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DIFFERENCES OF GENERALIZED COMPOSITION OPERATORS BETWEEN BLOCH TYPE SPACES

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Abstract. Let φ and ψ be analytic self-maps of the open unit disk D. Using pseudo-hyperbolic distance $\rho(\varphi,\psi)$, we characterize the boundedness and compactness of the differences of generalized composition operators

$$(C_\varphi^g - C_\psi^h)f(z) = \int_0^z \left[f'(\varphi(\xi))g(\xi) - f'(\psi(\xi))h(\xi)\right]d\xi, \quad z \in D$$

between two Bloch-type spaces on D. The results generalize the corresponding results on the single generalized composition operator and on the differences of generalized composition operators on the Bloch space.

1. Introduction

Let $D = \{z : |z| < 1\}$ be the open unit disk in the complex plane, $H(D)$ be the space of all analytic functions on D and $S(D)$ be the set of analytic self-maps of D. For $a \in D$, let σ_a be the Möbius transformation of D defined by

$$\sigma_a(z) = \frac{a-z}{1-\bar{a}z}.$$

For $w,z \in D$, the pseudo-hyperbolic distance $\rho(w,z)$ between z and w is given by

$$\rho(z,w) = \left|\sigma_w(z)\right|.$$

For $\varphi \in S(D)$ and $u \in H(D)$, we denote by uC_φ the weighted composition operator, which is defined by $(uC_\varphi f)(z) = u(z)f(\varphi(z))$. When $u(z) \equiv 1$, uC_φ becomes the composition operator C_φ, while if $\varphi(z) = z$, uC_φ becomes the multiplication operator M_u.

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During the past few decades much effort has been devoted to the research of such operators on different Banach spaces of analytic functions. The general idea is to explain the operator-theoretic behavior of uC_φ such as boundendness and compactness, by the function-theoretic properties of the symbols φ and u. For a comprehensive overview of the field, we refer to the books by Cowen and MacCluer [2] and Shapiro [13].

For $\alpha > 0$, recall that the Bloch type space $\mathscr{B}^\alpha$ is the space of all $f \in H(D)$ satisfying

$$b_\alpha(f) = \sup_{z\in D}(1-|z|^2)^\alpha |f'(z)| < \infty.$$

It is easy to see that $\mathscr{B}^\alpha$ is a Banch space with the norm $\|f\| = |f(0)| + b_\alpha(f)$. When $\alpha = 1$, we have $\mathscr{B}^1 = \mathscr{B}$, the Bloch space. When $0 < \alpha < 1$, the spaces $\mathscr{B}^\alpha$ can be identified with the analytic Lipschitz spaces $\mathrm{Lip}_{1-\alpha}$. See [15, 25] for details.

Motivated by the fact that composition operators and weighted composition operators naturally come from isometries of some function spaces, for $\varphi \in S(D)$ and $g \in H(D)$, in [6], Li and Stević defined the generalized composition operator C_φ^g as follows:

$$C_\varphi^g f(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi, \ \ f \in H(D), \ \ z \in D.$$

They characterized the boundedness and compactness of C_φ^g on Zygmund spaces and Bloch type spaces. See also [7, 8, 9, 14, 17, 16, 10, 18, 19, 20, 21, 22, 24, 26, 27] for the study of the operator C_φ^g and its generalizations.

The study of the differences of two composition operators was started on Hardy spaces. The primary motivation for this is to understand the topological structure of the set of composition operators $\mathscr{C}(H^2)$ on Hardy space H^2. After that, such related problems have been studied on several spaces of holomorphic functions by many authors. For example, in [11], Moorhouse characterized compact difference of two composition operators on the standard weighted Bergman spaces. In [4] and [12], the boundedness and compactness of the difference of two composition operators on the Bloch spaces were characterized. In [1], the authors investigated the boundedness and compactness of difference of two composition operators on weighted Banach spaces. In [23] and [3], the difference of two weighted composition operators was investigated also.

In [5], Li investigated the difference of two generalized composition operators on the Bloch space. In this note, we generalize the results in [5] and study the boundedness and compactness of the differences of generalized composition operators between Bloch type spaces.

Throughout this note, constants are denoted by C, they are positive and may differ from one occurrence to the other. $a \preceq b$ means that there is a positive constant C such that $a \leqslant Cb$. Moreover, if both $a \preceq b$ and $b \preceq a$ hold, then we write $a \asymp b$.

2. The boundedness of $C_\varphi^g - C_\psi^h$

To prove the main results of this paper, we need some lemmas. The first lemma was proved in [25].

LEMMA 1. *For every positive integer n, $f \in \mathscr{B}^\alpha$ if and only if $f^{(n)} \in \mathscr{B}^{\alpha+n}$, and the following asymptotic relationship holds*

$$\|f\|_{\mathscr{B}^\alpha} \asymp \sum_{k=0}^{n} |f^{(k)}(0)| + \sup_{z\in D}(1-|z|^2)^{\alpha+n}|f^{(n+1)}(z)|.$$

LEMMA 2. *Let $f \in \mathscr{B}^\alpha$. Then, for all $z,w \in D$,*

$$|(1-|z|^2)^\alpha f'(z) - (1-|w|^2)^\alpha f'(w)| \leqslant C\|f\|_{\mathscr{B}^\alpha}\rho(z,w). \tag{1}$$

Proof. In [12, 3], it has been shown that

$$|(1-|z|^2)^\alpha f(z) - (1-|w|^2)^\alpha f(w)| \leqslant C\rho(z,w)\sup_{z\in D}(1-|z|^2)^\alpha |f(z)|,$$

for every $f \in H(D)$. Then we obtain the desired inequality. □

REMARK 3. From the proof of Lemma 3.2 of [12], in fact, it follows that for every $f \in \mathscr{B}^\alpha$,

$$|(1-|z|^2)^\alpha f'(z) - (1-|w|^2)^\alpha f'(w)| \leqslant Cb_\alpha^r(f)\rho(z,w), \quad z,w \in D_r, \tag{2}$$

where $D_r = \{z \in D : |z| \leqslant r < 1\}$ and

$$b_\alpha^r(f) = \max\{\sup_{z\in D_r}(1-|z|^2)^\alpha |f'(z)|, \sup_{z\in D_r}(1-|z|^2)^{\alpha+1}|f''(z)|\}.$$

The following criterion for compactness follows from standard arguments similar to those outlined in Proposition 3.11 of [2]. We omit the details.

LEMMA 4. *Let $\varphi, \psi \in S(D)$ and $g,h \in H(D)$. Then $C_\varphi^g - C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is compact if and only if $C_\varphi^g - C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathscr{B}^\alpha$ which converges to zero uniformly on compact subsets of D, $\|(C_\varphi^g - C_\psi^h)f_k\|_{\mathscr{B}^\beta} \to 0$ as $k \to \infty$.*

In order to characterize the boundedness of $C_\varphi^g - C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}^\beta$, we will use the following three conditions in this section:

$$\sup_{z\in D}|M_g^\varphi(z)|\rho(\varphi(z),\psi(z)) < \infty; \tag{3}$$

$$\sup_{z\in D}|M_h^\psi(z)|\rho(\varphi(z),\psi(z)) < \infty; \tag{4}$$

$$\sup_{z\in D}|M_g^\varphi(z) - M_h^\psi(z)| < \infty, \tag{5}$$

where and henceforth

$$M_g^\varphi(z) = \frac{(1-|z|^2)^\beta g(z)}{(1-|\varphi(z)|^2)^\alpha}, \quad M_h^\psi(z) = \frac{(1-|z|^2)^\beta h(z)}{(1-|\psi(z)|^2)^\alpha}.$$

THEOREM 1. *Let $\varphi,\psi \in S(D)$ and $g,h \in H(D)$. Then the following statements are equivalent.*

(a) $C_\varphi^g - C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ *is bounded.*

(b) (3) *and* (5) *hold.*

(c) (4) *and* (5) *hold.*

Proof. $(a) \Rightarrow (b)$. Suppose that $C_\varphi^g - C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is bounded. Fix a point w in D such that $\varphi(w) \neq 0$, let

$$l_w(z) = \frac{1}{\alpha\overline{\varphi(w)}} \cdot \frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^\alpha},\ f_w(z) = \frac{\alpha l_w(z)}{\alpha+1}\Big(\sigma_{\varphi(w)}(z) + \frac{1}{\alpha\overline{\varphi(w)}}\Big),\ z \in D.$$

It is easy to check that $l_w(z), f_w(z) \in \mathscr{B}^\alpha$. Moreover,

$$l_w'(z) = \frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^{\alpha+1}}$$

and

$$\begin{aligned} f_w'(z) &= \frac{\alpha}{\alpha+1}\left[l_w'(z)\sigma_{\varphi(w)}(z) + l_w'(z)\frac{1}{\alpha\overline{\varphi(w)}} + l_w(z)\sigma_{\varphi(w)}'(z)\right] \\ &= \frac{\alpha}{\alpha+1}\left[\frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^{\alpha+1}} \cdot \frac{\varphi(w)-z}{1-\overline{\varphi(w)}z} + \frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^{\alpha+1}} \cdot \frac{1}{\alpha\overline{\varphi(w)}}\right. \\ &\quad \left. + \frac{1}{\alpha\overline{\varphi(w)}} \cdot \frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^\alpha} \cdot \frac{|\varphi(w)|^2-1}{(1-\overline{\varphi(w)}z)^2}\right] \\ &= \frac{(1-|\varphi(w)|^2)\left[\alpha\overline{\varphi(w)}(\varphi(w)-z) + (1-\overline{\varphi(w)}z) + (|\varphi(w)|^2-1)\right]}{(\alpha+1)\overline{\varphi(w)}(1-\overline{\varphi(w)}z)^{\alpha+2}} \\ &= \frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^{\alpha+1}}\sigma_{\varphi(w)}(z). \end{aligned}$$

Since the operator $C_\varphi^g - C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is bounded and note $(C_\varphi^g - C_\psi^h)l(0) = 0$, for any $f \in H(D)$, by Lemma 2, we have

$$\begin{aligned} \infty > \|(C_\varphi^g - C_\psi^h)l_w\|_{\mathscr{B}^\beta} &= \sup_{z\in D}(1-|z|^2)^\beta |l_w'(\varphi(z))g(z) - l_w'(\psi(z))h(z)| \\ &\geqslant (1-|w|^2)^\beta |l_w'(\varphi(w))g(w) - l_w'(\psi(w))h(w)| \\ &= \Big|M_g^\varphi(w) - M_h^\psi(w)\frac{(1-|\psi(w)|^2)^\alpha(1-|\varphi(w)|^2)}{(1-\overline{\varphi(w)}\psi(w))^{\alpha+1}}\Big|, \end{aligned} \tag{6}$$

and

$$\begin{aligned} \infty &> \|(C_\varphi^g - C_\psi^h)f_w\|_{\mathscr{B}^\beta} \\ &= \sup_{z\in D}(1-|z|^2)^\beta |f_w'(\varphi(z))g(z) - f_w'(\psi(z))h(z)| \end{aligned}$$

$$\begin{aligned}
&\geqslant (1-|w|^2)^{\beta}|f_w'(\varphi(w))g(w)-f_w'(\psi(w))h(w)|\\
&=(1-|w|^2)^{\beta}\frac{|h(w)|(1-|\varphi(w)|^2)\rho(\varphi(w),\psi(w))}{|1-\overline{\varphi(w)}\psi(w)|^{\alpha+1}}\\
&=|M_h^{\psi}(w)\frac{(1-|\psi(w)|^2)^{\alpha}(1-|\varphi(w)|^2)}{(1-\overline{\varphi(w)}\psi(w))^{\alpha+1}}|\rho(\varphi(w),\psi(w)).
\end{aligned}\tag{7}$$

Since the pseudo-hyperbolic metric $\rho<1$, multiplying (6) by $\rho(\varphi(w),\psi(w))$, from (7) we obtain

$$\left|M_g^{\varphi}(w)\right|\rho(\varphi(w),\psi(w))<\infty\tag{8}$$

holds for all $w\in D$ with $\varphi(w)\neq 0$.

If $\varphi(w)=0$, using test function $k_w(z)=(z-\psi(w))^2/2$, we see that

$$\begin{aligned}
\infty>\|(C_{\varphi}^{g}-C_{\psi}^{h})k_w\|_{\mathscr{B}^{\beta}}&\geqslant(1-|w|^2)^{\beta}|k_w'(\varphi(w))g(w)-k_w'(\psi(w))h(w)|\\
&=(1-|w|^2)^{\beta}\left|g(w)\psi(w)\right|.
\end{aligned}\tag{9}$$

Therefore, we get that (3) holds.

Using another triple test functions which come from $l_w(z)$, $f_w(z)$ and $k_w(z)$ by exchanging φ and ψ, we can get that (4) holds.

Next, we prove (5) holds. By (6), we also have

$$\begin{aligned}
\infty&>\|(C_{\varphi}^{g}-C_{\psi}^{h})l_w\|_{\mathscr{B}^{\beta}}\\
&\geqslant\left|M_g^{\varphi}(w)-M_h^{\psi}(w)\frac{(1-|\psi(w)|^2)^{\alpha}(1-|\varphi(w)|^2)}{(1-\overline{\varphi(w)}\psi(w))^{\alpha+1}}\right|\\
&=\left|M_g^{\varphi}(w)-M_h^{\psi}(w)+M_h^{\psi}(w)(1-\frac{(1-|\psi(w)|^2)^{\alpha}(1-|\varphi(w)|^2)}{(1-\overline{\varphi(w)}\psi(w))^{\alpha+1}})\right|\\
&\geqslant\left|M_g^{\varphi}(w)-M_h^{\psi}(w)\right|\\
&\quad-\left|M_h^{\psi}(w)\right|\cdot\left|l_w'(\varphi(w))(1-|\varphi(w)|^2)^{\alpha}-l_w'(\psi(w))(1-|\psi(w)|^2)^{\alpha}\right|.
\end{aligned}\tag{10}$$

From Lemma 2 and (4), we see that

$$\begin{aligned}
&\left|M_h^{\psi}(w)\right|\cdot\left|l_w'(\varphi(w))(1-|\varphi(w)|^2)^{\alpha}-l_w'(\psi(w))(1-|\psi(w)|^2)^{\alpha}\right|\\
&\leqslant C\|l_w\|_{\mathscr{B}^{\alpha}}|M_h^{\psi}(w)|\rho(\varphi(w),\psi(w))<\infty,
\end{aligned}$$

which with (10) implies $|M_g^{\varphi}(w)-M_h^{\psi}(w)|<\infty$ holds for all $w\in D$ with $\varphi(w)\neq 0$.

If $\varphi(w)=0$ and $|\psi(w)|\geqslant 1/2$, then $\rho(\varphi(w),\psi(w))=|\psi(w)|\geqslant 1/2$. From (3) and (4), we can deduce directly that $|M_g^{\varphi}(w)-M_h^{\psi}(w)|<\infty$ holds for all $w\in D$ with $\varphi(w)=0$ and $|\psi(w)|\geqslant 1/2$.

If $\varphi(w)=0$ and $|\psi(w)|<1/2$. Using test function $I(z)=z$, we see that

$$\begin{aligned}
\infty>\|(C_{\varphi}^{g}-C_{\psi}^{h})I\|_{\mathscr{B}^{\beta}}&\geqslant(1-|w|^2)^{\beta}|I'(\varphi(w))g(w)-I'(\psi(w))h(w)|\\
&=(1-|w|^2)^{\beta}\left|g(w)-h(w)\right|\\
&\geqslant|M_g^{\varphi}(w)-M_h^{\psi}(w)|-|M_h^{\psi}(w)|(1-(1-|\psi(w)|^2)^{\alpha}).
\end{aligned}\tag{11}$$

Since

$$|M_h^{\psi}(w)|(1-(1-|\psi(w)|^2)^{\alpha}) \leqslant C|M_h^{\psi}(w)\psi(w)| = C|M_h^{\psi}(w)|\rho(\varphi(w),\psi(w)),$$

from (4) and (11), we obtain $|M_g^{\varphi}(w)-M_h^{\psi}(w)|<\infty$ holds for all $w\in D$ with $\varphi(w)=0$ and $|\psi(w)|<1/2$.

Thus we conclude that $|M_g^{\varphi}(w)-M_h^{\psi}(w)|<\infty$ holds for all $w\in D$, which implies (5) holds.

$(b)\Rightarrow(c)$. Assume that the conditions (3) and (5) hold. Noticing that $\rho(\varphi(z),\psi(z)) <1$, we have

$$\begin{aligned}
&\big|M_h^{\psi}(z)\big|\rho(\varphi(z),\psi(z)) = \frac{(1-|z|^2)^{\beta}|h(z)|}{(1-|\psi(z)|^2)^{\alpha}}\rho(\varphi(z),\psi(z))\\
\leqslant\ & \frac{(1-|z|^2)^{\beta}|g(z)|}{(1-|\varphi(z)|^2)^{\alpha}}\rho(\varphi(z),\psi(z)) + \Big|\frac{(1-|z|^2)^{\beta}h(z)}{(1-|\psi(z)|^2)^{\alpha}} - \frac{(1-|z|^2)^{\beta}g(z)}{(1-|\varphi(z)|^2)^{\alpha}}\Big|\rho(\varphi(z),\psi(z))\\
\leqslant\ & \big|M_g^{\varphi}(z)\big|\rho(\varphi(z),\psi(z)) + \big|M_h^{\psi}(z)-M_g^{\varphi}(z)\big|\rho(\varphi(z),\psi(z)),
\end{aligned}$$

which implies (4) holds.

$(c)\Rightarrow(a)$. Suppose that (4) and (5) hold. For $f\in\mathscr{B}^{\alpha}$, by Lemmas 2 and 3, we have

$$\begin{aligned}
&\|(C_{\varphi}^{g}-C_{\psi}^{h})f\|_{\mathscr{B}^{\beta}} = \sup_{z\in D}(1-|z|^2)^{\beta}|[(C_{\varphi}^{g}-C_{\psi}^{h})f]'(z)|\\
=\ & \sup_{z\in D}(1-|z|^2)^{\beta}|f'(\varphi(z))g(z)-f'(\psi(z))h(z)|\\
\leqslant\ & \sup_{z\in D}|M_g^{\varphi}(z)f'(\varphi(z))(1-|\varphi(z)|^2)^{\alpha} - M_h^{\psi}(z)f'(\psi(z))(1-|\psi(z)|^2)^{\alpha}|\\
\leqslant\ & \sup_{z\in D}|M_g^{\varphi}(z)-M_h^{\psi}(z)||f'(\varphi(z))(1-|\varphi(z)|^2)^{\alpha}|\\
&+\sup_{z\in D}|M_h^{\psi}(z)||f'(\varphi(z))(1-|\varphi(z)|^2)^{\alpha} - f'(\psi(z))(1-|\psi(z)|^2)^{\alpha}|\\
\leqslant\ & C\|f\|_{\mathscr{B}^{\alpha}}\sup_{z\in D}|M_g^{\varphi}(z)-M_h^{\psi}(z)| + C\|f\|_{\mathscr{B}^{\alpha}}\sup_{z\in D}|M_h^{\psi}(z)|\rho(\varphi(z),\psi(z)).
\end{aligned}$$

Therefore conditions (4) and (5) imply that $C_{\varphi}^{g}-C_{\psi}^{h}:\mathscr{B}^{\alpha}\to\mathscr{B}^{\beta}$ is bounded. The proof is complete. □

From Theorem 1 with $h(z)\equiv 0$, we obtain the following corollary.

COROLLARY 2. *Let $\varphi\in S(D)$ and $g\in H(D)$. Then $C_{\varphi}^{g}:\mathscr{B}^{\alpha}\to\mathscr{B}^{\beta}$ is bounded if and only if*

$$\sup_{z\in D}\frac{(1-|z|^2)^{\beta}|g(z)|}{(1-|\varphi(z)|^2)^{\alpha}}<\infty.$$

Taking $\alpha=\beta=1$, from Theorem 1, we get the next corollary (see [5]).

COROLLARY 3. *Let $\varphi,\psi \in S(D)$ and $g,h \in H(D)$, then the following statements are equivalent*

(a) $C_\varphi^g - C_\psi^h : \mathscr{B} \to \mathscr{B}$ *is bounded*;

(b) $\sup_{z\in D} \left|\mathscr{D}_g^\varphi(z)\right| \rho(\varphi(z),\psi(z)) < \infty$ *and* $\sup_{z\in D} |\mathscr{D}_g^\varphi(z) - \mathscr{D}_h^\psi(z)| < \infty$;

(c) $\sup_{z\in D} \left|\mathscr{D}_h^\psi(z)\right| \rho(\varphi(z),\psi(z)) < \infty$ *and* $\sup_{z\in D} |\mathscr{D}_g^\varphi(z) - \mathscr{D}_h^\psi(z)| < \infty$,

where and henceforth

$$\mathscr{D}_g^\varphi(z) = \frac{(1-|z|^2)g(z)}{1-|\varphi(z)|^2}, \quad \mathscr{D}_h^\psi(z) = \frac{(1-|z|^2)h(z)}{1-|\psi(z)|^2}.$$

3. The compactness of $C_\varphi^g - C_\psi^h$

In order to characterize the compactness of $C_\varphi^g - C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}^\beta$, we will use the following three conditions in this section:

$$\lim_{|\varphi(z)|\to 1} \left|M_g^\varphi(z)\right| \rho(\varphi(z),\psi(z)) = 0; \tag{12}$$

$$\lim_{|\psi(z)|\to 1} \left|M_h^\psi(z)\right| \rho(\varphi(z),\psi(z)) = 0; \tag{13}$$

$$\lim_{|\varphi(z)|\to 1, |\psi(z)|\to 1} |M_g^\varphi(z) - M_h^\psi(z)| = 0. \tag{14}$$

THEOREM 4. *Let $\varphi,\psi \in S(D)$ and $g,h \in H(D)$ such that $C_\varphi^g, C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ are bounded. Then $C_\varphi^g - C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is compact if and only if $C_\varphi^g - C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is bounded and conditions* (12), (13) *and* (14) *hold.*

Proof. First, we prove the sufficiency. Assume $C_\varphi^g - C_\psi^h : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is bounded, then conditions (3), (4) and (5) hold. If conditions (12), (13) and (14) hold, then for any $\varepsilon > 0$, there exists $r \in (0,1)$ such that

$$\left|M_g^\varphi(z)\right| \rho(\varphi(z),\psi(z)) < \varepsilon \text{ when } |\varphi(z)| > r; \tag{15}$$

$$\left|M_h^\psi(z)\right| \rho(\varphi(z),\psi(z)) < \varepsilon \text{ when } |\psi(z)| > r; \tag{16}$$

$$|M_g^\varphi(z) - M_h^\psi(z)| < \varepsilon, \text{ when } |\varphi(z)|, |\psi(z)| > r. \tag{17}$$

Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $\mathscr{B}^\alpha$ such that $\|f_k\|_{\mathscr{B}^\alpha} \leqslant 1$ which converges to zero uniformly on compact subsets of D. In order to prove $C_\varphi^g - C_\psi^h$ is compact, using Lemma 4, we need only to show $\|(C_\varphi^g - C_\psi^h) f_k\|_{\mathscr{B}^\beta} \to 0$.

It is easy to see that

$$\|(C_\varphi^g - C_\psi^h) f_k\|_{\mathscr{B}^\alpha} = \sup_{z\in D} (1-|z|^2)^\beta |[(C_\varphi^g - C_\psi^h) f_k(z)]'|$$
$$= \sup_{z\in D} (1-|z|^2)^\beta |f_k'(\varphi(z))g(z) - f_k'(\psi(z))h(z)|$$

$$
\begin{aligned}
&= \sup_{z\in D}\left|M_g^{\varphi}(z)f_k'(\varphi(z))(1-|\varphi(z)|^2)^{\alpha}-M_h^{\psi}(z)f_k'(\psi(z))(1-|\psi(z)|^2)^{\alpha}\right|\\
&\leqslant \Big(\sup_{|\varphi(z)|\leqslant r,|\psi(z)|\leqslant r}+\sup_{|\varphi(z)|\leqslant r,|\psi(z)|>r}+\sup_{|\varphi(z)|>r,|\psi(z)|>r}+\sup_{|\varphi(z)|>r,|\psi(z)|\leqslant r}\Big)\\
&\quad \left|M_g^{\varphi}(z)f_k'(\varphi(z))(1-|\varphi(z)|^2)^{\alpha}-M_h^{\psi}(z)f_k'(\psi(z))(1-|\psi(z)|^2)^{\alpha}\right|. \qquad (18)
\end{aligned}
$$

We set

$$
M_g^{\varphi}(z)f_k'(\varphi(z))(1-|\varphi(z)|^2)^{\alpha}-M_h^{\psi}(z)f_k'(\psi(z))(1-|\psi(z)|^2)^{\alpha}=J_1^{(k)}+J_2^{(k)},
$$

where

$$
\begin{aligned}
J_1^{(k)} &= (M_g^{\varphi}(z)-M_h^{\psi}(z))f_k'(\varphi(z))(1-|\varphi(z)|^2)^{\alpha},\\
J_2^{(k)} &= M_h^{\psi}(z)\left[f_k'(\varphi(z))(1-|\varphi(z)|^2)^{\alpha}-f_k'(\psi(z))(1-|\psi(z)|^2)^{\alpha}\right].
\end{aligned}
$$

To estimate $|J_1^{(k)}+J_2^{(k)}|$, we distinguish four cases as last inequality in (18).

(i) If $|\varphi(z)|\leqslant r$ and $|\psi(z)|\leqslant r$, by (5), we have $|J_1^{(k)}|\leqslant C|f_k'(\varphi(z))|$. From Remark 3 and (4), we get

$$
|J_2^{(k)}|\leqslant C|M_h^{\psi}(z)|\rho(\varphi(z),\psi(z))b_{\alpha}^r(f_k)\leqslant Cb_{\alpha}^r(f_k).
$$

(ii) If $|\varphi(z)|\leqslant r$ and $|\psi(z)|>r$, with the same argument in case (i), we obtain $|J_1^{(k)}|\leqslant C|f_k'(\varphi(z))|$. Applying Lemma 3 and (16), we can get

$$
|J_2^{(k)}|\leqslant C\|f_k\|_{\mathscr{B}^{\alpha}}|M_h^{\psi}(z)|\rho(\varphi(z),\psi(z))\leqslant C\varepsilon.
$$

(iii) If $|\varphi(z)|>r$ and $|\psi(z)|>r$, by Lemma 2 and (17), we have

$$
|J_1^{(k)}|<C|M_g^{\varphi}(z)-M_h^{\psi}(z)|\|f_k\|_{\mathscr{B}^{\alpha}}<C\varepsilon.
$$

With the same argument in case (ii), we get $|J_2^{(k)}|\leqslant C\varepsilon$.

(iv) If $|\varphi(z)|>r$ and $|\psi(z)|\leqslant r$, we reset $J_1^{(k)}+J_2^{(k)}=-J_3^{(k)}-J_4^{(k)}$, where

$$
\begin{aligned}
J_3^{(k)} &= (M_h^{\psi}(z)-M_g^{\varphi}(z))f_k'(\psi(z))(1-|\psi(z)|^2)^{\alpha},\\
J_4^{(k)} &= M_g^{\varphi}(z)\left[f_k'(\psi(z))(1-|\psi(z)|^2)^{\alpha}-f_k'(\varphi(z))(1-|\varphi(z)|^2)^{\alpha}\right].
\end{aligned}
$$

Using (5) again, we have $|J_3^{(k)}|<C|f_k'(\psi(z))|$. Applying Lemma 3 and (15), we obtain

$$
|J_4^{(k)}|\leqslant C\|f_k\|_{\mathscr{B}^{\alpha}}|M_g^{\varphi}(z)|\rho(\varphi(z),\psi(z))\leqslant C\varepsilon.
$$

Therefore, from (18), we can get that

$$
\begin{aligned}
\|(C_{\varphi}^{g}-C_{\psi}^{h})f_k\|_{\mathscr{B}^{\beta}} &\leqslant Cb_{\alpha}^r(f_k)+C\sup_{|\varphi(z)|\leqslant r}|f_k'(\varphi(z))|\\
&\quad +C\varepsilon+C\sup_{|\psi(z)|\leqslant r}|f_k'(\psi(z))|. \qquad (19)
\end{aligned}
$$

In view of the fact that $\{z \in D : |z| \leqslant r\}$ is compact, and that

$$\begin{aligned} b^r_\alpha(f_k) &= \max\{\sup_{z\in D_r}(1-|z|^2)^\alpha |f_k'(z)|, \sup_{z\in D_r}(1-|z|^2)^{\alpha+1}|f_k''(z)|\} \\ &< \max\{\sup_{z\in D_r}|f_k'(z)|, \sup_{z\in D_r}|f_k''(z)|\}, \end{aligned}$$

then (19) implies that $\|(C^g_\varphi - C^h_\psi)f_k\|_{\mathscr{B}^\beta} \to 0$ as $k \to \infty$. So we obtain that $C^g_\varphi - C^h_\psi$ is compact by Lemma 4.

Next we assume that $C^g_\varphi - C^h_\psi : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is compact. Then $C^g_\varphi - C^h_\psi : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is bounded. Let $\{z_k\}$ be a sequence of points in D such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Define

$$l_k(z) = \frac{1}{\alpha\overline{\varphi(z_k)}} \cdot \frac{1-|\varphi(z_k)|^2}{(1-\overline{\varphi(z_k)}z)^\alpha}, \quad f_k(z) = \frac{\alpha l_k(z)}{\alpha+1}\left(\sigma_{\varphi(z_k)}(z) + \frac{1}{\alpha\overline{\varphi(z_k)}}\right), \ z \in D.$$

In view of $\rho < 1$, from (6) and (7), we can see that

$$\begin{aligned} \|(C^g_\varphi - C^h_\psi)l_k\|_{\mathscr{B}^\beta} \geqslant \Big| & M^\varphi_g(z_k)\rho(\varphi(z_k),\psi(z_k)) \\ & - M^\psi_h(z_k)\rho(\varphi(z_k),\psi(z_k)) \frac{(1-|\psi(z_k)|^2)^\alpha(1-|\varphi(z_k)|^2)}{(1-\overline{\varphi(z_k)}\psi(z_k))^{\alpha+1}}\Big|, \end{aligned} \tag{20}$$

$$\|(C^g_\varphi - C^h_\psi)f_k\|_{\mathscr{B}^\beta} \geqslant \left| M^\psi_h(z_k)\rho(\varphi(z_k),\psi(z_k)) \frac{(1-|\psi(z_k)|^2)^\alpha(1-|\varphi(z_k)|^2)}{(1-\overline{\varphi(z_k)}\psi(z_k))^{\alpha+1}}\right|. \tag{21}$$

Since $C^g_\varphi - C^h_\psi$ is compact, by Lemma 4, we have $\|(C^g_\varphi - C^h_\psi)l_k\|_{\mathscr{B}^\beta} \to 0$ and $\|(C^g_\varphi - C^h_\psi)f_k\|_{\mathscr{B}^\beta} \to 0$ as $k \to \infty$. From (20) and (21), we conclude that (12) holds. Changing test functions $l_k(z)$ and $f_k(z)$ by exchanging φ and ψ, we can prove that (13) holds.

From (10) in Section 3, we have

$$\begin{aligned} \|(C^g_\varphi - C^h_\psi)l_k\|_{\mathscr{B}^\beta} \geqslant & \left|M^\varphi_g(z_k) - M^\psi_h(z_k)\right| \\ & - \left|M^\psi_h(z_k)\left[l_k'(\varphi(z_k))(1-|\varphi(z_k)|^2)^\alpha - l_k'(\psi(z_k))(1-|\psi(z_k)|^2)^\alpha\right]\right|, \end{aligned}$$

and that

$$\begin{aligned} & \left|M^\psi_h(z_k)\left[l_k'(\varphi(z_k))(1-|\varphi(z_k)|^2)^\alpha - l_k'(\psi(z_k))(1-|\psi(z_k)|^2)^\alpha\right]\right| \\ \leqslant & C\|l_k\|_{\mathscr{B}^\alpha}|M^\psi_h(z_k)|\rho(\varphi(z_k),\psi(z_k)) \to 0 \end{aligned}$$

as $|\psi(z_k)| \to 1$ from Lemma 2 and (13), we get $\left|M^\varphi_g(z_k) - M^\psi_h(z_k)\right| \to 0$ as $|\varphi(z_k)| \to 1$ and $|\psi(z_k)| \to 1$, which implies (14) holds. The whole proof is complete. $\square$

From Theorem 4 with $h(z) \equiv 0$, we obtain the following corollary.

COROLLARY 5. *Let $\varphi \in S(D)$ and $g \in H(D)$. Then $C_\varphi^g : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is compact if and only if $C_\varphi^g : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is bounded and*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |g(z)|}{(1-|\varphi(z)|^2)^\alpha} = 0.$$

Taking $\alpha = \beta = 1$, from Theorem 4, we get the next corollary.

COROLLARY 6. *Let $\varphi, \psi \in S(D)$ and $g, h \in H(D)$ such that $C_\varphi^g, C_\psi^h : \mathscr{B} \to \mathscr{B}$ are bounded. Then $C_\varphi^g - C_\psi^h : \mathscr{B} \to \mathscr{B}$ is compact if and only if $C_\varphi^g - C_\psi^h : \mathscr{B} \to \mathscr{B}$ is bounded and the following three conditions hold*

$$\lim_{|\varphi(z)|\to 1} \left|\mathscr{D}_g^\varphi(z)\right| \rho(\varphi(z), \psi(z)) = 0;$$
$$\lim_{|\psi(z)|\to 1} \left|\mathscr{D}_h^\psi(z)\right| \rho(\varphi(z), \psi(z)) = 0;$$
$$\lim_{|\varphi(z)|\to 1, |\psi(z)|\to 1} |\mathscr{D}_g^\varphi(z) - \mathscr{D}_h^\psi(z)| = 0.$$

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