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TWO NEW FORMS OF HILBERT INTEGRAL INEQUALITY

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Abstract. In this paper we obtain two new forms of the Hilbert inequality. The first form is a sharper form of the classical Hilbert inequality and is connected to Hardy inequality. In the second one we introduce a differential form of Hilbert inequality.

1. Introduction

Let $f(x)$, $g(y) > 0$, $0 < \int_0^\infty f^p(x)dx < \infty$, and $0 < \int_0^\infty g^q(y)dy < \infty$, then the Hardy-Hilbert's inequality may be written as

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x)dx \right\}^{\frac{1}{q}} \tag{1.1}$$

where the constant $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ is the best possible [see 2].

Recently, many generalizations of (1.1) were given. Yang et al. [9] obtained the following extension of (1.1) as

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy < B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left\{ \int_0^\infty x^{p-1-\lambda} f^p(x)dt \right\}^{\frac{1}{p}} \left\{ \int_0^\infty y^{q-1-\lambda} g^q(y)dt \right\}^{\frac{1}{q}}, \tag{1.2}$$

where $\lambda > 0$ and the constant $B(\frac{\lambda}{p}, \frac{\lambda}{q})$ (the Beta function) is the best possible. The following general inequality was given in [5]

$$\int_0^\infty \int_0^\infty K(x,y)f(x)g(y)dxdy < k(pA_2) \left\{ \int_0^\infty x^{pqA_1-1} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty y^{pqA_2-1} g^q(x)dx \right\}^{\frac{1}{q}},$$

where $k(pA_2) = \int_0^\infty K(1,t)t^{-pA_2}dt$ is the best possible constant, $K(x,y) \geqslant 0$ is a homogeneous function of degree $-\lambda$ $(\lambda > 0)$, $A_1 \in (\frac{1-\lambda}{q}, \frac{1}{q})$, $A_2 \in (\frac{1-\lambda}{p}, \frac{1}{p})$ and $pA_2 + qA_1 = 2 - \lambda$.

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Suppose p and q are real numbers, such that $p>1$, $q>1$, $\frac{1}{p}+\frac{1}{q}\geqslant 1$ and let p' and q' be the conjugate exponents of p and q respectively, that is $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$. Let $\lambda=\frac{1}{p}+\frac{1}{q}$, then $0<\lambda\leqslant 1$. Such parameters p and q are called non-conjugate exponents. Considering p, q and λ as above, Hardy, Littlewood, and Polya obtained in [2] an extension of (1.1). Recently, A. Čižmešija et. al. introduced in [1] an unified treatment of the general Hilbert-type inequalities extended to the case of non-conjugate exponents. Refinements of some Hilbert-type inequalities by virtue of various methods is obtained in [6], [7] and [8]. A survey of some recent results concerning Hilbert and Hilbert- type inequalities can be found in [10].

If $p>1$, $f(x)>0$, and $F(x)=\int_0^x f(t)dt$, then the famous Hardy inequality [2] is given as

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx<\left(\frac{p}{p-1}\right)^p\int_0^\infty f^p(x)dx, \tag{1.3}$$

the constant $\left(\frac{p}{p-1}\right)^p$ is the best possible. A weighted form of (1.3) is giving also by Hardy [2] as

$$\int_0^\infty x^a\left(\frac{F(x)}{x}\right)^p dx<\left(\frac{p}{p-1-a}\right)^p\int_0^\infty x^a f^p(x)dx, \tag{1.4}$$

where $a<p-1$ and the constant $\left(\frac{p}{p-1-a}\right)^p$ is the best possible. Inequality (1.3) was discovered by Hardy while he was trying to introduce a simple proof of Hilbert inequality. For more information about inequality (1.3) and its history and development we refer the reader to the papers [3] and [4].

In this paper by estimating the double integral $\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy$, we introduce two new inequalities with a best constant factor, the first one contained in Theorem 1 gives a relation between Hardy inequality and Hilbert inequality, the second inequality contained in Theorem 2 gives a differential form of Hilbert inequality.

2. Preliminaries and Lemmas

Recall that the Gamma function $\Gamma(\theta)$ and the Beta function $B(\mu,\nu)$ are defined respectively by

$$\Gamma(\theta)=\int_0^\infty t^{\theta-1}e^{-t}dt,\quad \theta>0,$$

$$B(\mu,\nu)=\int_0^\infty \frac{t^{\mu-1}}{(t+1)^{\mu+\nu}}dt,\quad \mu,\ \nu>0.$$

LEMMA 2.1. *Let* $r > 1,\ \frac{1}{r}+\frac{1}{s}=1,\ \varphi > 0,\ \varphi \in L(0,\infty),\ \Phi(x)=\int_0^x \varphi(u)du$, *then for* $t,\ \alpha > 0$ *we have*

$$\int_0^\infty e^{-tx}\varphi(x)dx \leqslant t^{\frac{1}{r}-\alpha}\Gamma(\alpha s+1)^{\frac{1}{s}}\left\{\int_0^\infty x^{-\alpha r}e^{-tx}\Phi^r(x)dx\right\}^{\frac{1}{r}}. \tag{2.1}$$

Proof. Using integration by parts, we get

$$\int_0^\infty e^{-tx}\varphi(x)dx = t\int_0^\infty e^{-tx}\Phi(x)dx. \tag{2.2}$$

Applying Hölder inequality, we obtain

$$\begin{aligned}\int_0^\infty e^{-tx}\Phi(x)dx &= \int_0^\infty \left[x^{\alpha}e^{-\frac{tx}{s}}\right]\left[x^{-\alpha}e^{-\frac{tx}{r}}\Phi(x)\right]dx\\ &\leqslant \left(\int_0^\infty x^{\alpha s}e^{-tx}dx\right)^{\frac{1}{s}}\left(\int_0^\infty x^{-\alpha r}e^{-tx}\Phi^r(x)dx\right)^{\frac{1}{r}}\\ &= t^{\frac{-1}{s}-\alpha}\Gamma(\alpha s+1)^{\frac{1}{s}}\left(\int_0^\infty x^{-\alpha r}e^{-tx}\Phi^r(x)dx\right)^{\frac{1}{r}}.\end{aligned}$$

Substituting the last inequality in (2.2) we get (2.1). □

LEMMA 2.2. *Let* $r > 1,\ \frac{1}{r}+\frac{1}{s}=1,\ \varphi > 0$, *the derivatives* $\varphi', \varphi'', \ldots, \varphi^{(n)}$ *exists and positive and* $\varphi^{(n)} \in L(0,\infty)\ \ (n=0,1,\ldots)\ \left(\varphi^{(0)} := \varphi\right)$, *moreover, suppose that* $\varphi(0)=\varphi'(0)=\cdots=\varphi^{(n-1)}=0$, *then for* $t,\ \alpha > 0$ *we have*

$$\int_0^\infty e^{-tx}\varphi(x)dx \leqslant t^{-n-\frac{1}{r}-\alpha}\Gamma(\alpha s+1)^{\frac{1}{s}}\left\{\int_0^\infty x^{-\alpha r}e^{-tx}\left(\varphi^{(n)}(x)\right)^r dx\right\}^{\frac{1}{r}}. \tag{2.3}$$

Proof. Using integration by parts n times, we get

$$\int_0^\infty e^{-tx}\varphi(x)dx = \frac{1}{t}\int_0^\infty e^{-tx}\varphi'(x)dx = \cdots = \frac{1}{t^n}\int_0^\infty e^{-tx}\varphi^{(n)}(x)dx. \tag{2.4}$$

Applying Hölder inequality, we obtain

$$\begin{aligned}\int_0^\infty e^{-tx}\varphi^{(n)}(x)dx &= \int_0^\infty \left[x^{\alpha}e^{-\frac{tx}{s}}\right]\left[x^{-\alpha}e^{-\frac{tx}{r}}\varphi^{(n)}(x)\right]dx \\ &\leqslant \left(\int_0^\infty x^{\alpha s}e^{-tx}dx\right)^{\frac{1}{s}}\left(\int_0^\infty x^{-\alpha r}e^{-tx}\left(\varphi^{(n)}(x)\right)^r dx\right)^{\frac{1}{r}} \\ &= t^{\frac{-1}{s}-\alpha}\Gamma(\alpha s+1)^{\frac{1}{s}}\left(\int_0^\infty x^{-\alpha r}e^{-tx}\left(\varphi^{(n)}(x)\right)^r dx\right)^{\frac{1}{r}}.\end{aligned}$$

Substituting the last inequality in (2.4) we get (2.3). □

By the definition of the Gamma function, the following equality holds

$$\frac{1}{(x+y)^{\lambda}} = \frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-(x+y)t}dt. \tag{2.5}$$

3. Main Results

In this section, we introduce the main two results in this paper. Theorem 3.1 gives a new form of the Hilbert inequality which is sharper than the classical Hilbert inequality and is related to another classical inequality, that is, the famous Hardy inequality. In Theorem 3.2 we introduce another new form of the Hilbert inequality, namely, a differential form which is an extension of (1.2). Both of the obtained inequalities are with a best constant factor.

THEOREM 3.1. *Let* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda>0$, $f,\ g>0$, $f,\ g\in L(0,\infty)$, *define* $F(x)=\int_0^x f(u)du$ *and* $G(x)=\int_0^x g(u)du$. *If* $\int_0^\infty x^{-\lambda-1}F^p(x)dx<\infty$ *and* $\int_0^\infty y^{-\lambda-1}G^q(y)dy<\infty$, *then*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^{\lambda}}dxdy \leqslant C\left(\int_0^\infty x^{-\lambda-1}F^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{-\lambda-1}G^q(y)dy\right)^{\frac{1}{q}}, \tag{3.1}$$

where the constant $C=\frac{\lambda^2}{pq}B\left(\frac{\lambda}{q},\frac{\lambda}{p}\right)$ *is the best possible. In particular for* $\lambda=1$, $p=q=2$

$$\int_0^\infty\int_0^\infty \frac{f(x)f(y)}{x+y}dxdy \leqslant \frac{\pi}{4}\int_0^\infty\left(\frac{F(x)}{x}\right)^2 dx.$$

Proof. By using (2.5) and applying Hölder inequality, we have

$$\begin{aligned} I =: & \int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy \\ &= \frac{1}{\Gamma(\lambda)}\int_0^\infty\int_0^\infty f(x)g(y)\left(\int_0^\infty t^{\lambda-1}e^{-(x+y)t}dt\right)dxdy \\ &= \frac{1}{\Gamma(\lambda)}\int_0^\infty\left(t^{\frac{\lambda-1}{p}}\int_0^\infty e^{-xt}f(x)dx\right)\left(t^{\frac{\lambda-1}{q}}\int_0^\infty e^{-yt}g(y)dy\right)dt \\ &\leqslant \frac{1}{\Gamma(\lambda)}\left(\int_0^\infty t^{\lambda-1}\left(\int_0^\infty e^{-xt}f(x)dx\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^\infty t^{\lambda-1}\left(\int_0^\infty e^{-yt}g(y)dy\right)^q dt\right)^{\frac{1}{q}}. \end{aligned} \tag{3.2}$$

By Lemma 1.1 for $r=p,\ s=q,\ \alpha=\frac{\lambda}{pq}$ and then for $r=q,\ s=p,\ \alpha=\frac{\lambda}{pq}$, we obtain respectively,

$$\left(\int_0^\infty e^{-xt}f(x)dx\right)^p \leqslant t^{1-\frac{\lambda}{q}}\Gamma\left(\frac{\lambda}{p}+1\right)^{\frac{p}{q}}\int_0^\infty x^{-\frac{\lambda}{q}}e^{-tx}F^p(x)dx$$

$$\left(\int_0^\infty e^{-yt}g(y)dy\right)^q \leqslant t^{1-\frac{\lambda}{p}}\Gamma\left(\frac{\lambda}{q}+1\right)^{\frac{q}{p}}\int_0^\infty y^{-\frac{\lambda}{p}}e^{-ty}G^q(y)dy.$$

Substituting these two inequalities in (3.2) we have

$$\begin{aligned} I \leqslant & \frac{\Gamma(\frac{\lambda}{p}+1)^{\frac{1}{q}}\Gamma(\frac{\lambda}{q}+1)^{\frac{1}{p}}}{\Gamma(\lambda)}\left(\int_0^\infty x^{-\frac{\lambda}{q}}F^p(x)\left(\int_0^\infty t^{\frac{\lambda}{p}}e^{-xt}dt\right)dx\right)^{\frac{1}{p}} \\ & \times\left(\int_0^\infty y^{-\frac{\lambda}{p}}G^q(y)\left(\int_0^\infty t^{\frac{\lambda}{q}}e^{-yt}dt\right)dy\right)^{\frac{1}{q}}. \end{aligned}$$

Since $\int_0^\infty t^{\frac{\lambda}{p}}e^{-xt}dt = x^{\frac{-\lambda}{p}-1}\Gamma\left(\frac{\lambda}{p}+1\right)$ and $\int_0^\infty t^{\frac{\lambda}{q}}e^{-yt}dt = y^{-\frac{\lambda}{q}-1}\Gamma\left(\frac{\lambda}{q}+1\right)$, we find

$$I \leqslant C\left(\int_0^\infty x^{-\lambda-1}F^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{-\lambda-1}G^q(y)dy\right)^{\frac{1}{q}},$$

where the constant $C=\frac{\Gamma(\frac{\lambda}{p}+1)^{\frac{1}{q}}\Gamma(\frac{\lambda}{q}+1)^{\frac{1}{p}}\Gamma\left(\frac{\lambda}{p}+1\right)^{\frac{1}{p}}\Gamma\left(\frac{\lambda}{q}+1\right)^{\frac{1}{q}}}{\Gamma(\lambda)}=\frac{\lambda^2}{pq}B\left(\frac{\lambda}{q},\frac{\lambda}{p}\right)$, here we use the following formulas for the gamma function: $\Gamma(u+1)=u\Gamma(u)$ and $\frac{\Gamma(\frac{\lambda}{p})\Gamma(\frac{\lambda}{q})}{\Gamma(\lambda)}=$

$B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)$. Inequality (3.1) is proved. We need to show that the constant factor $\frac{\lambda^2}{pq}B\left(\frac{\lambda}{q},\frac{\lambda}{p}\right)$ contained in (3.1) is the best possible. To do that we define two functions $f_\varepsilon(x)=\frac{\lambda-\varepsilon}{p}x^{\frac{\lambda-\varepsilon}{p}-1}$ and $g_\varepsilon(x)=\frac{\lambda-\varepsilon}{q}x^{\frac{\lambda-\varepsilon}{q}-1}$ for $x\geqslant 1\,(0<\varepsilon<\lambda)$ and $f_\varepsilon\ (x)=g_\varepsilon(x)=0$ for $x\in(0,1)$. Then, we get $F_\varepsilon(x)=x^{\frac{\lambda-\varepsilon}{p}}-1$ and $G_\varepsilon(x)=x^{\frac{\lambda-\varepsilon}{q}}-1$ for $x\geqslant 1$, $F_\varepsilon\ (x)=\ G_\varepsilon(x)=0$ for $x\in(0,1)$. Suppose that $\frac{\lambda^2}{pq}B\left(\frac{\lambda}{q},\frac{\lambda}{p}\right)$ is not the best possible, then there exist $0<K<\frac{\lambda^2}{pq}B\left(\frac{\lambda}{q},\frac{\lambda}{p}\right)$ such that

$$\begin{aligned}
\int_1^\infty\int_1^\infty \frac{f_\varepsilon(x)g_\varepsilon(y)}{(x+y)^\lambda}dxdy &< K\left(\int_1^\infty x^{-\lambda-1}F_\varepsilon^p(x)dx\right)^{\frac{1}{p}}\left(\int_1^\infty y^{-\lambda-1}G_\varepsilon^q(y)dy\right)^{\frac{1}{q}}\\
&= K\left(\int_1^\infty x^{-\lambda-1}\left(x^{\frac{\lambda-\varepsilon}{p}}-1\right)^p dx\right)^{\frac{1}{p}}\left(\int_1^\infty y^{-\lambda-1}\left(x^{\frac{\lambda-\varepsilon}{q}}-1\right)^q dy\right)^{\frac{1}{q}}\\
&< \left(\int_1^\infty x^{-\lambda-1}\left(x^{\frac{\lambda-\varepsilon}{p}}\right)^p dx\right)^{\frac{1}{p}}\left(\int_1^\infty y^{-\lambda-1}\left(x^{\frac{\lambda-\varepsilon}{q}}\right)^q dy\right)^{\frac{1}{q}}=\frac{K}{\varepsilon}
\end{aligned}\tag{3.3}$$

On the other hand, we have

$$\begin{aligned}
\int_1^\infty\int_1^\infty \frac{f_\varepsilon(x)g_\varepsilon(y)}{(x+y)^\lambda}dxdy &= \frac{(\lambda-\varepsilon)^2}{pq}\int_1^\infty\int_1^\infty \frac{x^{\frac{\lambda-\varepsilon}{p}-1}y^{\frac{\lambda-\varepsilon}{q}-1}}{(x+y)^\lambda}dxdy\\
&= \frac{(\lambda-\varepsilon)^2}{pq}\int_1^\infty x^{-\varepsilon-1}\left\{\int_{\frac{1}{x}}^\infty \frac{u^{\frac{\lambda-\varepsilon}{q}-1}}{(u+1)^\lambda}du\right\}dx\\
&= \frac{(\lambda-\varepsilon)^2}{pq}\int_1^\infty x^{-\varepsilon-1}\left\{\int_0^\infty \frac{u^{\frac{\lambda-\varepsilon}{q}-1}}{(u+1)^\lambda}du-\int_0^{\frac{1}{x}} \frac{u^{\frac{\lambda-\varepsilon}{q}-1}}{(u+1)^\lambda}du\right\}dx\\
&= \frac{(\lambda-\varepsilon)^2}{pq}\frac{B\left(\frac{\lambda-\varepsilon}{q},\frac{\lambda-\varepsilon}{p}\right)}{\varepsilon}-\frac{(\lambda-\varepsilon)^2}{pq}\int_1^\infty x^{-\varepsilon-1}\int_0^{\frac{1}{x}}\frac{u^{\frac{\lambda-\varepsilon}{q}-1}}{(u+1)^\lambda}dudx\\
&> \frac{(\lambda-\varepsilon)^2}{pq}\frac{B\left(\frac{\lambda-\varepsilon}{q},\frac{\lambda-\varepsilon}{p}\right)}{\varepsilon}-\frac{(\lambda-\varepsilon)^2}{pq}\int_1^\infty x^{-\varepsilon-1}\int_0^{\frac{1}{x}}u^{\frac{\lambda-\varepsilon}{q}-1}dudy\\
&= \frac{(\lambda-\varepsilon)^2}{pq}\frac{B\left(\frac{\lambda-\varepsilon}{q},\frac{\lambda-\varepsilon}{p}\right)}{\varepsilon}-O(1).
\end{aligned}\tag{3.4}$$

Clearly, when $\varepsilon \to 0^+$ from (3.3) and (3.4) we obtain a contradiction. Thus the proof of the theorem is completed. $\square$

THEOREM 3.2. *If $f,\ g>0,\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ \lambda > n\max(p,q),\ n=0,1,...,$ and $f,\ g$ satisfies the conditions of Lemma 2.2 such that $\int_0^\infty x^{p(n+1)-\lambda-1} f^{(n)}(x)^p dx<\infty$, $\int_0^\infty y^{q(n+1)-\lambda-1} g^{(n)}(y)^q dy<\infty$, then:*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy \leqslant C\left(\int_0^\infty x^{p(n+1)-\lambda-1}\left(f^{(n)}(x)\right)^p dx\right)^{\frac{1}{p}} \times \left(\int_0^\infty y^{q(n+1)-\lambda-1}\left(g^{(n)}(y)\right)^q dy\right)^{\frac{1}{q}}, \tag{3.5}$$

where the constant factor $C=\frac{\Gamma\left(\frac{\lambda}{p}-n\right)\Gamma\left(\frac{\lambda}{q}-n\right)}{\Gamma(\lambda)}$ is the best possible. In particular for $n=1,\ \lambda=3,\ p=q=2$

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^3}dxdy \leqslant \frac{\pi}{2}\left(\int_0^\infty f'(x)^2dx\right)^{\frac{1}{2}}\left(\int_0^\infty g'(y)^2dy\right)^{\frac{1}{2}}.$$

Proof. Using (2.5) and applying Hölder inequality, we get

$$\begin{aligned} I =: &\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy \\ &= \frac{1}{\Gamma(\lambda)}\int_0^\infty\int_0^\infty f(x)g(y)\left(\int_0^\infty t^{\lambda-1}e^{-(x+y)t}dt\right)dxdy \\ &= \frac{1}{\Gamma(\lambda)}\int_0^\infty\left(t^{\frac{\lambda-1}{p}}\int_0^\infty e^{-xt}f(x)dx\right)\left(t^{\frac{\lambda-1}{q}}\int_0^\infty e^{-yt}g(y)dy\right)dt. \\ &\leqslant \frac{1}{\Gamma(\lambda)}\left(\int_0^\infty t^{\lambda-1}\left(\int_0^\infty e^{-xt}f(x)dx\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^\infty t^{\lambda-1}\left(\int_0^\infty e^{-yt}g(y)dy\right)^q dt\right)^{\frac{1}{q}}. \end{aligned} \tag{3.6}$$

By Lemma 2.2 for $r=p,\ s=q$, $\alpha=\frac{\lambda-p(n+1)}{pq}$ and then for $r=q,\ s=p$, $\alpha=$

$\frac{\lambda-q(n+1)}{pq}$ we obtain respectively,

$$\left(\int_0^\infty e^{-xt} f(x)dx\right)^p \leqslant t^{-n-\frac{\lambda}{q}} \Gamma\Big(\frac{\lambda}{p}-n\Big)^{\frac{p}{q}} \int_0^\infty x^{-\frac{\lambda-p(n+1)}{q}} e^{-tx} \left(f^{(n)}(x)\right)^p dx$$

$$\left(\int_0^\infty e^{-yt} g(y)dy\right)^q \leqslant t^{-n-\frac{\lambda}{p}} \Gamma\Big(\frac{\lambda}{q}-n\Big)^{\frac{q}{p}} \int_0^\infty y^{-\frac{\lambda-q(n+1)}{p}} e^{-ty} \left(g^{(n)}(y)\right)^q dy.$$

Substituting these two inequalities in (3.6) we have

$$I \leqslant \frac{\Gamma(\frac{\lambda}{p}-n)^{\frac{1}{q}} \Gamma(\frac{\lambda}{q}-n)^{\frac{1}{p}}}{\Gamma(\lambda)} \left(\int_0^\infty x^{-\frac{\lambda-p(n+1)}{q}} \left(f^{(n)}(x)\right)^p \left(\int_0^\infty t^{\frac{\lambda}{p}-n-1} e^{-xt} dt\right) dx\right)^{\frac{1}{p}}$$

$$\times \left(\int_0^\infty y^{-\frac{\lambda-q(n+1)}{p}} \left(g^{(n)}(y)\right)^q \left(\int_0^\infty t^{\frac{\lambda}{q}-n-1} e^{-yt} dt\right) dy\right)^{\frac{1}{q}}.$$

$$= C \left(\int_0^\infty x^{p(n+1)-\lambda-1} \left(f^{(n)}(x)\right)^p dx\right)^{\frac{1}{p}} \left(\int_0^\infty y^{q(n+1)-\lambda-1} \left(g^{(n)}(y)\right)^q dy\right)^{\frac{1}{q}},$$

where the constant $C = \frac{\Gamma(\frac{\lambda}{p}-n)^{\frac{1}{q}} \Gamma(\frac{\lambda}{q}-n)^{\frac{1}{p}} \Gamma(\frac{\lambda}{p}-n)^{\frac{1}{p}} \Gamma(\frac{\lambda}{q}-n)^{\frac{1}{q}}}{\Gamma(\lambda)} = \frac{\Gamma\left(\frac{\lambda}{p}-n\right)\Gamma\left(\frac{\lambda}{q}-n\right)}{\Gamma(\lambda)}$. Inequality (3.5) is proved. We need to show that the constant factor $\frac{\Gamma\left(\frac{\lambda}{p}-n\right)\Gamma\left(\frac{\lambda}{q}-n\right)}{\Gamma(\lambda)}$ contained in (3.5) is the best possible. Define two functions $f_\varepsilon(x) = \frac{x^{\frac{\lambda-\varepsilon}{p}-1}}{\left(\frac{\lambda-\varepsilon}{p}-n\right)_n} = \frac{\Gamma\left(\frac{\lambda-\varepsilon}{p}-n\right)}{\Gamma\left(\frac{\lambda-\varepsilon}{p}\right)} x^{\frac{\lambda-\varepsilon}{p}-1}$ and $g_\varepsilon(x) = \frac{x^{\frac{\lambda-\varepsilon}{q}-1}}{\left(\frac{\lambda-\varepsilon}{q}-n\right)_n} = \frac{\Gamma\left(\frac{\lambda-\varepsilon}{q}-n\right)}{\Gamma\left(\frac{\lambda-\varepsilon}{q}\right)} x^{\frac{\lambda-\varepsilon}{q}-1}$ for $x \geqslant 1\,(0<\varepsilon<\lambda)$ and $f_\varepsilon(x) = g_\varepsilon(x) = 0$ for $x \in (0,1)$, where $(\gamma)_r = \gamma(\gamma+1)\ldots(\gamma+r-1) = \frac{\Gamma(\gamma+r)}{\Gamma(\gamma)}$ is the Pochhammer symbol. Therefore, we find $f_\varepsilon{}^{(n)}(x) = x^{\frac{\lambda-\varepsilon}{p}-n-1}$ and $g_\varepsilon^{(n)}(x) = x^{\frac{\lambda-\varepsilon}{q}-n-1}$ for $x > 1$. Suppose that $\frac{\Gamma\left(\frac{\lambda}{p}-n\right)\Gamma\left(\frac{\lambda}{q}-n\right)}{\Gamma(\lambda)}$ is not the best possible, then there exist $0 < K < \frac{\Gamma\left(\frac{\lambda}{p}-n\right)\Gamma\left(\frac{\lambda}{q}-n\right)}{\Gamma(\lambda)}$ such that

$$\int_1^\infty \int_1^\infty \frac{f_\varepsilon(x) g_\varepsilon(x)}{(x+y)^\lambda} dxdy < K \left(\int_1^\infty x^{p(n+1)-\lambda-1} \left(f_\varepsilon{}^{(n)}(x)\right)^p dx\right)^{\frac{1}{p}}$$

$$\times \left(\int_1^\infty y^{q(n+1)-\lambda-1} \left(g_\varepsilon^{(n)}(y)\right)^q dy\right)^{\frac{1}{q}} = \frac{K}{\varepsilon}. \qquad (3.7)$$

On the other hand, we have

$$
\begin{aligned}
&\int_1^\infty\int_1^\infty \frac{f_\varepsilon(x)g_\varepsilon(x)}{(x+y)^\lambda}dxdy \\
&= \frac{\Gamma\left(\frac{\lambda-\varepsilon}{p}-n\right)\Gamma\left(\frac{\lambda-\varepsilon}{q}-n\right)}{\Gamma\left(\frac{\lambda-\varepsilon}{p}\right)\Gamma\left(\frac{\lambda-\varepsilon}{q}\right)}\int_1^\infty\int_1^\infty \frac{x^{\frac{\lambda-\varepsilon}{p}-1}y^{\frac{\lambda-\varepsilon}{q}-1}}{(x+y)^\lambda}dxdy \\
&= \frac{\Gamma\left(\frac{\lambda-\varepsilon}{p}-n\right)\Gamma\left(\frac{\lambda-\varepsilon}{q}-n\right)}{\Gamma\left(\frac{\lambda-\varepsilon}{p}\right)\Gamma\left(\frac{\lambda-\varepsilon}{q}\right)}\int_1^\infty x^{-\varepsilon-1}\left\{\int_{\frac{1}{x}}^\infty \frac{u^{\frac{\lambda-\varepsilon}{q}-1}}{(u+1)^\lambda}du\right\}dx \\
&= \frac{\Gamma\left(\frac{\lambda-\varepsilon}{p}-n\right)\Gamma\left(\frac{\lambda-\varepsilon}{q}-n\right)}{\Gamma\left(\frac{\lambda-\varepsilon}{p}\right)\Gamma\left(\frac{\lambda-\varepsilon}{q}\right)}\int_1^\infty x^{-\varepsilon-1}\left\{\int_0^\infty \frac{u^{\frac{\lambda-\varepsilon}{q}-1}}{(u+1)^\lambda}du-\int_0^{\frac{1}{x}} \frac{u^{\frac{\lambda-\varepsilon}{q}-1}}{(u+1)^\lambda}du\right\}dx \\
&= \frac{\Gamma\left(\frac{\lambda-\varepsilon}{p}-n\right)\Gamma\left(\frac{\lambda-\varepsilon}{q}-n\right)}{\Gamma\left(\frac{\lambda-\varepsilon}{p}\right)\Gamma\left(\frac{\lambda-\varepsilon}{q}\right)}\left[\frac{B\left(\frac{\lambda-\varepsilon}{q},\frac{\lambda-\varepsilon}{p}\right)}{\varepsilon}-\int_1^\infty x^{-\varepsilon-1}\int_0^{\frac{1}{x}} \frac{u^{\frac{\lambda-\varepsilon}{q}-1}}{(u+1)^\lambda}dudx\right] \\
&> \frac{\Gamma\left(\frac{\lambda-\varepsilon}{p}-n\right)\Gamma\left(\frac{\lambda-\varepsilon}{q}-n\right)}{\Gamma\left(\frac{\lambda-\varepsilon}{p}\right)\Gamma\left(\frac{\lambda-\varepsilon}{q}\right)}\left[\frac{B\left(\frac{\lambda-\varepsilon}{q},\frac{\lambda-\varepsilon}{p}\right)}{\varepsilon}-\int_1^\infty x^{-\varepsilon-1}\int_0^{\frac{1}{x}} u^{\frac{\lambda-\varepsilon}{q}-1}dudy\right] \\
&= \frac{\Gamma\left(\frac{\lambda-\varepsilon}{p}-n\right)\Gamma\left(\frac{\lambda-\varepsilon}{q}-n\right)}{\Gamma\left(\frac{\lambda-\varepsilon}{p}\right)\Gamma\left(\frac{\lambda-\varepsilon}{q}\right)}\left[\frac{B\left(\frac{\lambda-\varepsilon}{q},\frac{\lambda-\varepsilon}{p}\right)}{\varepsilon}-O(1)\right]. \qquad (3.8)
\end{aligned}
$$

Let $\varepsilon \to 0^+$, then by (3.7) and (3.8) we have

$$
K \geqslant \frac{\Gamma\left(\frac{\lambda}{p}-n\right)\Gamma\left(\frac{\lambda}{q}-n\right)}{\Gamma\left(\frac{\lambda}{p}\right)\Gamma\left(\frac{\lambda}{q}\right)}B\left(\frac{\lambda}{q},\frac{\lambda}{p}\right) = \frac{\Gamma\left(\frac{\lambda}{p}-n\right)\Gamma\left(\frac{\lambda}{q}-n\right)}{\Gamma(\lambda)}.
$$

The Theorem is proved. □

REMARK. If we apply the weighted Hardy inequality (1.4) to (3.1) we get Yang's inequality (1.2). Also if we put $n=0$ in (3.5) we get (1.2).

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