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WEIGHTED FORM OF A RECENT REFINEMENT OF THE DISCRETE JENSEN'S INEQUALITY

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Abstract. Recently, Xiao, Srivastava and Zhang (see [10]) have introduced a new refinement of the discrete Jensen's inequality for mid-convex functions. We give and discuss the weighted form of their results. This leads to some new inequlities and limit formulas. We illustrate the scope of the results by applying them to introduce and study some new quasi-arithmetic means.

1. Introduction and the main results

The different forms of Jensen's inequality have fundamental importance for many developments in mathematics. In this paper we consider the discrete Jensen's inequality:

THEOREM A. (see [1]) *Let C be a convex subset of a real vector space X, and $\{x_1,\dots,x_n\}$ be a finite subset of C, where $n \geqslant 1$ is fixed. Let $p_1,\dots,p_n$ be nonnegative numbers with $P_n := \sum_{j=1}^{n} p_j > 0$. If $f : C \to \mathbb{R}$ is either a convex or a mid-convex function and in the latter case the numbers p_j $(1 \leqslant j \leqslant n)$ are rational, then*

$$f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right) \leqslant \frac{1}{P_n}\sum_{j=1}^{n} p_j f(x_j). \tag{1}$$

The function $f : C \to \mathbb{R}$ is called convex if

$$f(\alpha x + (1-\alpha)y) \leqslant \alpha f(x) + (1-\alpha) f(y), \quad x,y \in C, \quad 0 \leqslant \alpha \leqslant 1,$$

and mid-convex if

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{2} f(x) + \frac{1}{2} f(y), \quad x,y \in C.$$

We denote by $\mathbb{N}$ and $\mathbb{N}_+$ the set of nonnegative integers, and positive integers, respectively.

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Discrete distribution: this means that $p_1,\ldots,p_n$ are nonnegative numbers with $\sum_{j=1}^{n} p_j = 1$.

Recently, Xiao, Srivastava and Zhang (see [10]) have introduced a new refinement of the discrete Jensen's inequality for mid-convex functions. Their results were motivated by the reformulation of the classical refinements in the papers [8] and [9]. As an illustration, consider the next inequality from [9]:

THEOREM B. *Let C be a convex subset of a real vector space X, and let $f : C \to \mathbb{R}$ be a mid-convex function. If $x_i \in C$ $(i = 1,\ldots,n)$, and*

$$\bar{B}_{k,n} := \frac{1}{\binom{n+k-1}{k}} \sum_{1 \leqslant i_1 \leqslant \ldots \leqslant i_k \leqslant n} f\left(\frac{x_{i_1} + \ldots + x_{i_k}}{k}\right), \quad k \in \mathbb{N}_+,$$

then

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leqslant \ldots \leqslant \bar{B}_{k+1,n} \leqslant \bar{B}_{k,n} \leqslant \ldots \leqslant \bar{B}_{1,n} = \frac{1}{n}\sum_{i=1}^{n} f(x_i).$$

The expression $\bar{B}_{k,n}$ can be rewritten in the form

$$\bar{B}_{k,n} = \frac{1}{\binom{n+k-1}{k}} \sum_{\substack{i_1+\ldots+i_n=k \\ i_j \in \mathbb{N};\quad 1 \leqslant j \leqslant n}} f\left(\frac{1}{k}\sum_{j=1}^{k} i_j x_j\right).$$

Inspired by this interpretation of $\bar{B}_{k,n}$, Xiao, Srivastava and Zhang have obtained the following result:

THEOREM C. (see [10]) *Let C be a convex subset of a real vector space X, and $\{x_1,\ldots,x_n\}$ be a finite subset of C, where $n \geqslant 1$ is fixed. If $f : C \to \mathbb{R}$ is a mid-convex function, and*

$$F_{k,n} := \frac{1}{\binom{n+k-2}{k-1}} \sum_{\substack{i_1+\ldots+i_n=n+k-1 \\ i_j \in \mathbb{N}_+;\quad 1 \leqslant j \leqslant n}} f\left(\frac{1}{n+k-1}\sum_{j=1}^{n} i_j x_j\right), \quad k \in \mathbb{N}_+, \tag{2}$$

then

(a) $f\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right) = F_{1,n} \leqslant \ldots \leqslant F_{k,n} \leqslant F_{k+1,n} \leqslant \ldots \leqslant \frac{1}{n}\sum_{j=1}^{n} f(x_j).$

(b) $F_{k,n} \leqslant \bar{B}_{k,n}, \quad k \in \mathbb{N}_+.$

The limit of the constructed increasing sequence is also determined. We recall this result too:

THEOREM D. (see [10]) *Let C be a convex subset of a real vector space X, and $\{x_1,\ldots,x_n\}$ be a finite subset of C, where $n \geqslant 1$ is fixed. Suppose $f : C \to \mathbb{R}$ is a mid-convex function. Define the function g on the set*

$$E_n := \left\{(t_1,\ldots,t_{n-1}) \in \mathbb{R}^{n-1} \mid \sum_{j=1}^{n-1} t_j \leqslant 1, \quad t_j \geqslant 0, \quad j = 1,\ldots,n-1\right\} \tag{3}$$

by

$$g(t_1,\ldots,t_{n-1}) := f\left(\sum_{j=1}^{n-1} t_j x_j + \left(1-\sum_{j=1}^{n-1} t_j\right) x_n\right).$$

If g is integrable over E_n, then

$$\lim_{k\to\infty} F_{k,n} = \lim_{k\to\infty} \bar{B}_{k,n} = (n-1)! \int_{E_n} g(t_1,\ldots,t_{n-1})\, dt_1 \ldots dt_{n-1}.$$

In Theorem C and D the discrete uniform distribution is used. The purpose of this paper to give and discuss the weighted versions of Theorem C and D for convex and mid-convex functions.

There are many papers on the refinements of the discrete Jensen's inequality. In particular, the weighted version of Theorem B has been discovered by Horváth and Pečarić [4]:

THEOREM E. *Let C be a convex subset of a real vector space X, and $\{x_1,\ldots,x_n\}$ be a finite subset of C, where $n \geqslant 1$ is fixed. Let $p_1,\ldots,p_n$ be a discrete distribution, where p_j $(1 \leqslant j \leqslant n)$ is positive. If $f : C \to \mathbb{R}$ is either a convex or a mid-convex function and in the latter case the numbers p_j $(1 \leqslant j \leqslant n)$ are rational, and*

$$B_{k,n} := \frac{1}{\binom{n+k-1}{k-1}} \sum_{1\leqslant i_1\leqslant\ldots\leqslant i_k\leqslant n} \left(\sum_{j=1}^{k} p_{i_j}\right) f\left(\frac{1}{\sum_{j=1}^{k} p_{i_j}} \sum_{j=1}^{k} p_{i_j} x_{i_j}\right)$$

$$= \frac{1}{\binom{n+k-1}{k-1}} \sum_{\substack{i_1+\ldots+i_n=k \\ i_j\in\mathbb{N};\quad 1\leqslant j\leqslant n}} \left(\sum_{j=1}^{n} i_j p_j\right) \left(\frac{1}{\sum_{j=1}^{n} i_j p_j} \sum_{j=1}^{n} i_j p_j x_j\right), \quad k \in \mathbb{N}_+,$$

then

$$f\left(\sum_{j=1}^{n} p_j x_j\right) \leqslant \ldots \leqslant B_{k+1,n} \leqslant B_{k,n} \leqslant \ldots \leqslant B_{1,n} = \sum_{j=1}^{n} p_j f(x_j), \quad k \in \mathbb{N}_+.$$

This result makes it possible to obtain the generalized form of Theorem C (b). A method has been developed to refine the discrete Jensen's inequality by Horváth [3]. The results in [3] include those considered in [4], but the method can not be applied to solve the present problem (for further details see [2]). In [2], a different approach led to a parameter dependent refinement, whose construction is similar to (2) in Theorem C. However, the treatment of the problem in [2] is totally different from that in [10].

Our main results in this paper are as follows. First, we generalize Theorem B. Moreover, we compare the expressions $F_{k,n}$, $B_{k,n}$ and $G_{k,n}$ (see 4).

THEOREM 1. *Let C be a convex subset of a real vector space X, and $\{x_1,\ldots,x_n\}$ be a finite subset of C, where $n \geqslant 1$ is fixed. Let $p_1,\ldots,p_n$ be a discrete distribution. Assume $f : C \to \mathbb{R}$ is either a convex or a mid-convex function and in the latter case the numbers p_j $(1 \leqslant j \leqslant n)$ are rational. Define*

$$G_{k,n} := \frac{1}{\binom{n+k-1}{k-1}} \sum_{\substack{i_1+\ldots+i_n=n+k-1 \\ i_j\in\mathbb{N}_+;\quad 1\leqslant j\leqslant n}} \left(\sum_{j=1}^{n} i_j p_j\right) f\left(\frac{1}{\sum_{j=1}^{n} i_j p_j}\sum_{j=1}^{n} i_j p_j x_j\right), \quad k\in\mathbb{N}_+. \tag{4}$$

Then

(a) $f\left(\sum_{j=1}^{n} p_j x_j\right) = G_{1,n} \leqslant \ldots \leqslant G_{k,n} \leqslant G_{k+1,n} \leqslant \ldots \leqslant \sum_{j=1}^{n} p_j f(x_j)$.

(b) $F_{k,n} \leqslant G_{k,n}, \quad k \in \mathbb{N}_+$.

(c) *If the numbers $p_1,\ldots,p_n$ are positive, then*

$$G_{k,n} \leqslant B_{k,n}, \quad k \in \mathbb{N}_+.$$

REMARK 2. It is easy to see that in case $p_j = \frac{1}{n}$ $(1 \leqslant j \leqslant n)$

$$G_{k,n} = F_{k,n}, \quad k \in \mathbb{N}_+,$$

so $G_{k,n}$ is the weighted form of $F_{k,n}$.

Next, we extend Theorem D.

THEOREM 3. *Let C be a convex subset of a real vector space X, and $\{x_1,\ldots,x_n\}$ be a finite subset of C, where $n \geqslant 2$ is fixed. Let $p_1,\ldots,p_n$ be a discrete distribution with positive p_j's $(1 \leqslant j \leqslant n)$. Assume $f : C \to \mathbb{R}$ is convex. Define the function h on the set E_n (see 3) by*

$$\begin{aligned} h(t_1,\ldots,t_{n-1}) :=& \left(\sum_{j=1}^{n-1} t_j p_j + \left(1-\sum_{j=1}^{n-1} t_j\right) p_n\right) \\ &\times f\left(\frac{1}{\sum_{j=1}^{n-1} t_j p_j + \left(1-\sum_{j=1}^{n-1} t_j\right) p_n}\left(\sum_{j=1}^{n-1} t_j p_j x_j + \left(1-\sum_{j=1}^{n-1} t_j\right) p_n x_n\right)\right). \end{aligned} \tag{5}$$

(a) *The function h is convex on E_n, and it is Riemann integrable over E_n.*

(b) $\lim_{k\to\infty} G_{k,n} = \lim_{k\to\infty} B_{k,n} = n! \int_{E_n} h(t_1,\ldots,t_{n-1})\, dt_1 \ldots dt_{n-1}$.

2. Discussion and applications

Xiao, Srivastava and Zhang seems to have regarded it as evident that the proof of Theorem D is valid for every integral concept. What does integrable mean in Theorem D? The proof of Theorem 3 actually uses the Riemann integrability of h over E_n, but then f is essentially convex as the following result shows.

For a fixed subset $\{x_1,\dots,x_n\}$ of C, only the restriction of f to the set

$$H := \left\{ \sum_{j=1}^{n} \alpha_j x_j \in C \mid \sum_{j=1}^{n} \alpha_j = 1, \quad \alpha_j \geqslant 0,\ j=1,\dots,n \right\}$$

is important in Theorem 1 and 3.

LEMMA 4. *Let C be a convex subset of a real vector space X, and $\{x_1,\dots,x_n\}$ be a finite subset of C, where $n \geqslant 2$ is fixed. Let $p_1,\dots,p_n$ be a discrete distribution with positive p_j's $(1 \leqslant j \leqslant n)$. Assume $f : C \to \mathbb{R}$ is mid-convex. If the function h in (5) is Riemann integrable over E_n, then f is convex on the set*

$$\hat{H} := \left\{ \sum_{j=1}^{n} \alpha_j x_j \in C \mid \sum_{j=1}^{n} \alpha_j = 1, \quad \alpha_j > 0,\ j=1,\dots,n \right\}.$$

Proof. Let $p := \min\{p_1,\dots,p_n\}$. Then $p > 0$ and

$$\sum_{j=1}^{n-1} t_j p_j + \left(1 - \sum_{j=1}^{n-1} t_j\right) p_n \geqslant p, \quad (t_1,\dots,t_{n-1}) \in E_n.$$

Therefore, recalling the definition of h

$$f\left(\frac{1}{\sum_{j=1}^{n-1} t_j p_j + \left(1 - \sum_{j=1}^{n-1} t_j\right) p_n} \left(\sum_{j=1}^{n-1} t_j p_j x_j + \left(1 - \sum_{j=1}^{n-1} t_j\right) p_n x_n \right) \right) \tag{6}$$
$$\leqslant \frac{1}{p} h(t_1,\dots,t_{n-1}), \quad (t_1,\dots,t_{n-1}) \in E_n.$$

By Lemma 9, the function

$$(t_1,\dots,t_{n-1}) \to \left(\frac{t_1 p_1}{\sum_{j=1}^{n-1} t_j p_j + \left(1 - \sum_{j=1}^{n-1} t_j\right) p_n}, \right.$$
$$\left. \dots, \frac{t_{n-1} p_{n-1}}{\sum_{j=1}^{n-1} t_j p_j + \left(1 - \sum_{j=1}^{n-1} t_j\right) p_n}, \frac{\left(1 - \sum_{j=1}^{n-1} t_j\right) p_n}{\sum_{j=1}^{n-1} t_j p_j + \left(1 - \sum_{j=1}^{n-1} t_j\right) p_n} \right)$$

maps E_n onto the set

$$\left\{(\alpha_1,\ldots,\alpha_n)\in\mathbb{R}^n \mid \sum_{j=1}^{n}\alpha_j=1,\quad \alpha_j\geqslant 0,\ j=1,\ldots,n\right\},$$

and hence (6) and the Riemann integrability of h over E_n (h is bounded on E_n) show that f is bounded above on H.

Since f is mid-convex, the function $\bar{h}$ defined on E_n by

$$\bar{h}(t_1,\ldots,t_{n-1}):=f\left(\sum_{j=1}^{n-1}t_jx_j+\left(1-\sum_{j=1}^{n-1}t_j\right)x_n\right)$$

is also mid-convex on E_n. Because f is bounded above on H, $\bar{h}$ is bounded above on E_n. These two properties of $\bar{h}$, together with the Bernstein-Doetsch theorem (see [5]) give that $\bar{h}$ is convex on the interior of E_n, and therefore f is convex on $\hat{H}$.

The proof is complete. □

As an application we introduce some new quasi-arithmetic means (about means see [7]) and study their monotonicity and convergence.

DEFINITION 5. Let $I\subset\mathbb{R}$ be an interval, $x_j\in I$ $(1\leqslant j\leqslant n)$, $p_1,\ldots,p_n$ be a discrete distribution, and let φ, $\psi: I\to\mathbb{R}$ be continuous and strictly monotone functions. We define the quasi-arithmetic means with respect to (4) by

$$M_{\psi,\varphi}(k):=\psi^{-1}\left(\frac{1}{\binom{n+k-1}{k-1}}\sum_{\substack{i_1+\ldots+i_n=n+k-1\\ i_j\in\mathbb{N}_+;\quad 1\leqslant j\leqslant n}}\left(\sum_{j=1}^{n}i_jp_j\right)\right.$$
$$\left.\times(\psi\circ\varphi^{-1})\left(\frac{1}{\sum_{j=1}^{n}i_jp_j}\sum_{j=1}^{n}i_jp_j\varphi(x_j)\right)\right),\quad k\in\mathbb{N}_+. \tag{7}$$

Some other means are also needed.

DEFINITION 6. Let $I\subset\mathbb{R}$ be an interval, $x_j\in I$ $(1\leqslant j\leqslant n)$, and $p_1,\ldots,p_n$ be a discrete distribution. For a continuous and strictly monotone function $z: I\to\mathbb{R}$ we introduce the following mean

$$M_z:=z^{-1}\left(\sum_{j=1}^{n}p_jz(x_j)\right). \tag{8}$$

We now prove the monotonicity of the means (7) and give limit formulas.

PROPOSITION 7. *Let $I \subset \mathbb{R}$ be an interval, let $x_j \in I$ $(1 \leqslant j \leqslant n)$, let $p_1, \ldots, p_n$ be a discrete distribution, and let φ, $\psi : I \to \mathbb{R}$ be continuous and strictly monotone functions. Then*

(a) $M_\varphi = M_{\psi,\varphi}(1) \leqslant \ldots \leqslant M_{\psi,\varphi}(k) \leqslant \ldots \leqslant M_\psi, \quad k \in \mathbb{N}_+,$
if either $\psi \circ \varphi^{-1}$ is convex and ψ is increasing or $\psi \circ \varphi^{-1}$ is concave and ψ is decreasing.

(b) $M_\varphi = M_{\psi,\varphi}(1) \geqslant \ldots \geqslant M_{\psi,\varphi}(k) \geqslant \ldots \geqslant M_\psi, \quad k \in \mathbb{N}_+,$
if either $\psi \circ \varphi^{-1}$ is convex and ψ is decreasing or $\psi \circ \varphi^{-1}$ is concave and ψ is increasing.

(c) *Moreover, in both cases*

$$\lim_{k\to\infty} M_{\psi,\varphi}(k) = \psi^{-1}\left(n! \int\limits_{E_n} h(t_1, \ldots, t_{n-1})\, dt_1 \ldots dt_{n-1} \right),$$

where the function h is defined on the set E_n (see 3) by

$$h(t_1, \ldots, t_{n-1}) := \left(\sum_{j=1}^{n-1} t_j p_j + \left(1 - \sum_{j=1}^{n-1} t_j\right) p_n \right) \times$$
$$\times \left(\psi \circ \varphi^{-1}\right) \left(\frac{1}{\sum_{j=1}^{n-1} t_j p_j + \left(1 - \sum_{j=1}^{n-1} t_j\right) p_n} \left(\sum_{j=1}^{n-1} t_j p_j \varphi(x_j) + \left(1 - \sum_{j=1}^{n-1} t_j\right) p_n \varphi(x_n) \right) \right).$$

Proof. Theorem 1 (a) can be applied to the function $\psi \circ \varphi^{-1}$, if it is convex ($-\psi \circ \varphi^{-1}$, if it is concave) and the n-tuples $(\varphi(x_1), \ldots, \varphi(x_n))$, then upon taking ψ^{-1}, we get (a) and (b). (c) comes from Theorem 3 (b_1). □

As a special case we consider the following example.

EXAMPLE 8. If $I :=]0, \infty[$, $\psi := \ln$ and $\varphi(x) := x$ $(x \in]0, \infty[)$, then by Proposition 7 (b), we have the following sharpened version of the weighted arithmetic mean – geometric mean inequality: for every $x_j > 0$ $(1 \leqslant j \leqslant n)$ and $k \in \mathbb{N}_+$

$$\sum_{j=1}^{n} p_j x_j \geqslant \prod_{\substack{i_1+\ldots+i_n=n+k-1 \\ i_j \in \mathbb{N}_+;\quad 1 \leqslant j \leqslant n}} \left(\frac{\sum_{j=1}^{n} i_j p_j x_j}{\sum_{j=1}^{n} i_j p_j} \right)^{\frac{1}{\binom{n+k-1}{k-1}} \sum_{j=1}^{n} i_j p_j} \geqslant \prod_{j=1}^{n} x_j^{p_j}.$$

Moreover, by Proposition 7 (c)

$$\lim_{k\to\infty}\prod_{\substack{i_1+\ldots+i_n=n+k-1\\ i_j\in\mathbb{N}_+;\quad 1\leqslant j\leqslant n}}\left(\frac{\sum_{j=1}^{n} i_j p_j x_j}{\sum_{j=1}^{n} i_j p_j}\right)^{\frac{1}{\binom{n+k-1}{k-1}}\sum_{j=1}^{n} i_j p_j}$$

$$=\exp\left(n!\int_{E_n} h(t_1,\ldots,t_{n-1})\,dt_1\ldots dt_{n-1}\right),$$

where the function h is defined on the set E_n (see 3) by

$$h(t_1,\ldots,t_{n-1})$$
$$:=\left(\sum_{j=1}^{n-1} t_j p_j+\left(1-\sum_{j=1}^{n-1} t_j\right)p_n\right)$$
$$\times\ln\left(\frac{1}{\sum_{j=1}^{n-1} t_j p_j+\left(1-\sum_{j=1}^{n-1} t_j\right)p_n}\left(\sum_{j=1}^{n-1} t_j p_j x_j+\left(1-\sum_{j=1}^{n-1} t_j\right)p_n x_n\right)\right).$$

3. Preliminary results and the proofs

LEMMA 9. *Let $p_1,\ldots,p_n$ be a discrete distribution with positive p_j's $(1\leqslant j\leqslant n)$, and let $q_1,\ldots,q_n$ be another discrete distribution. Then there is a discrete distribution $t_1,\ldots,t_n$ such that*

$$\frac{t_i p_i}{\sum_{j=1}^{n} t_j p_j}=q_i,\quad i=1,\ldots,n. \tag{9}$$

Proof. At this proof the Perron-Frobenius theory comes into play (see [6]). Suppose $q_j>0$ $(1\leqslant j\leqslant n)$. Consider the $n\times n$ matrix

$$A:=\begin{pmatrix} q_1 & q_1 & \ldots & q_1\\ q_2 & q_2 & \ldots & q_2\\ \vdots & \vdots & \vdots & \vdots\\ q_n & q_n & \ldots & q_n\end{pmatrix}.$$

Since A is positive and $\sum_{j=1}^{n} q_j=1$, the Perron-Frobenius eigenvalue of A is 1. Then there exists an eigenvector $(v_1,\ldots,v_n)$ of A corresponding to the eigenvalue 1 such that $v_j > 0 \ (1 \leqslant j \leqslant n)$. It follows that $(v_1, \ldots, v_n)$ is a positive solution of the system of equations

$$\frac{x_i}{\sum_{j=1}^{n} x_j} = q_i, \quad i = 1, \ldots, n. \tag{10}$$

It is easy to see that we can abandon the supplementary hypothesis on $q_j \ (1 \leqslant j \leqslant n)$: if $q_j \geqslant 0 \ (1 \leqslant j \leqslant n)$, then (10) has a nonnegative solution $(v_1, \ldots, v_n)$ different from $(0, \ldots, 0)$. In this case

$$\left(\frac{v_1}{p_1}, \ldots, \frac{v_n}{p_n} \right)$$

is a solution of (9). We have from this that

$$t_i = \frac{1}{\sum_{j=1}^{n} \frac{v_j}{p_j}} \frac{v_i}{p_i}, \quad i = 1, \ldots, n$$

is appropriate.

The proof is complete. □

Proof of Theorem 1. We introduce the following set:

$$S_{k,n} := \left\{ (i_1, \ldots, i_n) \in \mathbb{N}_+^n \mid \sum_{j=1}^{n} i_j = n + k - 1 \right\}, \quad k \in \mathbb{N}_+.$$

(a) Since $S_{1,n} = \{(1, \ldots, 1)\}$

$$f\left(\sum_{v=1}^{n} p_v x_v \right) = G_{1,n}.$$

Next, we prove that

$$G_{k,n} \leqslant G_{k+1,n}, \quad k \in \mathbb{N}_+.$$

Let $k \in \mathbb{N}_+$ be fixed. First we note that

$$\binom{n+k-1}{k-1} = \binom{n+k}{k} \frac{k}{n+k},$$

and therefore

$$k = \sum_{u=1}^{n} (i_u - 1), \quad (i_1, \ldots, i_n) \in S_{k+1,n}$$

implies

$$G_{k+1,n} = \frac{1}{\binom{n+k-1}{k-1}} \frac{1}{n+k} \sum_{(i_1,\ldots,i_n) \in S_{k+1,n}} \left(\sum_{u=1}^{n} (i_u - 1) \left(\sum_{v=1}^{n} i_v p_v \right) f\left(\frac{1}{\sum_{v=1}^{n} i_v p_v} \sum_{v=1}^{n} i_v p_v x_v \right) \right).$$

By introducing

$$j_u := i_u - 1, \quad u = 1, \ldots, n, \quad (i_1, \ldots, i_n) \in S_{k+1,n},$$

we have that

$$G_{k+1,n} = \frac{1}{\binom{n+k-1}{k-1}} \frac{1}{n+k} \sum_{(j_1,\ldots,j_n) \in S_{k,n}} \left(\sum_{u=1}^{n} j_u \left(\sum_{v=1}^{n} j_v p_v + p_u \right) \right.$$

$$\left. \times f \left(\frac{1}{\left(\sum_{v=1}^{n} j_v p_v + p_u \right)} \left(\sum_{v=1}^{n} j_v p_v x_v + p_u x_u \right) \right) \right). \tag{11}$$

It is easy to observe that

$$\sum_{u=1}^{n} j_u \left(\sum_{v=1}^{n} j_v p_v + p_u \right) = (n+k) \sum_{v=1}^{n} j_v p_v, \quad (j_1, \ldots, j_n) \in S_{k,n}, \tag{12}$$

and

$$\sum_{u=1}^{n} j_u \left(\sum_{v=1}^{n} j_v p_v x_v + p_u x_u \right) = (n+k) \sum_{v=1}^{n} j_v p_v x_v, \quad (j_1, \ldots, j_n) \in S_{k,n}. \tag{13}$$

With the help of the discrete Jensen's inequality (either Theorem A (a) or (b)) (11), (12) and (13) yield

$$G_{k+1,n} \geqslant \frac{1}{\binom{n+k-1}{k-1}} \frac{1}{n+k} \sum_{(j_1,\ldots,j_n) \in S_{k,n}} \left((n+k) \sum_{v=1}^{n} j_v p_v \right.$$

$$\left. \times f \left(\frac{1}{(n+k) \sum_{v=1}^{n} j_v p_v} \sum_{u=1}^{n} j_u \left(\sum_{v=1}^{n} j_v p_v x_v + p_u x_u \right) \right) \right)$$

$$= \frac{1}{\binom{n+k-1}{k-1}} \sum_{(j_1,\ldots,j_n) \in S_{k,n}} \left(\sum_{v=1}^{n} j_v p_v \right) f \left(\frac{1}{\sum_{v=1}^{n} j_v p_v} \sum_{v=1}^{n} j_v p_v x_v \right) = G_{k,n}.$$

It remained to prove that

$$G_{k,n} \leqslant \sum_{v=1}^{n} p_v f(x_v), \quad k \in \mathbb{N}_+.$$

We can apply the discrete Jensen's inequality (either Theorem A (a) or (b)) again, which insures

$$
\begin{aligned}
G_{k,n} &= \frac{1}{\binom{n+k-1}{k-1}} \sum_{(i_1,\ldots,i_n)\in S_{k,n}} \left(\sum_{v=1}^{n} i_v p_v\right) f\left(\frac{1}{\sum_{v=1}^{n} i_v p_v} \sum_{v=1}^{n} i_v p_v x_v\right) \\
&\leqslant \frac{1}{\binom{n+k-1}{k-1}} \sum_{(i_1,\ldots,i_n)\in S_{k,n}} \sum_{v=1}^{n} i_v p_v f(x_v) \\
&= \frac{1}{\binom{n+k-1}{k-1}} \sum_{v=1}^{n} \sum_{(i_1,\ldots,i_n)\in S_{k,n}} i_v p_v f(x_v), \quad k \in \mathbb{N}_+.
\end{aligned}
$$

Since the set $S_{k,n}$ has $\binom{n+k-2}{k-1}$ elements

$$
\begin{aligned}
\frac{1}{\binom{n+k-1}{k-1}} \sum_{v=1}^{n} \sum_{(i_1,\ldots,i_n)\in S_{k,n}} i_v p_v f(x_v) &= \frac{1}{\binom{n+k-1}{k-1}} \sum_{v=1}^{n} \binom{n+k-1}{k-1} p_v f(x_v) \\
&= \sum_{v=1}^{n} p_v f(x_v), \quad k \in \mathbb{N}_+.
\end{aligned}
$$

(b) Let $\pi_i(j)$ be the unique integer from $\{1,\ldots,n\}$ for which

$$
\pi_i(j) \equiv i + j - 1 \pmod{n}, \quad i,j = 1,\ldots,n.
$$

Then the functions π_i $(i = 1,\ldots,n)$ are permutations of the numbers $1,\ldots,n$. Clearly, $\sum_{j=1}^{n} p_{\pi_i(j)} = 1$ $(i = 1,\ldots,n)$, and $\pi_i(j) = \pi_j(i)$ $(i,j = 1,\ldots,n)$.

Fix $k \in \mathbb{N}_+$. The previous establishments imply

$$
\begin{aligned}
F_{k,n} &= \frac{1}{\binom{n+k-2}{k-1}} \sum_{(i_1,\ldots,i_n)\in S_{k,n}} f\left(\frac{1}{n+k-1} \sum_{v=1}^{n} \left(\sum_{u=1}^{n} p_{\pi_v(u)} i_v x_v\right)\right) \\
&= \frac{1}{\binom{n+k-2}{k-1}} \sum_{(i_1,\ldots,i_n)\in S_{k,n}} f\left(\frac{1}{n+k-1} \sum_{u=1}^{n} \left(\sum_{v=1}^{n} p_{\pi_u(v)} i_v x_v\right)\right) \\
&= \frac{1}{\binom{n+k-2}{k-1}} \sum_{(i_1,\ldots,i_n)\in S_{k,n}} f\left(\frac{1}{n+k-1} \sum_{u=1}^{n} \left(\sum_{w=1}^{n} p_{\pi_u(w)} i_w \sum_{v=1}^{n} \frac{p_{\pi_u(v)} i_v}{\sum_{w=1}^{n} p_{\pi_u(w)} i_w} x_v\right)\right).
\end{aligned}
\tag{14}
$$

Noting that

$$
\sum_{u=1}^{n} \left(\sum_{w=1}^{n} p_{\pi_u(w)} i_w\right) = \sum_{w=1}^{n} i_w = n + k - 1,
$$

the discrete Jensen's inequality (either Theorem A (a) or (b)) can be applied in (14), and we get

$$F_{k,n} \leqslant \frac{1}{\binom{n+k-2}{k-1}(n+k-1)} \times \sum_{(i_1,\dots,i_n)\in S_{k,n}} \sum_{u=1}^{n} \left(\sum_{w=1}^{n} p_{\pi_u(w)} i_w f\left(\sum_{v=1}^{n} \frac{p_{\pi_u(v)} i_v}{\sum\limits_{w=1}^{n} p_{\pi_u(w)} i_w} x_v \right)\right)$$
$$= \frac{1}{\binom{n+k-1}{k}} \frac{1}{n} \sum_{u=1}^{n} \left(\sum_{(i_1,\dots,i_n)\in S_{k,n}} \sum_{w=1}^{n} p_{\pi_u(w)} i_w f\left(\sum_{v=1}^{n} \frac{p_{\pi_u(v)} i_v}{\sum\limits_{w=1}^{n} p_{\pi_u(w)} i_w} x_v \right)\right).$$

Since π_u $(u=1,\dots,n)$ is a permutation of the numbers $1,\dots,n$, and $\pi_u(S_{k,n})=S_{k,n}$ $(u=1,\dots,n)$ we can see that for every fixed $u\in\{1,\dots,n\}$

$$\sum_{(i_1,\dots,i_n)\in S_{k,n}} \sum_{w=1}^{n} p_{\pi_u(w)} i_w f\left(\sum_{v=1}^{n} \frac{p_{\pi_u(v)} i_v}{\sum\limits_{w=1}^{n} p_{\pi_u(w)} i_w} x_v \right)$$
$$= \sum_{(i_1,\dots,i_n)\in S_{k,n}} \left(\sum_{v=1}^{n} i_v p_v\right) f\left(\frac{1}{\sum\limits_{v=1}^{n} i_v p_v} \sum_{v=1}^{n} i_v p_v x_v \right).$$

(c) Fix $k\in\mathbb{N}_+$. By the definition of $G_{k+1,n}$

$$G_{k+1,n} = \frac{1}{\binom{n+k}{k}} \sum_{(i_1,\dots,i_n)\in S_{k+1,n}} \left(\sum_{v=1}^{n} i_v p_v\right) f\left(\frac{1}{\sum\limits_{v=1}^{n} i_v p_v} \sum_{v=1}^{n} i_v p_v x_v \right)$$
$$= \frac{1}{\binom{n+k}{k}} \sum_{(i_1,\dots,i_n)\in S_{k+1,n}} \left(\sum_{v=1}^{n} (i_v-1) p_v + \sum_{v=1}^{n} p_v\right)$$
$$\times f\left(\frac{1}{\sum\limits_{v=1}^{n} (i_v-1) p_v + \sum\limits_{v=1}^{n} p_v} \left(\sum_{v=1}^{n} (i_v-1) p_v x_v + \sum_{v=1}^{n} p_v x_v\right)\right)$$

$$= \frac{1}{\binom{n+k}{k}} \sum_{\substack{j_1+\ldots+j_n=k \\ j_l\in\mathbb{N};\quad 1\leqslant l\leqslant n}} \left(\sum_{v=1}^{n} j_v p_v + 1\right)$$

$$\times f\left(\frac{1}{\sum\limits_{v=1}^{n} j_v p_v + 1}\left(\sum_{v=1}^{n} j_v p_v \frac{\sum\limits_{v=1}^{n} j_v p_v x_v}{\sum\limits_{v=1}^{n} j_v p_v} + \sum_{v=1}^{n} p_v x_v\right)\right).$$

In this situation the discrete Jensen's inequality (either Theorem A (a) or (b)) implies that

$$G_{k+1,n} \leqslant \frac{1}{\binom{n+k}{k}} \sum_{\substack{j_1+\ldots+j_n=k \\ j_l\in\mathbb{N};\quad 1\leqslant l\leqslant n}} \left(\left(\sum_{v=1}^{n} j_v p_v\right) f\left(\frac{1}{\sum\limits_{v=1}^{n} j_v p_v} \sum_{v=1}^{n} j_v p_v x_v\right) + f\left(\sum_{v=1}^{n} p_v x_v\right)\right)$$

$$= \frac{1}{\binom{n+k}{k}} \sum_{\substack{j_1+\ldots+j_n=k \\ j_l\in\mathbb{N};\ 1\leqslant l\leqslant n}} \left(\sum_{v=1}^{n} j_v p_v\right) f\left(\frac{1}{\sum\limits_{v=1}^{n} j_v p_v} \sum_{v=1}^{n} j_v p_v x_v\right) + \frac{\binom{n+k-1}{k}}{\binom{n+k}{k}} f\left(\sum_{v=1}^{n} p_v x_v\right).$$

From this, by means of Theorem E, we get

$$G_{k+1,n} \leqslant \left(\frac{1}{\binom{n+k}{k}}\left(1 + \frac{\binom{n+k-1}{k}}{\binom{n+k-1}{k-1}}\right)\right) \sum_{\substack{j_1+\ldots+j_n=k \\ j_l\in\mathbb{N};\quad 1\leqslant l\leqslant n}} \left(\sum_{v=1}^{n} j_v p_v\right) f\left(\frac{1}{\sum\limits_{v=1}^{n} j_v p_v} \sum_{v=1}^{n} j_v p_v x_v\right)$$

$$= B_{k,n}.$$

Combining this and (a) yields finally

$$G_{k,n} \leqslant G_{k+1,n} \leqslant B_{k,n}, \quad k \in \mathbb{N}_+.$$

The proof is complete. □

Proof of Theorem 3. (a) E_n is obviously a convex set, and by using the convexity of f, some elementary computation shows that h is convex. Since f is bounded on the convex set

$$\left\{\sum_{j=1}^{n} \alpha_j x_j \in X \mid \sum_{j=1}^{n} \alpha_j = 1, \quad \alpha_j \geqslant 0, \quad j = 1,\ldots,n\right\},$$

h is bounded too. The convexity of h implies that it is continuous on the interior of E_n. The previous two establishments, together with the fact that the measure of the boundary of E_n is 0, yield that h is Riemann integrable over E_n.

(b) Fix $k \in \mathbb{N}_+$.

By the definition of $G_{k,n}$, elementary considerations show that

$$G_{k,n} = \frac{1}{\binom{n+k-1}{k-1}} \sum_{\substack{i_1+\ldots+i_n=n+k-1 \\ i_j\in\mathbb{N}_+;\quad 1\leqslant j\leqslant n}} \left(\sum_{j=1}^{n} i_j p_j\right) f\left(\frac{1}{\sum_{j=1}^{n} i_j p_j}\sum_{j=1}^{n} i_j p_j x_j\right)$$

$$= n!\frac{(n+k-2)^{n-2}}{k(k+1)\ldots(n+k-3)}\cdot\frac{1}{(n+k-2)^{n-1}}$$

$$\times\sum_{i_1=1}^{k}\sum_{i_2=1}^{k+1-i_1}\sum_{i_3=1}^{k+2-(i_1+i_2)}\cdots\sum_{i_{n-1}=1}^{n+k-2-(i_1+\ldots+i_{n-2})}\left(\sum_{j=1}^{n-1}\frac{i_j}{n+k-1}p_j\right.$$

$$\left.+\left(1-\sum_{j=1}^{n-1}\frac{i_j}{n+k-1}\right)p_n\right)$$

$$\times f\left(\frac{\sum_{j=1}^{n-1}\frac{i_j}{n+k-1}p_j x_j+\left(1-\sum_{j=1}^{n-1}\frac{i_j}{n+k-1}\right)p_n x_n}{\sum_{j=1}^{n-1}\frac{i_j}{n+k-1}p_j+\left(1-\sum_{j=1}^{n-1}\frac{i_j}{n+k-1}\right)p_n}\right).$$

Since h is Riemann integrable, the result for the sequence $\left(G_{k,n}\right)$ follows from this and from

$$\frac{i-1}{n+k-2}<\frac{i}{n+k-1}<\frac{i}{n+k-2},\quad i=1,\ldots,k.$$

Similarly, according to the definition of $B_{k,n}$, we have

$$B_{k,n} = \frac{1}{\binom{n+k-1}{k-1}} \sum_{\substack{i_1+\ldots+i_n=k \\ i_j\in\mathbb{N};\quad 1\leqslant j\leqslant n}} \left(\sum_{j=1}^{n} i_j p_j\right)\left(\frac{1}{\sum_{j=1}^{n} i_j p_j}\sum_{j=1}^{n} i_j p_j x_j\right)$$

$$= n!\frac{(k+1)^{n-1}}{(k+1)\ldots(n+k-1)}\cdot\frac{1}{(k+1)^{n-1}}$$

$$\times\sum_{i_1=0}^{k}\sum_{i_2=0}^{k-i_1}\sum_{i_3=0}^{k-(i_1+i_2)}\cdots\sum_{i_{n-1}=0}^{k-(i_1+\ldots+i_{n-2})}\left(\sum_{j=1}^{n-1}\frac{i_j}{k}p_j+\left(1-\sum_{j=1}^{n-1}\frac{i_j}{k}\right)p_n\right)$$

$$\times f\left(\frac{\sum_{j=1}^{n-1}\frac{i_j}{k}p_j x_j+\left(1-\sum_{j=1}^{n-1}\frac{i_j}{k}\right)p_n x_n}{\sum_{j=1}^{n-1}\frac{i_j}{k}p_j+\left(1-\sum_{j=1}^{n-1}\frac{i_j}{k}\right)p_n}\right).$$

By taking into account the Riemann integrability of h and

$$\frac{i}{k+1} < \frac{i}{k} < \frac{i+1}{k+1}, \quad i = 0, \ldots, k,$$

we have the result for the sequence $\left(B_{k,n}\right)$. □

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