Mathematical Inequalities & Applications
Volume 17, Number 3 (2014), 865–877
doi:10.7153/mia-17-63

SHEPHARD TYPE PROBLEMS FOR L_p–CENTROID BODIES

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(*Communicated by I. Perić*)

Abstract. Lutwak and Zhang proposed the notion of the L_p-centroid body. In this article, based on the definition of L_p-dual affine surface area, we research Shephard type problems for the L_p-centroid body

1. Introduction and main results

Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors, the set of convex bodies whose centroid lie at the origin and the set of origin-symmetric convex bodies in $\mathbb{R}^n$, we write $\mathscr{K}_o^n$, $\mathscr{K}_c^n$ and $\mathscr{K}_{os}^n$, respectively. S_o^n and S_{os}^n respectively denote the set of star bodies (about the origin) and the set of origin-symmetric star bodies in $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, and $V(K)$ denotes the n-dimensional volume of body K. For the standard unit ball B in $\mathbb{R}^n$, denote $\omega_n = V(B)$.

In 1997, Lutwak and Zhang in [3] introduced the concept of L_p-centroid body as follows: For each compact star-shaped about the origin $K \subset \mathbb{R}^n$, real number $p \geqslant 1$, the L_p-centroid body, $\Gamma_p K$, of K is the origin-symmetric convex body whose support function is defined by

$$h^p_{\Gamma_p K}(u) = \frac{1}{c_{n,p} V(K)} \int_K |u \cdot x|^p dx, \tag{1.1}$$

for any $u \in S^{n-1}$.

Here the integration is with respect to Lebesgue and $c_{n,p} = \omega_{n+p}/\omega_2 \omega_n \omega_{p-1}$.

Using polar coordinates in (1.1), we easily get

$$h^p_{\Gamma_p K}(u) = \frac{1}{(n+p) c_{n,p} V(K)} \int_{S^{n-1}} |u \cdot v|^p \rho_K(v)^{n+p} dv, \tag{1.2}$$

for any $u \in S^{n-1}$.

Lutwak, Yang and Zhang have made a series of studies about the L_p-centroid body, and many scholars were attracted. The L_p-centroid body have got many results from these articles (see [2–5, 12, 13, 17]). Particularly, Grinberg and Zhang gave the following Shephard problems for the L_p-centroid body in [2].

Mathematics subject classification (2010): 52A20, 52A40.
Keywords and phrases: L_p-centroid body, Shephard type problems, L_p-dual affine surface area.
Research is supported in part by the Natural Science Foundation of China (Grant No. 10671117) and Science Foundation of China Three Gorges University.

THEOREM 1.A. *For $K,L \in S_o^n$, if $\Gamma_p K \subseteq \Gamma_p L$ and L is the polar of L_p-projection body, then*

$$V(K) \leqslant V(L),$$

with equality if and only if $K = L$.

THEOREM 1.B. *For $K \in \mathscr{F}_{os}^2$, if $K \notin \mathscr{L}_p$, then there exists $L \in \mathscr{K}_{os}^n$ such that $\Gamma_p K \subset \Gamma_p L$, but*

$$V(K) > V(L).$$

Here $\mathscr{F}_{os}^2$ denotes the set of centered convex bodies whose support functions are of C^2 and have positive continuous curvature functions, and $\mathscr{L}_p$ denotes the set of L_p-balls (see [2]).

In 1996, Lutwak in [6] introduced the concept of L_p-affine surface area as follows: For $K \in \mathscr{K}_o^n$ and $p \geqslant 1$, the L_p-affine surface area, $\Omega_p(K)$, of K is defined by

$$n^{-\frac{p}{n}} \Omega_p(K)^{\frac{n+p}{n}} = \inf\{nV_p(K,Q^*)V(Q)^{\frac{p}{n}} : Q \in S_o^n\},$$

where $V_p(M,N)$ denotes the L_p-mixed volume of $M,N \in \mathscr{K}_o^n$.

Further, Wang and Leng in [10] defined ith L_p-mixed affine surface area, $\Omega_{p,i}(K)$, of K (for $i=0$, $\Omega_{p,i}(K)$ is just the L_p-affine surface area $\Omega_p(K)$) and extended Lutwak's some results. Regarding the study of L_p-affine surface area, many results have been obtained in these articles (see [6, 10, 11, 15, 16]).

According to the notion of L_p-affine surface area. In 2008, Wang and He in [14], associated with the L_p-dual mixed volume, gave the notion of the L_p-dual affine surface area. For $K \in S_o^n$ and $1 \leqslant p < n$, the L_p-dual affine surface area, $\widetilde{\Omega}_{-p}(K)$, of K is defined by

$$n^{\frac{p}{n}} \widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}} = \inf\{n\widetilde{V}_{-p}(K,Q^*)V(Q)^{-\frac{p}{n}} : Q \in \mathscr{K}_c^n\}, \tag{1.3}$$

where $\widetilde{V}_{-p}(M,N)$ denotes the L_p-dual mixed volume of $M,N \in S_o^n$.

In this paper, associated with definition (1.3) of the L_p-dual affine surface area, we will research the Shephard-type problems for the L_p-centroid bodies. For the convenience of our work, we improve the definition (1.3) from $Q \in \mathscr{K}_c^n$ to $Q \in S_{os}^n$:

For $K \in S_o^n$ and $1 \leqslant p < n$, L_p-dual affine surface area, $\widetilde{\Omega}_{-p}(K)$, of K is defined by

$$n^{\frac{p}{n}} \widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}} = \inf\{n\widetilde{V}_{-p}(K,Q^*)V(Q)^{-\frac{p}{n}} : Q \in S_{os}^n\}. \tag{1.4}$$

Let Z_p^n denote the set of L_p-projection bodies, then $Z_p^n \subseteq S_{os}^n$. If $Q \in Z_p^n$ in (1.4), write $\widetilde{\Omega}_{-p}^{\circ}(K)$ by

$$n^{\frac{p}{n}} \widetilde{\Omega}_{-p}^{\circ}(K)^{\frac{n-p}{n}} = \inf\{n\widetilde{V}_{-p}(K,Q^*)V(Q)^{-\frac{p}{n}} : Q \in Z_p^n\}. \tag{1.5}$$

According to equality (1.5), we first give an affirmative form of the Shephard-type problems for the L_p-centroid bodies.

THEOREM 1.1. *For $K,L \in S_o^n$, $1 \leqslant p < n$, if $\Gamma_p K \subseteq \Gamma_p L$, then*

$$\frac{\widetilde{\Omega}^{\circ}_{-p}(K)^{\frac{n-p}{n}}}{V(K)} \leqslant \frac{\widetilde{\Omega}^{\circ}_{-p}(L)^{\frac{n-p}{n}}}{V(L)},$$

with equality if and only if $\Gamma_p K = \Gamma_p L$.

As the application of Theorem 1.1, together with Theorem 1.A, we obtain a special affirmation form for the Shephard-type problems of L_p-centroid bodies.

THEOREM 1.2. *For $K,L \in S_o^n$, $1 \leqslant p < n$, if $\Gamma_p K \subseteq \Gamma_p L$ and L is the polar of L_p-projection body, then*

$$\widetilde{\Omega}^{\circ}_{-p}(K) \leqslant \widetilde{\Omega}^{\circ}_{-p}(L),$$

with equality if and only if $K = L$.

Next, combining with definition (1.4) of the L_p-dual affine surface area, we get an improved form of the Shephard-type problems for the L_p-centroid bodies.

THEOREM 1.3. *For $K \in S_o^n$, $L \in S_{os}^n$ and $1 \leqslant p < n$, if $\Gamma_p K = \Gamma_p L$, then*

$$\widetilde{\Omega}_{-p}(K) \leqslant \widetilde{\Omega}_{-p}(L),$$

with equality if and only if $K = L$.

Finally, we obtain a negative form of the Shephard-type problems for the L_p-centroid bodies.

THEOREM 1.4. *For $L \in S_o^n$ and $1 \leqslant p < n$, if L is not origin-symmetric star body, then there exists $K \in S_{os}^n$, such that*

$$\Gamma_p K \subset \Gamma_p L,$$

but

$$\widetilde{\Omega}_{-p}(K) > \widetilde{\Omega}_{-p}(L).$$

The proofs of Theorems 1.1–1.4 will be completed in section 4 of this paper.

2. Preliminaries

2.1. Support function, radial function and polar of convex bodies

If $K \in \mathscr{K}^n$, then its support function, $h_K = h(K,\cdot) : \mathbb{R}^n \to (-\infty,\infty)$, is defined by (see [1, 8])

$$h(K,x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n, \tag{2.1}$$

where $x \cdot y$ denotes the standard inner product of x and y.

From the definition of the support function, we easily obtain for $c > 0$ and any $u \in S^{n-1}$

$$h(cK,u) = ch(K,u). \tag{2.2}$$

If K is a compact star-shaped (about the origin) in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \setminus \{0\} \to [0,\infty)$, is defined by (see [1, 8])

$$\rho(K,u) = \max\{\lambda \geqslant 0 : \lambda \cdot u \in K\}, \ \ u \in S^{n-1}. \tag{2.3}$$

Given $c > 0$, we can get for any $u \in S^{n-1}$

$$\rho(cK,u) = c\rho(K,u). \tag{2.4}$$

If ρ_K is continuous and positive, then K will be called a star body. Two star bodies K, L are said to be dilates (of one another) if $\rho_K(u) / \rho_L(u)$ is independent of $u \in S^{n-1}$.

If $K \in \mathscr{K}_o^n$, the polar body , K^*, of K is defined by (see [1, 8])

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leqslant 1, y \in K\}. \tag{2.5}$$

From (2.5), we easily have $(K^*)^* = K$, and

$$h_{K^*} = \frac{1}{\rho_K}, \quad \rho_{K^*} = \frac{1}{h_K}. \tag{2.6}$$

2.2. L_p-mixed volume

For $K,L \in \mathscr{K}_o^n$, $p \geqslant 1$ and $\lambda,\mu \geqslant 0$ (not both zero), the Firey L_p-combination (also called the L_p-Minkowski combination), $\lambda \cdot K +_p \mu \cdot L \in \mathscr{K}_o^n$, of K and L is defined by (see [7])

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot)^p = \lambda h(K,\cdot)^p + \mu h(L,\cdot)^p, \tag{2.7}$$

where the operation "$+_p$" is called Firey addition and $\lambda \cdot K$ denotes the Firey scalar multiplication. From (2.2) and (2.7), we can get

$$\lambda \cdot K = \lambda^{\frac{1}{p}} K.$$

If $K,L \in \mathscr{K}_o^n$, then for $p \geqslant 1$, the L_p-mixed volume, $V_p(K,L)$, of K and L is defined by (see [7])

$$\frac{n}{p} V_p(K,L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

Corresponding to each $K,L \in \mathscr{K}_o^n$, there is a positive Borel measure, $S_p(K,\cdot)$, on S^{n-1}, such that (see [7])

$$V_p(K,L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(u) dS_p(K,\cdot),$$

where $S_p(K,\cdot)$ is called the L_p-surface area measure of $K \in \mathscr{K}_o^n$.

2.3. L_p-dual mixed volume

For $K,L \in S_o^n$, $p \geqslant 1$ and $\lambda,\mu \geqslant 0$ (not both zero), the L_p-harmonic radial combination, $\lambda \star K +_{-p} \mu \star L \in S_o^n$, of K and L is defined by (see [6])

$$\rho(\lambda \star K +_{-p} \mu \star L, \cdot)^{-p} = \lambda \rho(K,\cdot)^{-p} + \mu \rho(L,\cdot)^{-p}, \tag{2.8}$$

where the operation "$+_{-p}$" is called L_p-harmonic radial addition and $\lambda \star K$ denotes the L_p-harmonic radial scalar multiplication. From (2.4) and (2.8), we can obtain

$$\lambda \star K = \lambda^{-\frac{1}{p}} K.$$

Associated with the L_p-harmonic radial combination of star bodies, Lutwak in [6] introduced the notion of L_p-dual mixed volume as follows: For $K,L \in S_o^n$, $p \geqslant 1$ and $\varepsilon > 0$, the L_p-dual mixed volume, $\widetilde{V}_{-p}(K,L)$, of K and L is defined by (see [6])

$$\frac{n}{-p}\widetilde{V}_{-p}(K,L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_{-p} \varepsilon \star L) - V(K)}{\varepsilon}. \tag{2.9}$$

The definition above and Hospital's role give the following integral representation of L_p-dual mixed volume (see [6]):

$$\widetilde{V}_{-p}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n+p}(u)\rho_L^{-p}(u)du, \tag{2.10}$$

where the integration is with respect to spherical Lebesgue measure S on S^{n-1}.

From the formula (2.10), we get

$$\widetilde{V}_{-p}(K,K) = V(K) = \frac{1}{n}\int_{S^{n-1}} \rho_K^n(u)du. \tag{2.11}$$

The Minkowski's inequality for the L_p-dual mixed volume can be stated that (see [6]):

THEOREM 2.A. *If $K,L \in S_o^n$, $p \geqslant 1$, then*

$$\widetilde{V}_{-p}(K,L) \geqslant V(K)^{\frac{n+p}{n}} V(L)^{-\frac{p}{n}}, \tag{2.12}$$

with equality if and only if K and L are dilates.

2.4. L_p-projection body

For $K \in \mathscr{K}_o^n$, $p \geqslant 1$, the L_p-projection body, $\Pi_p K$, of K is the origin-symmetric convex body whose support function is given by (see [4])

$$h_{\Pi_p K}^p(u) = \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} |u \cdot v|^p dS_p(K,v),$$

where $u,v \in S^{n-1}$, and $S_p(K,\cdot)$ is the L_p-surface area measure of K.

3. L_p-harmonic Blaschke combination

In order to prove our results, we need the concept of L_p-harmonic Blaschke combination.

For $K,L \in S_o^n$, $p \geqslant 1$ and $\lambda,\mu \geqslant 0$ (not both zero), the L_p-harmonic Blaschke combination, $\lambda * K \hat{+}_p \mu * L \in S_o^n$, of K and L is defined by

$$\frac{\rho(\lambda * K\hat{+}_p\mu * L,\cdot)^{n+p}}{V(\lambda * K\hat{+}_p\mu * L)} = \lambda\frac{\rho(K,\cdot)^{n+p}}{V(K)} + \mu\frac{\rho(L,\cdot)^{n+p}}{V(L)}, \tag{3.1}$$

where the operation "$\hat{+}_p$" is called L_p-harmonic Blaschke addition and $\lambda * K$ denotes L_p-harmonic Blaschke scalar multiplication. From (2.4) and (3.1), we easily have

$$\lambda * K = \lambda^{\frac{1}{p}} K.$$

Taking $\lambda = \mu = 1$ in $\lambda * K\hat{+}_p\mu * L$, then $K\hat{+}_pL$ is just L_p-harmonic Blaschke addition, which was introduced in [17], of $K,L \in S_o^n$.

Let $\lambda = \mu = \frac{1}{2}$, $L = -K$, then L_p-harmonic Blaschke body is defined by

$$\widehat{\nabla}_pK = \frac{1}{2} * K\hat{+}_p\frac{1}{2} * (-K). \tag{3.2}$$

Obviously, the L_p-harmonic Blaschke body $\widehat{\nabla}_pK$ is origin-symmetric.

THEOREM 3.1. *If $K,L \in S_o^n$, $p \geqslant 1$, $\lambda,\mu \geqslant 0$ (not both zero), then*

$$V(\lambda * K\hat{+}_p\mu * L)^{\frac{p}{n}} \geqslant \lambda V(K)^{\frac{p}{n}} + \mu V(L)^{\frac{p}{n}}, \tag{3.3}$$

with equality if and only if K and L are dilates.

Proof. From (2.10), (2.12) and (3.1), we have for any $Q \in S_o^n$,

$$\begin{aligned}\frac{\widetilde{V}_{-p}(\lambda * K\hat{+}_p\mu * L,Q)}{V(\lambda * K\hat{+}_p\mu * L)} &= \lambda\frac{\widetilde{V}_{-p}(K,Q)}{V(K)} + \mu\frac{\widetilde{V}_{-p}(L,Q)}{V(L)} \\ &\geqslant [\lambda V(K)^{\frac{p}{n}} + \mu V(L)^{\frac{p}{n}}]V(Q)^{-\frac{p}{n}}.\end{aligned} \tag{3.4}$$

Taking $Q = \lambda * K\hat{+}_p\mu * L$ in (3.4), and from (2.11), we can get (3.3).

Associated with the equality condition of (2.12), we see that equality holds in (3.3) if and only if K and L are dilates. □

Taking $\lambda = \mu = \frac{1}{2}$, $L = -K$ in (3.3), we easily get the following result.

COROLLARY 3.1. *If $K \in S_o^n$, $p \geqslant 1$, then*

$$V(\widehat{\nabla}_pK) \geqslant V(K), \tag{3.5}$$

with equality if and only if K is an origin-symmetric.

Further, we also give a Brunn-Minkowski type inequality for L_p-harmonic Blaschke combination as well as its a corollary.

THEOREM 3.2. *If* $K,L \in S_o^n$, $\lambda,\mu \geqslant 0$ *(not both zero) and* $1 \leqslant p < n$*, then*

$$\frac{\widetilde{\Omega}_{-p}(\lambda * K \hat{+}_p \mu * L)^{\frac{n-p}{n}}}{V(\lambda * K \hat{+}_p \mu * L)} \geqslant \lambda \frac{\widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}}}{V(K)} + \mu \frac{\widetilde{\Omega}_{-p}(L)^{\frac{n-p}{n}}}{V(L)}, \tag{3.6}$$

with equality if and only if K *and* L *are dilates.*

Proof. From the definition (1.4), we get

$$\begin{aligned}
&n^{\frac{p}{n}} \widetilde{\Omega}_{-p}(\lambda * K \hat{+}_p \mu * L)^{\frac{n-p}{n}} \\
&= \inf\{n\widetilde{V}_{-p}(\lambda * K \hat{+}_p \mu * L, Q^*) V(Q)^{-\frac{p}{n}} : Q \in S_{os}^n\} \\
&= \inf\Big\{ \Big[\int_{S^{n-1}} \rho(\lambda * K \hat{+}_p \mu * L, u)^{n+p} \rho(Q^*, u)^{-p} du \Big] V(Q)^{-\frac{p}{n}} : Q \in S_{os}^n \Big\}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{n^{\frac{p}{n}} \widetilde{\Omega}_{-p}(\lambda * K \hat{+}_p \mu * L)^{\frac{n-p}{n}}}{V(\lambda * K \hat{+}_p \mu * L)} \\
&= \inf\Big\{ \Big[\int_{S^{n-1}} \frac{\rho(\lambda * K \hat{+}_p \mu * L, u)^{n+p}}{V(\lambda * K \hat{+}_p \mu * L)} \rho(Q^*, u)^{-p} du \Big] V(Q)^{-\frac{p}{n}} : Q \in S_{os}^n \Big\} \\
&= \inf\Big\{ \Big[\int_{S^{n-1}} \Big[\lambda \frac{\rho(K,u)^{n+p}}{V(K)} + \mu \frac{\rho(L,u)^{n+p}}{V(L)}\Big] \rho(Q^*, u)^{-p} du \Big] V(Q)^{-\frac{p}{n}} : Q \in S_{os}^n \Big\} \\
&= \inf\Big\{ \frac{\lambda}{V(K)} \Big[\int_{S^{n-1}} \rho(K,u)^{n+p} \rho(Q^*, u)^{-p} du \Big] V(Q)^{-\frac{p}{n}} \\
&\quad + \frac{\mu}{V(L)} \Big[\int_{S^{n-1}} \rho(L,u)^{n+p} \rho(Q^*, u)^{-p} du \Big] V(Q)^{-\frac{p}{n}} : Q \in S_{os}^n \Big\} \\
&\geqslant \frac{\lambda}{V(K)} \inf\{n\widetilde{V}_{-p}(K, Q^*) V(Q)^{-\frac{p}{n}} : Q \in S_{os}^n\} \\
&\quad + \frac{\mu}{V(L)} \inf\{n\widetilde{V}_{-p}(L, Q^*) V(Q)^{-\frac{p}{n}} : Q \in S_{os}^n\}.
\end{aligned}$$

This give (3.6).

The equality of (3.6) holds if and only if $\lambda * K \hat{+}_p \mu * L$ are dilates with K and L, respectively. This mean that the equality holds in (3.6) if and only if K and L are dilates. □

COROLLARY 3.2. *If* $K \in S_o^n$, $1 \leqslant p < n$*, then*

$$\widetilde{\Omega}_{-p}(\widehat{\nabla}_p K) \geqslant \widetilde{\Omega}_{-p}(K), \tag{3.7}$$

with equality if and only if K *is an origin-symmetric.*

Proof. Taking $\lambda = \mu = \frac{1}{2}$, $L = -K$ in (3.6), and combining with (3.2), we get

$$\frac{\widetilde{\Omega}_{-p}(\widehat{\nabla}_p K)^{\frac{n-p}{n}}}{V(\widehat{\nabla}_p K)} \geqslant \frac{1}{2} \frac{\widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}}}{V(K)} + \frac{1}{2} \frac{\widetilde{\Omega}_{-p}(-K)^{\frac{n-p}{n}}}{V(K)}. \tag{3.8}$$

For any $Q \in S^n_{os}$, then $Q^* \in \mathscr{K}^n_{os}$, and using $\rho_{Q^*}(u) = \rho_{-Q^*}(u) = \rho(Q^*, -u)$, we get

$$\widetilde{V}_{-p}(-K, Q^*) = \widetilde{V}_{-p}(K, Q^*). \tag{3.9}$$

Associated with (1.4), and from (3.9), we easily have

$$\widetilde{\Omega}_{-p}(-K) = \widetilde{\Omega}_{-p}(K). \tag{3.10}$$

Thus from (3.8) and (3.10), we know

$$\left(\frac{\widetilde{\Omega}_{-p}(\widehat{\nabla}_p K)}{\widetilde{\Omega}_{-p}(K)}\right)^{\frac{n-p}{n}} \geqslant \frac{V(\widehat{\nabla}_p K)}{V(K)}. \tag{3.11}$$

Since $1 \leqslant p < n$, thus combining with (3.5) and (3.11), this yields (3.7).

Associated with the equality condition of (3.5), we see that equality holds in (3.7) if and only if K is an origin-symmetric. □

4. The proofs of Theorems

In this section, we complete the proofs of Theorems 1.1–1.4. Here the proof of Theorem 1.1 require a Lemma as follows:

LEMMA 4.1. [4] *If $K \in S^n_o$, $p \geqslant 1$, then for any $Q \in \mathscr{K}^n_o$*

$$V_p(Q, \Gamma_p K) = \frac{\omega_n}{V(K)} \widetilde{V}_{-p}(K, \Pi^*_p Q).$$

Proof of Theorem 1.1. Since $\Gamma_p K \subseteq \Gamma_p L$, thus for any $Q \in \mathscr{K}^n_o$

$$V_p(Q, \Gamma_p K) \leqslant V_p(Q, \Gamma_p L), \tag{4.1}$$

with equality if and only if $\Gamma_p K = \Gamma_p L$. Therefore, from (4.1) and Lemma 4.1, we have

$$\frac{\widetilde{V}_{-p}(K, \Pi^*_p Q)}{V(K)} \leqslant \frac{\widetilde{V}_{-p}(L, \Pi^*_p Q)}{V(L)}. \tag{4.2}$$

Let $M = \Pi_p Q$, then $M \in Z^n_p$. From (1.5) and (4.2), we get

$$\begin{aligned} \frac{n^{\frac{p}{n}} \widetilde{\Omega}^\circ_{-p}(K)^{\frac{n-p}{n}}}{V(K)} &= \inf\left\{\frac{n\widetilde{V}_{-p}(K, M^*)}{V(K)} V(M)^{-\frac{p}{n}} : M \in Z^n_p\right\} \\ &\leqslant \inf\left\{\frac{n\widetilde{V}_{-p}(L, M^*)}{V(L)} V(M)^{-\frac{p}{n}} : M \in Z^n_p\right\} \\ &= \frac{n^{\frac{p}{n}} \widetilde{\Omega}^\circ_{-p}(L)^{\frac{n-p}{n}}}{V(L)}, \end{aligned} \tag{4.3}$$

i.e.

$$\frac{\widetilde{\Omega}^{\circ}_{-p}(K)^{\frac{n-p}{n}}}{V(K)} \leqslant \frac{\widetilde{\Omega}^{\circ}_{-p}(L)^{\frac{n-p}{n}}}{V(L)}. \tag{4.4}$$

According to the equality condition of (4.1), thus we know that the equality holds in (4.4) if and only if $\Gamma_p K = \Gamma_p L$. □

Proof of Theorem 1.2. For $K, L \in S^n_o$, if $\Gamma_p K \subseteq \Gamma_p L$, then from Theorem 1.1, we know

$$\frac{\widetilde{\Omega}^{\circ}_{-p}(K)^{\frac{n-p}{n}}}{V(K)} \leqslant \frac{\widetilde{\Omega}^{\circ}_{-p}(L)^{\frac{n-p}{n}}}{V(L)},$$

i.e.,

$$\left(\frac{\widetilde{\Omega}^{\circ}_{-p}(K)}{\widetilde{\Omega}^{\circ}_{-p}(L)}\right)^{\frac{n-p}{n}} \leqslant \frac{V(K)}{V(L)}. \tag{4.5}$$

Since L is the polar of L_p-projection body, thus from Theorem 1.A and (4.5), and notice that $1 \leqslant p < n$, we may get

$$\widetilde{\Omega}^{\circ}_{-p}(K) \leqslant \widetilde{\Omega}^{\circ}_{-p}(L). \tag{4.6}$$

According to the equality condition of Theorem 1.A, we see that equality holds in (4.6) if and only if $K = L$. □

For the proof of Theorem 1.3, we also need to establish the following lemmas.

LEMMA 4.2. *If $K \in S^n_o$, $p \geqslant 1$, then*

$$\Gamma_p(\hat{\nabla}_p K) = \Gamma_p K. \tag{4.7}$$

Proof. From (1.2) and (3.2), we get

$$\begin{aligned}
&h^p(\Gamma_p(\hat{\nabla}_p K), u)\\
&= h^p\left(\Gamma_p\left(\frac{1}{2} * K \hat{+}_p \frac{1}{2} * (-K)\right), u\right)\\
&= \frac{1}{(n+p)c_{n,p}} \int_{S^{n-1}} |u \cdot v|^p \frac{\rho(\frac{1}{2} * K \hat{+}_p \frac{1}{2} * (-K), v)^{n+p}}{V(\frac{1}{2} * K \hat{+}_p \frac{1}{2} * (-K))} dv\\
&= \frac{1}{(n+p)c_{n,p}} \int_{S^{n-1}} |u \cdot v|^p \left[\frac{1}{2}\frac{\rho(K, v)^{n+p}}{V(K)} + \frac{1}{2}\frac{\rho(-K, v)^{n+p}}{V(-K)}\right] dv\\
&= \frac{1}{2} h^p(\Gamma_p K, u) + \frac{1}{2} h^p(\Gamma_p(-K), u).
\end{aligned} \tag{4.8}$$

From (1.2), we easily know $\Gamma_p(-K) = \Gamma_p K$, so combining with (4.8), then for any $u \in S^{n-1}$,

$$h^p(\Gamma_p(\hat{\nabla}_p K), u) = h^p(\Gamma_p K, u).$$

This yields (4.7). □

LEMMA 4.3. *For $K,L \in S_o^n$, $p \geqslant 1$, then $\Gamma_p K = \Gamma_p L$ if and only if for any $Q \in S_{os}^n$,*

$$\frac{\widetilde{V}_{-p}(K,Q)}{V(K)} = \frac{\widetilde{V}_{-p}(L,Q)}{V(L)}. \tag{4.9}$$

Proof. From (2.6) and (2.10), we get

$$\begin{aligned}\frac{\widetilde{V}_{-p}(K,Q)}{V(K)} &= \frac{1}{nV(K)}\int_{S^{n-1}} \rho_K(v)^{n+p}\rho_Q(v)^{-p}dv \\ &= \frac{1}{nV(K)}\int_{S^{n-1}} \rho_K(v)^{n+p}h_{Q^*}^p(v)dv.\end{aligned} \tag{4.10}$$

Since $Q \in S_{os}^n$, so $Q^* \in \mathscr{K}_{os}^n$. Thus taking $Q^* = [-u,u]$, then we know for every $v \in S^{n-1}$,

$$h(Q^*,v) = |u \cdot v|.$$

From (1.2) and (4.10), we have

$$\begin{aligned}h_{\Gamma_p K}^p(u) &= \frac{1}{(n+p)c_{n,p}V(K)}\int_{S^{n-1}} |u\cdot v|^p\rho_K(v)^{n+p}dv \\ &= \frac{n\widetilde{V}_{-p}(K,[-u,u]^*)}{(n+p)c_{n,p}V(K)}.\end{aligned} \tag{4.11}$$

Therefore, for $K,L \in S_o^n$ and any $Q \in S_{os}^n$, if

$$\frac{\widetilde{V}_{-p}(K,Q)}{V(K)} = \frac{\widetilde{V}_{-p}(L,Q)}{V(L)},$$

then we have

$$\Gamma_p K = \Gamma_p L.$$

In turn, according to (4.7), we may assume that $K,L \in S_{os}^n$, because we can replace K and L by $\hat{\nabla}_p K$ and $\hat{\nabla}_p L$, respectively. Thus from (1.2), we know

$$h_{\Gamma_p K}^p(u) = \frac{1}{(n+p)c_{n,p}V(K)}\int_{S^{n-1}} |u\cdot v|^p\rho_K(v)^{n+p}dv. \tag{4.12}$$

Let

$$f(v) = \frac{1}{V(K)}\rho_K(v)^{n+p},$$

since $K \in S_{os}^n$, thus $\rho_K(v) = \rho_K(-v)$ for any $v \in S^{n-1}$, this gives $f(v)$ is a finite even Borel measure on S^{n-1}. From (4.12), we have

$$h_{\Gamma_p K}^p(u) = \frac{1}{(n+p)c_{n,p}}\int_{S^{n-1}} |\, u\cdot v \,|^p f(v)dv, \tag{4.13}$$

for any $u \in S^{n-1}$.

Similarly, if $L \in S^n_{os}$, then for any $u \in S^{n-1}$,

$$h^p_{\Gamma_p L}(u) = \frac{1}{(n+p)c_{n,p}} \int_{S^{n-1}} |\, u \cdot v \,|^p \, g(v) dv, \tag{4.14}$$

where

$$g(v) = \frac{1}{V(L)} \rho_L(v)^{n+p}$$

is also a finite even Borel measure on S^{n-1}.

Therefore, if $\Gamma_p K = \Gamma_p L$, then by (4.13) and (4.14), we obtain

$$\frac{1}{(n+p)c_{n,p}} \int_{S^{n-1}} |\, u \cdot v \,|^p \, [f(v) - g(v)] dv = 0. \tag{4.15}$$

Let $\mu(v) = f(v) - g(v)$, then (4.15) may be written as

$$\frac{1}{(n+p)c_{n,p}} \int_{S^{n-1}} |\, u \cdot v \,|^p \, \mu(v) dv = 0. \tag{4.16}$$

Notice that $\mu(v)$ is a continuous finite even Borel measure on S^{n-1}, therefore together with (4.16), we obtain $\mu(v) = 0$, i.e., $f(v) - g(v) = 0$. This show that for any $v \in S^{n-1}$,

$$\frac{1}{V(K)} \rho_K(v)^{n+p} = \frac{1}{V(L)} \rho_L(v)^{n+p}. \tag{4.17}$$

But we know for any $Q \in S^n_o$,

$$\frac{\widetilde{V}_{-p}(K,Q)}{V(K)} = \frac{1}{nV(K)} \int_{S^{n-1}} \rho_K(v)^{n+p} \rho_Q^{-p}(v) dv,$$

$$\frac{\widetilde{V}_{-p}(L,Q)}{V(L)} = \frac{1}{nV(L)} \int_{S^{n-1}} \rho_L(v)^{n+p} \rho_Q^{-p}(v) dv.$$

Hence, associated with (4.17), we have that for any $Q \in S^n_{os}$,

$$\frac{\widetilde{V}_{-p}(K,Q)}{V(K)} = \frac{\widetilde{V}_{-p}(L,Q)}{V(L)}. \qquad \square$$

Proof of Theorem 1.3. According to (1.4), we know

$$\frac{n^{\frac{p}{n}} \widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}}}{V(K)} = \inf \left\{ n \frac{\widetilde{V}_{-p}(K,Q^*)}{V(K)} V(Q)^{-\frac{p}{n}} : Q \in S^n_{os} \right\}. \tag{4.18}$$

Since $\Gamma_p K = \Gamma_p L$, thus from Lemma 4.3, we get for any $Q \in S^n_{os}$

$$\frac{\widetilde{V}_{-p}(K,Q^*)}{V(K)} = \frac{\widetilde{V}_{-p}(L,Q^*)}{V(L)}. \tag{4.19}$$

Thus from (4.18) and (4.19), we can get

$$\frac{\widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}}}{V(K)} = \frac{\widetilde{\Omega}_{-p}(L)^{\frac{n-p}{n}}}{V(L)},$$

i.e.,

$$\left(\frac{\widetilde{\Omega}_{-p}(K)}{\widetilde{\Omega}_{-p}(L)}\right)^{\frac{n-p}{n}} = \frac{V(K)}{V(L)}. \tag{4.20}$$

And since $L \in S^n_{os}$, thus taking $Q = L$ in (4.9), and associated with (2.12), we obtain

$$V(K) = \widetilde{V}_{-p}(K,L) \geqslant V(K)^{\frac{n+p}{n}} V(L)^{-\frac{p}{n}}, \tag{4.21}$$

i.e.,

$$V(K) \leqslant V(L). \tag{4.22}$$

Combining with (4.20) and (4.22), we get

$$\widetilde{\Omega}_{-p}(K) \leqslant \widetilde{\Omega}_{-p}(L) \tag{4.23}$$

According to the equality condition of (4.22), we see that equality holds in (4.23) if and only if $K = L$. $\square$

Proof of Theorem 1.4. Since L is not an origin-symmetric, so from Corollary 3.2, we know

$$\widetilde{\Omega}_{-p}(\widehat{\nabla}_p L) > \widetilde{\Omega}_{-p}(L).$$

Choose $\varepsilon > 0$, such that $\widetilde{\Omega}_{-p}((1-\varepsilon)\widehat{\nabla}_p L) > \widetilde{\Omega}_{-p}(L)$, thus let $K = (1-\varepsilon)\widehat{\nabla}_p L$, then

$$\widetilde{\Omega}_{-p}(K) > \widetilde{\Omega}_{-p}(L).$$

But from Lemma 4.2, and notice that $\Gamma_p(1-\varepsilon)K = (1-\varepsilon)\Gamma_p K$, we can get

$$\Gamma_p K = \Gamma_p(1-\varepsilon)\widehat{\nabla}_p L = (1-\varepsilon)\Gamma_p \widehat{\nabla}_p L = (1-\varepsilon)\Gamma_p L \subset \Gamma_p L. \quad \square$$

Acknowledgement. The authors wish to thank the referees for their very valuable and helpful comments and suggestions on the original version of this paper.

REFERENCES

[1] R. J. GARDNER, *Geometric Tomography*, Second ed., Cambridge Univ. Press, Cambridge, 2006.

[2] E. GRINBERG AND G. Y. ZHANG, *Convolutions transforms and convex bodies*, Pron. London Math. Soc., **78** (1999), 3: 77–115.

[3] E. LUTWAK AND G. Y. ZHANG, *Blaschke-Santaló inequalities*, J. Differential Geom., **47** (1997), 1: 1–16.

[4] E. LUTWAK, D. YANG AND G. Y. ZHANG, *L_p-affine isoperimetric inequalities*, J. Differential Geom., **56** (2000), 1: 111–132.

[5] E. LUTWAK, D. YANG AND G. Y. ZHANG, *The Cramer-Rao inequality for star bodies*, Duke. Math. J., **112** (2002), 1: 59–81.
[6] E. LUTWAK, *the Brunn-Minkowski-Firey theory II: affine and geominimal surface areas*, Adv. Math., **118** (1996), 2: 244–294.
[7] E. LUTWAK, *The Brunn-Minkowski-Firey theory I: mixed volumes and the minkowski problem*, J. Differential Geom., **38** (1993), 1: 131–150.
[8] R. SCHNEIDER, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge Univ., Press, Cambridge, 1993.
[9] C. SCHÜTT AND E. WERNER, *Surface bodies and p-affine surface area*, Adv. Math., **187** (2004), 98–145.
[10] W. D. WANG AND G. S. LENG, *L_p-mixed affine surface area*, J. Math. Anal. Appl., **335** (2007), 1: 341–354.
[11] W. D. WANG AND G. S. LENG, *Some affine isoperimetric inequalities associated with L_p-affine surface area*, Houston Journal of Mathematics, **34** (2008), 2: 443–453.
[12] W. D. WANG, F. H. LU AND G. S. LENG, *A type of monotonicity on the L_p-centroid body and L_p-projection body*, Mathematical inequalities & Application, **8** (2005), 4: 735–742.
[13] W. D. WANG AND G. S. LENG, *On the monotonicity of L_p-centroid body*, J. Sys. Sci. & Math. Scis., **28** (2008), 2: 154–162. (in chinese)
[14] W. WANG AND B. W. HE, *L_p-dual affine surface area*, J. Math. Anal. Appl., **348** (2008), 746–751.
[15] E. WERNER AND D. YE, *New L_p-affine isoperimetric inequalities*, Adv. Math., **218** (2008), 3: 762–780.
[16] E. WERNER AND D. YE, *Inequalities for mixed p-affine surface area*, Math. Ann., **347** (2010), 3: 703–737.
[17] J. YUAN, L. Z. ZHAO AND G. S. LENG, *Inequalities for L_p-centroid body*, Tai wan. J. Math., **29**(2008), 2: 209–220.

(Received August 15, 2012)

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