Thus, we get

$$|u_k(x)|^{p_k} < D_k(x) \int_a^b r_k(s) \left|u_k'(s)\right|^{p_k} ds \tag{2.7}$$

for $k = 1,2$ and $x \in (a,b)$. For $k = 1$ in the inequality (2.7), we get

$$|u_1(x)|^{p_1} < D_1(x) \int_a^b f_1(s) |u_1(s)|^{\alpha_1} |u_2(s)|^{\alpha_2} ds \tag{2.8}$$

from (2.2). If we take the $\frac{\alpha_1}{p_1}$ and $\frac{\beta_1}{p_1}$-th powers of both side of inequality (2.8), we have

$$|u_1(x)|^{\alpha_1} < D_1^{\alpha_1/p_1}(x) \left(\int_a^b f_1^+(s) |u_1(s)|^{\alpha_1} |u_2(s)|^{\alpha_2} ds \right)^{\alpha_1/p_1} \tag{2.9}$$

and

$$|u_1(x)|^{\beta_1} < D_1^{\beta_1/p_1}(x) \left(\int_a^b f_1^+(s) |u_1(s)|^{\alpha_1} |u_2(s)|^{\alpha_2} ds \right)^{\beta_1/p_1}, \tag{2.10}$$

respectively. Multiplying both sides of (2.9) by $f_1^+(x) |u_2(x)|^{\alpha_2}$, integrating from a to b, we have

$$\left(\int_a^b f_1^+(s) |u_1(s)|^{\alpha_1} |u_2(s)|^{\alpha_2} ds \right)^{1-\alpha_1/p_1} < \int_a^b f_1^+(s) |u_2(s)|^{\alpha_2} D_1^{\alpha_1/p_1}(s) ds. \tag{2.11}$$

Similarly, for $k = 2$ in the inequality (2.7), we get

$$|u_2(x)|^{p_2} < D_2(x) \int_a^b f_2(s) |u_1(s)|^{\beta_1} |u_2(s)|^{\beta_2} ds \tag{2.12}$$

from (2.3). If we take the $\frac{\alpha_2}{p_2}$ and $\frac{\beta_2}{p_2}$-th powers of both side of inequality (2.12), we have

$$|u_2(x)|^{\alpha_2} < D_2^{\alpha_2/p_2}(x) \left(\int_a^b f_2^+(s) |u_1(s)|^{\beta_1} |u_2(s)|^{\beta_2} ds \right)^{\alpha_2/p_2} \tag{2.13}$$

and

$$|u_2(x)|^{\beta_2} < D_2^{\beta_2/p_2}(x)\left(\int_a^b f_2^+(s)\,|u_1(s)|^{\beta_1}\,|u_2(s)|^{\beta_2}\,ds\right)^{\beta_2/p_2}, \tag{2.14}$$

respectively. Multiplying both sides of (2.14) by $f_2^+(x)\,|u_1(x)|^{\beta_1}$, integrating from a to b, we have

$$\left(\int_a^b f_2^+(s)\,|u_1(s)|^{\beta_1}\,|u_2(s)|^{\beta_2}\,ds\right)^{1-\beta_2/p_2} < \int_a^b f_2^+(s)\,|u_1(s)|^{\beta_1}\,D_2^{\beta_2/p_2}(s)\,ds. \tag{2.15}$$

By using (2.13) in (2.11) and (2.10) in (2.15), we have

$$\left(\int_a^b f_1^+(s)\,|u_1(s)|^{\alpha_1}\,|u_2(s)|^{\alpha_2}\,ds\right)^{1-\alpha_1/p_1} < M_1\left(\int_a^b f_2^+(s)\,|u_1(s)|^{\beta_1}\,|u_2(s)|^{\beta_2}\,ds\right)^{\alpha_2/p_2} \tag{2.16}$$

and

$$\left(\int_a^b f_2^+(s)\,|u_1(s)|^{\beta_1}\,|u_2(s)|^{\beta_2}\,ds\right)^{1-\beta_2/p_2} < M_2\left(\int_a^b f_1^+(s)\,|u_1(s)|^{\alpha_1}\,|u_2(s)|^{\alpha_2}\,ds\right)^{\beta_1/p_1}, \tag{2.17}$$

where

$$M_1 = \int_a^b f_1^+(s)\,D_1^{\alpha_1/p_1}(s)\,D_2^{\alpha_2/p_2}(s)\,ds \quad \text{and} \quad M_2 = \int_a^b f_2^+(s)\,D_1^{\beta_1/p_1}(s)\,D_2^{\beta_2/p_2}(s)\,ds,$$

respectively. If we take the e_1 and e_2-th powers of both side of inequalities (2.16) and (2.17), and multiplying the resulting equations, we obtain

$$\begin{aligned}
&\left[\left(\int_a^b f_1^+(s)\,|u_1(s)|^{\alpha_1}\,|u_2(s)|^{\alpha_2}\,ds\right)^{1-\alpha_1/p_1}\right]^{e_1} \\
&\quad\times\left[\left(\int_a^b f_2^+(s)\,|u_1(s)|^{\beta_1}\,|u_2(s)|^{\beta_2}\,ds\right)^{1-\beta_2/p_2}\right]^{e_2} \\
&< \left[M_1\left(\int_a^b f_2^+(s)\,|u_1(s)|^{\beta_1}\,|u_2(s)|^{\beta_2}\,ds\right)^{\alpha_2/p_2}\right]^{e_1} \\
&\quad\times\left[M_2\left(\int_a^b f_1^+(s)\,|u_1(s)|^{\alpha_1}\,|u_2(s)|^{\alpha_2}\,ds\right)^{\beta_1/p_1}\right]^{e_2}.
\end{aligned} \tag{2.18}$$

It is easy to see that by using similar technique to the proof of Theorem 2.1 given by Tang and He [32], we obtain the following inequalities

$$0 < \int_a^b f_1^+(s)\,|u_1(s)|^{\alpha_1}\,|u_2(s)|^{\alpha_2}\,ds \quad \text{and} \quad 0 < \int_a^b f_2^+(s)\,|u_1(s)|^{\beta_1}\,|u_2(s)|^{\beta_2}\,ds. \tag{2.19}$$

Now, we choose e_1 and e_2 such that $0 < \int_a^b f_1^+(s)|u_1(s)|^{\alpha_1}|u_2(s)|^{\alpha_2}ds$ and $0 < \int_a^b f_2^+(s)|u_1(s)|^{\beta_1}|u_2(s)|^{\beta_2}ds$ cancel out in inequality (2.18), i.e. solve the homogeneous linear system

$$\left.\begin{aligned} (1-\tfrac{\alpha_1}{p_1})e_1 - \quad \tfrac{\beta_1}{p_1}e_2 \quad &= 0,\\ \tfrac{\alpha_2}{p_2}e_1 \quad - \left(1-\tfrac{\beta_2}{p_2}\right)e_2 &= 0.\end{aligned}\right\} \tag{2.20}$$

We observe that by hypotheses $\frac{\alpha_1}{p_1}+\frac{\alpha_2}{p_2}=1$ and $\frac{\beta_1}{p_1}+\frac{\beta_2}{p_2}=1$, this system admits a nontrivial solution, indeed all equations are equivalent to

$$\left(1-\frac{\alpha_1}{p_1}\right)e_1 = \frac{\beta_1}{p_1}e_2 \quad \text{and} \quad \frac{\alpha_2}{p_2}e_1 = \left(1-\frac{\beta_2}{p_2}\right)e_2.$$

Hence, we may take $e_1 = \frac{\beta_1}{p_1}$ and $e_2 = \frac{\alpha_2}{p_2}$, and we get inequality (2.1) which completes the proof. □

For system (1.1), another main result of this section is the following theorem.

THEOREM 2.2. *If $f_k \in C([a,b],\mathbb{R})$ for $k=1,2$ and $(u_1(x),u_2(x))$ is a nontrivial solution on $[a,b]$ for the problem (1.1)–(1.3), then the inequality*

$$1<\left(\int_a^b f_1^+(s)E_1^{\alpha_1/p_1}(s)E_2^{\alpha_2/p_2}(s)ds\right)^{\beta_1/p_1}\left(\int_a^b f_2^+(s)E_1^{\beta_1/p_1}(s)E_2^{\beta_2/p_2}(s)ds\right)^{\alpha_2/p_2} \tag{2.21}$$

holds.

Proof. Let $u_k(a)=0=u_k(b)$ for $k=1,2$ where $a,b\in\mathbb{R}$ with $a<b$ are consecutive zeros, and u_k for $k=1,2$ are not identically zero on $[a,b]$. As in the proof of Theorem 2.1, we have (2.2)–(2.5). If we take $k=1$ in inequalities (2.4) and (2.5), then we have

$$|u_1(x)|^{p_1} \leqslant \xi_1^{p_1-1}(x)\int_a^x r_1(s)\left|u_1'(s)\right|^{p_1}ds \tag{2.22}$$

and

$$|u_1(x)|^{p_1} \leqslant \eta_1^{p_1-1}(x)\int_x^b r_1(s)\left|u_1'(s)\right|^{p_1}ds \tag{2.23}$$

for $x\in[a,b]$. Multiplying the inequalities (2.22) and (2.23) by $\eta_1^{p_1-1}(x)$ and $\xi_1^{p_1-1}(x)$, respectively, we obtain

$$\eta_1^{p_1-1}(x)|u_1(x)|^{p_1} \leqslant (\xi_1(x)\eta_1(x))^{p_1-1}\int_a^x r_1(s)\left|u_1'(s)\right|^{p_1}ds \tag{2.24}$$

and

$$\xi_1^{p_1-1}(x)|u_1(x)|^{p_1} \leqslant (\xi_1(x)\eta_1(x))^{p_1-1}\int_x^b r_1(s)\left|u_1'(s)\right|^{p_1}ds \tag{2.25}$$

for $x\in[a,b]$. Thus, adding the inequalities (2.24) and (2.25), we have

$$|u_1(x)|^{p_1}\left(\xi_1^{p_1-1}(x)+\eta_1^{p_1-1}(x)\right) \leqslant (\xi_1(x)\eta_1(x))^{p_1-1}\int_a^b r_1(s)\left|u_1'(s)\right|^{p_1}ds \tag{2.26}$$

for $x \in [a,b]$. It is easy to see that the function $\xi_1^{p_1-1}(x) + \eta_1^{p_1-1}(x)$ takes the minimum value at $c_1 \in (a,b)$ such that $\xi_1(c_1) = \eta_1(c_1)$. Thus, we get

$$|u_1(x)|^{p_1}\left(\xi_1^{p_1-1}(c_1) + \eta_1^{p_1-1}(c_1)\right) \leqslant (\xi_1(x)\eta_1(x))^{p_1-1}\int_a^b r_1(s)\left|u_1'(s)\right|^{p_1} ds. \quad (2.27)$$

Since $\xi_1(c_1) + \eta_1(c_1) = \xi_1(x) + \eta_1(x)$, $\forall x, c_1 \in (a,b)$, and $\xi_1(c_1) = \dfrac{\xi_1(x) + \eta_1(x)}{2} = \frac{1}{2}\int_a^b r_1^{1/(1-p_1)}(s)ds$, we have

$$|u_1(x)|^{p_1}\left[2^{2-p_1}(\xi_1(x) + \eta_1(x))^{p_1-1}\right] = |u_1(x)|^{p_1}\left[2\xi_1^{p_1-1}(c_1)\right] \leqslant$$

$$(\xi_1(x)\eta_1(x))^{p_1-1}\int_a^b r_1(s)\left|u_1'(s)\right|^{p_1} ds \quad (2.28)$$

and hence from (2.2)

$$|u_1(x)|^{p_1} \leqslant E_1(x)\int_a^b f_1(s)\,|u_1(s)|^{\alpha_1}\,|u_2(s)|^{\alpha_2}\,ds \quad (2.29)$$

for $x \in [a,b]$. After that by using similar technique to the proof of Theorem 2.1 given by Tang and He [32], it can be showed that the equality case in (2.29) does not hold. Thus, we get

$$|u_1(x)|^{p_1} < E_1(x)\int_a^b f_1(s)\,|u_1(s)|^{\alpha_1}\,|u_2(s)|^{\alpha_2}\,ds \quad (2.30)$$

for $x \in (a,b)$. The rest of the proof is the same as in the proof of Theorem 2.1, and hence is omitted. □

REMARK 2.1. It is easy to see from the inequality (1.39) that if we take $1 < p_k < 2$ for $k = 1,2$, then inequality (2.21) is weaker than inequality (2.1). Hence, Theorem 2.2 is better than Theorem 2.1. Similarly, from the inequality (1.40), if $p_k > 2$ for $k = 1,2$, then Theorem 2.1 is better than Theorem 2.2. In addition, if $p_k = 2$ for $k = 1,2$, then Theorem 2.1 coincides with Theorem 2.2.

In general case, the results of Tang and He [32] can not be compared with our results, but they only compared with each other in the special cases as follows.

REMARK 2.2. Let the condition (1.21) holds. Thus, Theorem 2.1 coincides with Theorem K (or Theorem 2.2 given in Tang and He [32]) for the problem (1.22)–(1.23). Moreover, from the inequality (1.39), if we take $1 < p_1 < 2$, then Theorem 2.2 gives a better result than Theorem K for the problem (1.22)–(1.23).

By using the inequality (1.41) in (2.1) or (1.42) in (2.21), we obtain the following result from Theorem 2.1 or 2.2.

COROLLARY 2.1. *If* $f_k \in C([a,b],\mathbb{R})$ *for* $k = 1,2$ *and* $(u_1(x), u_2(x))$ *is a non-*

trivial solution on $[a,b]$ *for the problem* (1.1)–(1.3)*, then the inequality*

$$\left(\int_a^b r_1^{1/(1-p_1)}(s)ds\right)^{\frac{\beta_1(p_1-1)}{p_1}}\left(\int_a^b r_2^{1/(1-p_2)}(s)ds\right)^{\frac{\alpha_2(p_2-1)}{p_2}}$$
$$\times\left(\int_a^b f_1^+(s)\,ds\right)^{\frac{\beta_1}{p_1}}\left(\int_a^b f_2^+(s)\,ds\right)^{\frac{\alpha_2}{p_2}}>2^{\alpha_2+\beta_1} \tag{2.31}$$

holds.

REMARK 2.3. It is clear that Corollary 2.1 coincides with Theorem H given by [7]. In fact, Theorem 2.1 or 2.2 is a generalization of Theorem H.

3. Lyapunov-type inequalities for system (1.4)

For system (1.4), one of the main results of this section is the following theorem.

THEOREM 3.1. *If* $f_k\in C([a,b],\mathbb{R})$ *for* $k=1,2,...,n$ *and* $(u_1(x),u_2(x),...,u_n(x))$ *is a nontrivial solution on* $[a,b]$ *for the problem* (1.4)–(1.6)*, then the inequality*

$$1<\prod_{k=1}^n\left(\int_a^b f_k^+(s)\prod_{i=1}^n D_i^{\alpha_i/p_i}(s)\,ds\right)^{\alpha_k/p_k} \tag{3.1}$$

holds.

Proof. Let $u_k(a)=0=u_k(b)$ for $k=1,2,...,n$ where $n\in\mathbb{N}$, $a,b\in\mathbb{R}$ with $a<b$ are consecutive zeros and u_k for $k=1,2,...,n$ *are* not identically zero on $[a,b]$. By using $u_k(a)=0$ and Hölder's inequality, we get

$$|u_k(x)|\leqslant\int_a^x|u_k'(s)|\,ds\leqslant\left(\int_a^x r_k^{1/(1-p_k)}(s)\,ds\right)^{(p_k-1)/p_k}\left(\int_a^x r_k(s)\,|u_k'(s)|^{p_k}\,ds\right)^{1/p_k}$$

for $k=1,2,...,n$ and $x\in[a,b]$. Thus, we get

$$|u_k(x)|^{p_k}\,\xi_k^{1-p_k}(x)\leqslant\int_a^x r_k(s)\,|u_k'(s)|^{p_k}\,ds \tag{3.2}$$

for $k=1,2,...,n$. Similarly, by using $u_k(b)=0$ and Hölder's inequality, we get

$$|u_k(x)|^{p_k}\,\eta_k^{1-p_k}(x)\leqslant\int_x^b r_k(s)\,|u_k'(s)|^{p_k}\,ds \tag{3.3}$$

for $k=1,2,...,n$ and $x\in[a,b]$. Adding (3.2) and (3.3), we have

$$|u_k(x)|^{p_k}\leqslant D_k(x)\int_a^b r_k(s)\,|u_k'(s)|^{p_k}\,ds \tag{3.4}$$

for $k = 1,2,...,n$ and $x \in [a,b]$. After that by using similar technique to the proof of Theorem 3.1 given by Tang and He [32], it can be showed that the equality case in (3.4) does not hold. Thus, we get

$$|u_k(x)|^{p_k} < A_k^{p_k/\alpha_k} D_k(x), \tag{3.5}$$

where $x \in (a,b)$ and $A_k = \left(\int_a^b r_k(s)\left|u_k'(s)\right|^{p_k} ds\right)^{\alpha_k/p_k}$ for $k = 1,2,...,n$. If we take the $\frac{\alpha_k}{p_k}$-th power of both side of inequality (3.5), we obtain

$$|u_k(x)|^{\alpha_k} < A_k D_k^{\alpha_k/p_k}(x). \tag{3.6}$$

Multiplying both sides of (3.6) by $f_k^+(x) \prod_{i=1(\neq k)}^{n} |u_i(x)|^{\alpha_i}$ for $k = 1,2,...,n$, integrating from a to b, we have

$$\int_a^b f_k^+(s) \prod_{i=1}^{n} |u_i(s)|^{\alpha_i} ds < \int_a^b A_k D_k^{\alpha_k/p_k}(s) f_k^+(s) \prod_{i=1(\neq k)}^{n} |u_i(s)|^{\alpha_i} ds \tag{3.7}$$

for $k = 1,2,...,n$. On the other hand, multiplying the k-th equation of system (1.4) by u_k and integrating from a to b, we get

$$\int_a^b r_k(s)\left|u_k'(s)\right|^{p_k} ds = \int_a^b f_k(s) \prod_{i=1}^{n} |u_i(s)|^{\alpha_i} ds \leqslant \int_a^b f_k^+(s) \prod_{i=1}^{n} |u_i(s)|^{\alpha_i} ds \tag{3.8}$$

for $k = 1,2,...,n$. By using (3.8) in (3.7), we have

$$\int_a^b r_k(s)\left|u_k'(s)\right|^{p_k} ds < \int_a^b A_k D_k^{\alpha_k/p_k}(s) f_k^+(s) \prod_{i=1(\neq k)}^{n} |u_i(s)|^{\alpha_i} ds \tag{3.9}$$

for $k = 1,2,...,n$. Therefore, by using (3.6) in (3.9), we have

$$\left(\int_a^b r_k(s)\left|u_k'(s)\right|^{p_k} ds\right)^{1-\alpha_k/p_k} < \prod_{i=1(\neq k)}^{n} A_i \int_a^b f_k^+(s) \prod_{i=1}^{n} D_i^{\alpha_i/p_i}(s) ds \tag{3.10}$$

for $k = 1,2,...,n$. If we take the e_k-th power of both side of inequalities (3.10) for $k = 1,2,\ldots,n$, and multiplying the resulting equations, we obtain

$$\prod_{k=1}^{n}\left(\int_a^b r_k(s)\left|u_k'(s)\right|^{p_k} ds\right)^{e_k(1-\alpha_k/p_k)} < \prod_{k=1}^{n}\left(\prod_{i=1(\neq k)}^{n} A_i \int_a^b f_k^+(s) \prod_{i=1}^{n} D_i^{\alpha_i/p_i}(s) ds\right)^{e_k}$$

and hence

$$\prod_{k=1}^{n}\left(\int_a^b r_k(s)\left|u_k'(s)\right|^{p_k} ds\right)^{e_k(1-\alpha_k/p_k)}$$
$$< \prod_{k=1}^{n}\left(\int_a^b r_k(s)\left|u_k'(s)\right|^{p_k} ds\right)^{\frac{\alpha_k}{p_k}\sum_{i=1(\neq k)}^{n} e_i} \prod_{k=1}^{n}\left(\int_a^b f_k^+(s) \prod_{i=1}^{n} D_i^{\alpha_i/p_i}(s) ds\right)^{e_k}. \tag{3.11}$$

It is easy to see that by using similar technique to the proof of Theorem 3.1 given by Tang and He [32], we obtain the following inequalities

$$\int_a^b r_k(s)\left|u_k'(s)\right|^{p_k} ds > 0 \tag{3.12}$$

for $k = 1,2,...,n$. Now, we choose e_k such that $\int_a^b r_k(s)\left|u_k'(s)\right|^{p_k} ds$ for $k = 1,2,...,n$ cancel out in inequality (3.11), i.e. solve the homogeneous linear system

$$\left.\begin{array}{lllll} (p_1-\alpha_1)e_1 & -\ \alpha_1 e_2 & -\ \alpha_1 e_3 \ldots - & \alpha_1 e_n & = 0 \\ -\alpha_2 e_1 & +\ (p_2-\alpha_2)e_2 & -\ \alpha_2 e_3 \ldots - & \alpha_2 e_n & = 0 \\ \ldots & & & & \\ -\alpha_n e_1 & -\ \alpha_n e_2 & -\ \alpha_n e_3 \ldots + & (p_n-\alpha_n)e_n & = 0 \end{array}\right\}. \tag{3.13}$$

We observe that by hypothesis $\sum_{k=1}^{n}\frac{\alpha_k}{p_k} = 1$, this system admits a nontrivial solution, indeed all equations are equivalent to

$$\frac{\alpha_k}{p_k}\left(\sum_{i=1(\neq k)}^{n} e_i\right) = e_k\left(\sum_{i=1(\neq k)}^{n}\frac{\alpha_i}{p_i}\right)$$

for $k = 1,2,...,n$. Hence, we may take $e_k = \frac{\alpha_k}{p_k}$ for $k = 1,2,...,n$, and we get inequality (3.1) which completes the proof. □

For system (1.4), another main result of this section is the following theorem.

THEOREM 3.2. *If $f_k \in C([a,b],\mathbb{R})$ for $k = 1,2,...,n$ and $(u_1(x), u_2(x), ..., u_n(x))$ is a nontrivial solution on $[a,b]$ for the problem (1.4)–(1.6), then the inequality*

$$1 < \prod_{k=1}^{n}\left(\int_a^b f_k^+(s)\prod_{i=1}^{n}E_i^{\alpha_i/p_i}(s)\,ds\right)^{\alpha_k/p_k} \tag{3.14}$$

holds.

Proof. Let $u_k(a) = 0 = u_k(b)$ for $k = 1,2,...,n$ where $n \in \mathbb{N}$, $a,b \in \mathbb{R}$ with $a < b$ are consecutive zeros and u_k for $k = 1,2,...,n$ are not identically zero on $[a,b]$. As in the proof of Theorem 3.1, we have inequalities (3.2) and (3.3). Multiplying the inequalities (3.2) and (3.3) by $\eta_k^{p_k-1}(x)$ and $\xi_k^{p_k-1}(x)$, $k = 1,2,...,n$, respectively, we obtain

$$\eta_k^{p_k-1}(x)\left|u_k(x)\right|^{p_k} \leqslant (\xi_k(x)\eta_k(x))^{p_k-1}\int_a^x r_k(s)\left|u_k'(s)\right|^{p_k} ds \tag{3.15}$$

and

$$\xi_k^{p_k-1}(x)\,|u_k(x)|^{p_k} \leqslant (\xi_k(x)\eta_k(x))^{p_k-1}\int_x^b r_k(s)\left|u_k'(s)\right|^{p_k} ds \tag{3.16}$$

for $k=1,2,\ldots,n$ and $x\in[a,b]$. Thus, adding the inequalities (3.15) and (3.16), we have

$$|u_k(x)|^{p_k}\left(\xi_k^{p_k-1}(x)+\eta_k^{p_k-1}(x)\right) \leqslant (\xi_k(x)\eta_k(x))^{p_k-1}\int_a^b r_k(s)\left|u_k'(s)\right|^{p_k} ds \tag{3.17}$$

for $k=1,2,\ldots,n$ and $x\in[a,b]$. It is easy to see that the functions $\xi_k^{p_k-1}(x)+\eta_k^{p_k-1}(x)$ take the minimum values at $c_k\in(a,b)$ such that $\xi_k(c_k)=\eta_k(c_k)$ for $k=1,2,\ldots,n$. Thus, we get

$$|u_k(x)|^{p_k}\left(\xi_k^{p_k-1}(c_k)+\eta_k^{p_k-1}(c_k)\right) \leqslant (\xi_k(x)\eta_k(x))^{p_k-1}\int_a^b r_k(s)\left|u_k'(s)\right|^{p_k} ds \tag{3.18}$$

for $k=1,2,\ldots,n$. Since $\xi_k(c_k)+\eta_k(c_k)=\xi_k(x)+\eta_k(x)$, $\forall x,c_k\in(a,b)$, and $\xi_k(c_k)=\dfrac{\xi_k(x)+\eta_k(x)}{2}=\frac{1}{2}\int_a^b r_k^{1/(1-p_k)}(s)ds$, we have

$$|u_k(x)|^{p_k}\left[2^{2-p_k}(\xi_k(x)+\eta_k(x))^{p_k-1}\right]=|u_k(x)|^{p_k}\left[2\xi_k^{p_k-1}(c_k)\right]\leqslant$$

$$(\xi_k(x)\eta_k(x))^{p_k-1}\int_a^b r_k(s)\left|u_k'(s)\right|^{p_k} ds \tag{3.19}$$

and hence

$$|u_k(x)|^{p_k}\leqslant E_k(x)\int_a^b r_k(s)\left|u_k'(s)\right|^{p_k} ds \tag{3.20}$$

for $k=1,2,\ldots,n$ and $x\in[a,b]$. After that by using similar technique to the proof of Theorem 3.1 given by Tang and He [32], it can be showed that the equality case in (3.20) does not hold. Thus, we get

$$|u_k(x)|^{p_k}<A_k^{p_k/\alpha_k}E_k(x), \tag{3.21}$$

where $x\in(a,b)$ and $A_k=\left(\int_a^b r_k(s)\left|u_k'(s)\right|^{p_k} ds\right)^{\alpha_k/p_k}$ for $k=1,2,\ldots,n$. The rest of the proof is the same as in the proof of Theorem 3.1, and hence is omitted. □

REMARK 3.1. By using the inequalities (1.39) and (1.40), it is easy to see that Theorem 3.1 is better than Theorem 3.2 when $p_k>2$ for $k=1,2,\ldots,n$, and Theorem 3.2 is better than Theorem 3.1 when $1<p_k<2$ for $k=1,2,\ldots,n$. In addition, if $p_k=2$ for $k=1,2,\ldots,n$, then Theorem 3.1 coincides with Theorem 3.2.

REMARK 3.2. Let $n=1$. Thus, Theorem 3.1 coincides with Theorem L given by Tang and He [32] for the problem (1.22)–(1.23). Moreover, from the inequality (1.39), if we take $1<p_1<2$, then Theorem 3.2 gives a better result than Theorem L for the problem (1.22)–(1.23).

REMARK 3.3. Note that (1.1) is a special case of (1.4) where $n=2$, $\beta_1=\alpha_1$ and $\beta_2=\alpha_2$. Under these conditions, we can see that the inequality (3.1) (or (3.14)) of Theorem 3.1 (or 3.2) reduces to the inequality (2.1) (or (2.21)) of Theorem 2.1 (or 2.2).

If we use the second mean value theorem for integrals in the inequalities (3.1) and (3.14), we obtain the following results from Theorems 3.1 and 3.2, respectively.

COROLLARY 3.1. *If $f_k \in C([a,b],\mathbb{R})$ for $k=1,2,...,n$ and $(u_1(x),u_2(x),...,u_n(x))$ is a nontrivial solution on $[a,b]$ for the problem (1.4)–(1.6), then there exist some points $d_k \in (a,b)$ for $k=1,2,...,n$ such that*

$$\prod_{k=1}^{n}\left(\prod_{i=1}^{n} D_i^{-\alpha_i/p_i}(d_k)\right)^{\alpha_k/p_k} < \prod_{k=1}^{n}\left(\int_a^b f_k^+(s)\,ds\right)^{\alpha_k/p_k} \tag{3.22}$$

holds.

COROLLARY 3.2. *If $f_k \in C([a,b],\mathbb{R})$ for $k=1,2,...,n$ and $(u_1(x),u_2(x),...,u_n(x))$ is a nontrivial solution on $[a,b]$ for the problem (1.4)–(1.6), then there exist some points $d_k \in (a,b)$ for $k=1,2,...,n$ such that*

$$\prod_{k=1}^{n}\left(\prod_{i=1}^{n} E_i^{-\alpha_i/p_i}(d_k)\right)^{\alpha_k/p_k} < \prod_{k=1}^{n}\left(\int_a^b f_k^+(s)ds\right)^{\alpha_k/p_k} \tag{3.23}$$

holds.

REMARK 3.4. Note that if we take $r_k(x)=1$ and $d_k=c_i$ where $|u_i(c_i)| = \max_{a<x<b}|u_i(x)|$ for $i,k=1,2,...,n$ in the inequality (3.22), then Corollary 3.1 coincides with Theorem I given by [8]. Therefore, Theorem 3.1 is a generalization of Theorem I. In addition, by using the inequality (1.39), if we take $1<p_k<2$, $r_k(x)=1$ and $d_k=c_i$ where $|u_i(c_i)| = \max_{a<x<b}|u_i(x)|$ for $i,k=1,2,...,n$, then Corollary 3.2 gives a better result than Theorem I.

By using the inequality (1.41) in Theorem 3.1 or (1.42) in Theorem 3.2, we obtain the following result.

COROLLARY 3.3. *If $f_k \in C([a,b],\mathbb{R})$ for $k=1,2,...,n$ and $(u_1(x),u_2(x),...,u_n(x))$ is a nontrivial solution on $[a,b]$ for the problem (1.4)–(1.6), then the inequality*

$$2^{\sum_{k=1}^{n}\alpha_k}\prod_{k=1}^{n}\left(\int_a^b r_k^{1/(1-p_k)}(s)ds\right)^{\alpha_k(1-p_k)/p_k} < \prod_{k=1}^{n}\left(\int_a^b f_k^+(s)\,ds\right)^{\alpha_k/p_k} \tag{3.24}$$

holds.

REMARK 3.5. If we take $n=2$, then Corollary 3.3 coincides with Corollary 2.1 with $\beta_i=\alpha_i$ for $i=1,2$.

REMARK 3.6. Let $n=1$ and $r_1(x)=1$. If we compare Theorem 3.2 with Theorem D given by Sim and Lee [31], then the positivity condition on the function f_1 in

Theorem D can be dropped for the problem (1.17)–(1.18). Thus, Theorem 3.2 generalizes and extends Theorem D. In addition to this if $p_1 = 2$, then Theorem 3.1 or 3.2 reduces to Theorem B for the problem (1.10)–(1.11). Moreover, from the inequality (1.40), Theorem 3.1 with $p_1 > 2$ gives a better result than Theorem D for the problem (1.17)–(1.18).

REMARK 3.7. Since $|f(x)| \geqslant f^+(x)$, the functions $f_k^+(x)$ for $k = 1,2,...,n$ in the above results can also be replaced by $|f_k(x)|$ for $k = 1,2,...,n$.

4. Lower bounds for generalized eigenvalues

Now, we present an application of the Lyapunov-type inequality obtained for system (1.4).

We obtain the following result which gives lower bounds for the n-th component of any generalized eigenvalue $(\lambda_1,\lambda_2,...,\lambda_n)$ of system (1.7). The proof of the following theorem is based on above generalization of the Lyapunov-type inequality, as in that of Theorem 9 of Çakmak and Tiryaki [8] and hence is omitted.

THEOREM 4.1. *There exist a function* $h_1(\lambda_1,\lambda_2,...,\lambda_{n-1})$ *such that*

$$h_1(\lambda_1,\lambda_2,...,\lambda_{n-1}) < \lambda_n \tag{4.1}$$

for any generalized eigenvalue $(\lambda_1,\lambda_2,...,\lambda_n)$ *of the problem* (1.7)–(1.9), *where*

$$h_1(\lambda_1,\lambda_2,...,\lambda_{n-1}) = \frac{1}{\alpha_n}\left[\prod_{k=1}^{n-1}(\lambda_k\alpha_k)^{\alpha_k/p_k}\int_a^b r(s)\prod_{i=1}^{n} D_i(s)^{\alpha_i/p_i}(s)ds\right]^{-p_n/\alpha_n} \tag{4.2}$$

from the inequality (3.1) *or*

$$h_1(\lambda_1,\lambda_2,...,\lambda_{n-1}) = \frac{1}{\alpha_n}\left[\prod_{k=1}^{n-1}(\lambda_k\alpha_k)^{\alpha_k/p_k}\int_a^b r(s)\prod_{i=1}^{n} E_i^{\alpha_i/p_i}(s)\,ds\right]^{-p_n/\alpha_n} \tag{4.3}$$

from the inequality (3.14).

REMARK 4.1. By using the inequalities (1.39) and (1.40), it is easy to see that Theorem 4.1 with (4.2) gives a better lower bound than Theorem 4.1 with (4.3) when $p_k > 2$ for $k = 1,2,...,n$, and Theorem 4.1 with (4.3) gives a better lower bound than Theorem 4.1 with (4.2) when $1 < p_k < 2$ for $k = 1,2,...,n$. In addition, if $p_k = 2$ for $k = 1,2,...,n$, then Theorem 4.1 with (4.2) is exactly the same as Theorem 4.1 with (4.3).

REMARK 4.2. Let $n = 2$ and $r_k(x) = 1$ for $k = 1,2$. If we compare Theorem 4.1 with Theorem G given by Napoli and Pinasco [21], we obtain $h_1(\lambda_1) > k_1(\lambda_1)$ since the inequality (1.41) or (1.42) holds. Therefore, Theorem 4.1 gives a better lower bound than Theorem G.

REMARK 4.3. Note that after the second mean value theorem for integrals in (4.2) and (4.3) are used, if we take $r_k(x) = 1$ and $d_k = c_i$ where $|u_i(c_i)| = \max_{a<x<b} |u_i(x)|$ for $i,k = 1,2,...,n$ in (4.2), then Theorem 4.1 with (4.2) reduces to Theorem J given by [8]. In addition, if we take $1 < p_k < 2$, $r_k(x) = 1$ and $d_k = c_i$ for $i,k = 1,2,...,n$ in (4.3), by using the inequality (1.39), then Theorem 4.1 with (4.3) gives a better lower bound than Theorem J.

REMARK 4.4. Since h_1 is a continuous function, then $h_1(\lambda_1,\lambda_2,...,\lambda_{n-1}) \to +\infty$ as any eigenvalue of $\lambda_k \to 0^+$ for $k = 1,2,...,n-1$. Therefore, there exists a ball centered in the origin such that the generalized spectrum is contained in its exterior. Also, by rearranging terms in (4.1) we obtain from (4.2)

$$\left[\prod_{k=1}^{n} \alpha_k^{\alpha_k/p_k} \int_a^b r(s) \prod_{i=1}^{n} D_i^{\alpha_i/p_i}(s)\, ds\right]^{-1} < \prod_{k=1}^{n} \lambda_k^{\alpha_k/p_k} \tag{4.4}$$

or from (4.3)

$$\left[\prod_{k=1}^{n} \alpha_k^{\alpha_k/p_k} \int_a^b r(s) \prod_{i=1}^{n} E_i^{\alpha_i/p_i}(s)\, ds\right]^{-1} < \prod_{k=1}^{n} \lambda_k^{\alpha_k/p_k}. \tag{4.5}$$

It is clear that when the interval collapses, left-hand side of (4.4) or (4.5) goes to infinity.

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