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THE SHARP INEQUALITIES RELATED TO WILKER TYPE

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Abstract. Wu and Srivastava have shown that for $t \in (0,\pi/2)$ the inequalities

$$\left(\frac{\sqrt{\cos^{2q} t+8}+\cos^q t}{4}\right)^{1/q} < \frac{\sin t}{t} < \left(\frac{\sqrt{\cos^{2p} t+8}+\cos^p t}{4}\right)^{1/p}$$

hold if $q<0$ and $p \geqslant 1$. In this paper we find the largest $q=3/5$ and the smallest $p=\frac{\ln 2}{2(\ln\pi-\ln 2)}$ such that these inequalities hold. Moreover, our results also imply a type of new inequalities for trigonometric functions and give an anwser for a problem posed by Zhu.

1. Introduction and main results

Wilker [12] proposed two open problems, the first of which states that if $t \in (0,\pi/2)$ then

$$\left(\frac{\sin t}{t}\right)^2 + \frac{\tan t}{t} > 2, \tag{1.1}$$

which was proved by Sumner et al. in [11].

Wilker inequality (1.1) and the second one have attracted great interest of many mathematicians and have produced a batch of Wilker type ones by various generalizing and improving as well as different methods and ideas (see [1], [2], [3], [7], [9], [8], [15], [16], [13], [14], [18], [19], [20], [21], [22] and related references therein).

We now focus on the Wu and Srivastava's results. In 2007, they proved a weighted and exponential generalization of Wilker inequality in [13, Theorem 1]. As an application, they obtained that for $0<t<\pi/2$ the inequality

$$\left(\frac{\sin t}{t}\right)^p > \frac{4\cos^p t}{1+\sqrt{1+8\cos^{2p} t}} \tag{1.2}$$

holds true if $p>0$ or $p \leqslant -1$ [13, Theorem 2], which was used to solve an open problem posed by Sándor and Bencze in [10]. Additionally, by replacing p with $-p$ they derived that the inequality

$$\left(\frac{\sin t}{t}\right)^p < \frac{\sqrt{\cos^{2p} t+8}+\cos^p t}{4} \tag{1.3}$$

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holds for $t \in (0, \pi/2)$ if $p \geqslant 1$ [13, Corollary 4].

In fact, the right hand side of (1.2) can be written as

$$\frac{4}{\cos^{-p} t + \sqrt{\cos^{-2p} t + 8}}$$

and inequality (1.2) is equivalent to

$$\left(\frac{\sin t}{t}\right)^{-p} < \frac{\cos^{-p} t + \sqrt{\cos^{-p} t + 8}}{4}.$$

Hence, Wu and Srivastava have shown actually that the inequality (1.3) holds true for $t \in (0, \pi/2)$ if $p < 0$ or $p \geqslant 1$. In other words, the inequality

$$\frac{\sin t}{t} > \left(\frac{\sqrt{\cos^{2p} t + 8} + \cos^{p} t}{4}\right)^{1/p} \tag{1.4}$$

holds for $t \in (0, \pi/2)$ if $p < 0$ and its reverse holds if $p \geqslant 1$.

The purpose of this paper is to determine the smallest (or largest) p such that inequality (1.4) (or its reverse) holds for $t \in (0, \pi/2)$. Our main results are the following two theorems.

THEOREM 1. *The inequality (1.4) holds for $t \in (0, \pi/2)$ if and only if $p \leqslant 3/5$. Moreover, the double inequality*

$$\left(\frac{\sqrt{\cos^{6/5} t + 8} + \cos^{3/5} t}{4}\right)^{5/3} < \frac{\sin t}{t} < \frac{2^{11/6}}{\pi}\left(\frac{\sqrt{\cos^{6/5} t + 8} + \cos^{3/5} t}{4}\right)^{5/3} \tag{1.5}$$

with the best possible constant $2^{11/6}/\pi \approx 1.1343$.

THEOREM 2. *The reverse inequality of (1.4) holds for $t \in (0, \pi/2)$ if and only if $p \geqslant p_0 = \frac{\ln 2}{2(\ln \pi - \ln 2)} \approx 0.76746$. Moreover, we have*

$$\alpha\left(\frac{\sqrt{\cos^{2p_0} t + 8} + \cos^{p_0} t}{4}\right)^{1/p_0} < \frac{\sin t}{t} < \left(\frac{\sqrt{\cos^{2p_0} t + 8} + \cos^{p_0} t}{4}\right)^{1/p_0}, \tag{1.6}$$

where $\alpha = t_1^{-1}\left(\frac{\sqrt{\cos^{2p_0} t_1 + 8} + \cos^{p_0} t_1}{4}\right)^{-1/p_0} \sin t_1 \approx 0.98213$ is the best possible constant, here $t_1 \approx 1.4427$ is the unique root of the equation

$$\frac{(\sin t \cos t)\sqrt{\cos^{2p_0} t + 8}}{\sqrt{\cos^{2p_0} t + 8}\cos^2 t + \cos^{p_0} t \sin^2 t} - t = 0$$

on $(0, \pi/2)$.

2. Lemmas

In order to prove Theorems 1 and 2, we need the following lemmas.

LEMMA 1. *Let H be the function defined on $(-\infty,\infty)\times(0,\infty)$ by*

$$H(p,a)=\left(\frac{\sqrt{a^{2p}+8}+a^{p}}{4}\right)^{1/p} \text{ if } p\neq 0 \text{ and } H(0,a)=a^{1/3}. \tag{2.1}$$

Then H is increasing with respect to p on $(-\infty,\infty)$.

Proof. Differentiation yields

$$\begin{aligned} p^{2}\frac{1}{H}\frac{\partial H}{\partial p} &= \frac{pa^{p}}{\sqrt{a^{2p}+8}}\ln a-\ln\frac{\sqrt{a^{2p}+8}+a^{p}}{4}:=h(p),\\ h'(p) &= \frac{8pa^{p}\ln^{2}a}{\left(\sqrt{a^{2p}+8}\right)^{3}}, \end{aligned}$$

hence, $h(p)\geqslant h(0)=0$, then $\partial H/\partial p>0$, which is the desired result. □

LEMMA 2. *Let G_p be the function defined on $(0,\pi/2)$ by*

$$\begin{aligned} G_{p}(t) &= \ln H(p,\cos t)-\ln\frac{\sin t}{t} \qquad (2.2)\\ &= \begin{cases} \frac{1}{p}\ln\frac{\sqrt{\cos^{2p}t+8}+\cos^{p}t}{4}-\ln\frac{\sin t}{t} & \text{if } p\neq 0,\\ \frac{1}{3}\ln\cos t-\ln\frac{\sin t}{t} & \text{if } p=0. \end{cases} \end{aligned}$$

Then

$$\lim_{t\to 0^{+}}\frac{G_{p}(t)}{t^{4}}=\frac{1}{135}(5p-3), \tag{2.3}$$

$$\lim_{t\to\pi/2^{-}}G_{p}(t)=\begin{cases} -\frac{1}{2p}\ln 2+\ln\frac{\pi}{2} & \text{if } p>0,\\ -\infty & \text{if } p\leqslant 0. \end{cases} \tag{2.4}$$

Proof. Expanding in power series yields

$$G_{p}(t)=\frac{1}{135}t^{4}(5p-3)+O(t^{6}),$$

which implies (2.3).

The first limit relation (2.4) is evident if $p>0$. To obtain the second one, it suffices to note that $H(p,\cos t)$ can be written as

$$H(p,\cos t)=\left(\frac{\sqrt{1+8\cos^{-2p}t}+1}{4}\right)^{1/p}\cos t \text{ if } p<0 \text{ and } H(0,\cos t)=(\cos t)^{1/3},$$

which proves the lemma. □

3. Proof of Theorem 1

Proof of Theorem 1.
Necessity. If (1.4) holds for $t \in (0, \pi/2)$, then by Lemma 2 we have

$$\lim_{t \to 0^+} \frac{G_p(t)}{t^4} = \frac{1}{135}(5p-3) \leqslant 0.$$

Solving the inequality for p yields $p \leqslant 3/5$.

Sufficiency. Due to Lemma 1, it suffices to prove $G_p(t) < 0$ for all $t \in (0, \pi/2)$ if $p = 3/5$.

Differentiation leads to

$$\begin{aligned} G'_p(t) &= \frac{1}{t} - \frac{(\cos^2 t)\sqrt{\cos^{2p} t + 8} + \cos^p t \sin^2 t}{(\cos t \sin t)\sqrt{\cos^{2p} t + 8}} \\ &= \frac{(\cos^2 t)\sqrt{\cos^{2p} t + 8} + \cos^p t \sin^2 t}{t(\cos t \sin t)\sqrt{\cos^{2p} t + 8}} g_1(t), \end{aligned} \tag{3.1}$$

where

$$g_1(t) = \frac{(\sin t \cos t)\sqrt{\cos^{2p} t + 8}}{\sqrt{\cos^{2p} t + 8}\cos^2 t + \cos^p t \sin^2 t} - t. \tag{3.2}$$

Differentiating $g_1(t)$ gives

$$g'_1(t) = \frac{-\sin^2 t \times g_2(\cos^2 t)}{\left(\sqrt{\cos^{2p} t + 8}\cos^2 t + \cos^p t \sin^2 t\right)^2 \sqrt{\cos^{2p} t + 8}}, \tag{3.3}$$

where

$$g_2(x) = x^{p/2}\left(2x^{p+1} + x^p + 8(p+2)x + 8(1-p)\right) - \left(8x + 2x^{p+1} - x^p\right)\sqrt{x^p + 8}, \tag{3.4}$$

here $x = \cos^2 t \in (0,1)$.

Clearly, if we can prove $g_2(x) > 0$ for $x \in (0,1)$, then $g_2(\cos^2 t) > 0$ for $t \in (0, \pi/2)$, and then g_1 is decreasing on $(0, \pi/2)$. It follows that $g_1(t) < 0$, then $G'_{3/5}(t) < 0$, which leads us to $G_{3/5}(t) < G_{3/5}(0^+) = 0$, thus the sufficiency will be complete. For this end, we have to deal with $g_2(x)$. We define

$$g_{21}(x) = 2x^{p+1} + x^p + 8(p+2)x + 8(1-p), \tag{3.5}$$

$$g_{22}(x) = 8 + 2x^p - x^{p-1}. \tag{3.6}$$

Then $g_2(x)$ can be written as

$$g_2(x) = x^{p/2} g_{21}(x) - x\sqrt{x^p + 8}\, g_{22}(x). \tag{3.7}$$

Since

$$g'_{22}(x) = 2px^{p-1} + (1-p)x^{p-2} > 0$$

if $p \in (0,1)$ and $g_{22}(0^+) = -\infty$, $g_{22}(1^-) = 9 > 0$, there is a $x_{01} \in (0,1)$ such that $g_{22}(x) < 0$ for $x \in (0, x_{01})$ and $g_{22}(x) > 0$ for $x \in (x_{01}, 1)$.

We now distinguish two cases to prove that $g_2(x) > 0$ for $x \in (0,1)$.

Case 1: $x \in (0, x_{01})$. In this case, we see that $g_{22}(x) < 0$, which together with

$$g_{21}(x) > g_{21}\left(0^+\right) = 8(1-p) > 0$$

if $p \in (0,1)$ yields $g_2(x) > 0$ for $x \in (0, x_{01})$.

Case 2: $x \in (x_{01}, 1)$. In this case, since $g_{21}(x), g_{22}(x) > 0$, we have $\operatorname{sgn} g_2(x) = \operatorname{sgn} g_3(x)$, where g_3 is defined on $(x_{01}, 1)$ by the formula

$$g_3(x) = \frac{x^{p/2} g_{21}(x) + x\sqrt{x^p + 8}\, g_{22}(x)}{8x^p} g_2(x), \tag{3.8}$$

here g_{21}, g_{22} are defined by (3.5), (3.6), respectively. Simplifying yields

$$\begin{aligned} g_3(x) = {} & 8(p-1)^2 + 16\left(-p^2 - p + 3\right)x \\ & + 8\left(p^2 + 4p - 1\right)x^2 + 4px^{p+2} + 2(7-p)x^{p+1} \\ & + x^{2p+1} - 64x^{2-p} - (2p-1)x^p. \end{aligned} \tag{3.9}$$

When $p = 3/5$, we have

$$25g_3(x) = 816x + 352x^2 - 5x^{3/5} - 1600x^{7/5} + 320x^{8/5} + 25x^{11/5} + 60x^{13/5} + 32.$$

With $x \to x^5$ and factoring yield

$$\begin{aligned} 25g_3\left(x^5\right) &= 60x^{13} + 25x^{11} + 352x^{10} + 320x^8 - 1600x^7 + 816x^5 - 5x^3 + 32 \\ &= (x-1)^2 (60x^{11} + 120x^{10} + 205x^9 + 642x^8 + 1079x^7 + 1836x^6 \\ &\quad + 993x^5 + 150x^4 + 123x^3 + 96x^2 + 64x + 32) > 0. \end{aligned}$$

Case 1 and 2 indicate that $g_2(x) > 0$ for $x \in (0,1)$ and the sufficiency follows.

Using the monotonicity of $G_{3/5}$, we get

$$\ln \frac{\pi}{2^{11/6}} = G_{3/5}\left(\pi/2^-\right) < G_{3/5}(t) < G_{3/5}\left(0^+\right) = 0,$$

which implies (1.5) and the proof of Theorem 1 is finished. □

4. Proof of Theorem 2

Proof of Theorem 2.

Necessity. If the reverse inequality of (1.4) holds for all $t \in (0, \pi/2)$, then by Lemma 2 we have

$$\lim_{t \to \pi/2^-} G_p(t) = -\frac{1}{2p} \ln 2 + \ln \frac{\pi}{2} \geqslant 0 \text{ if } p > 0,$$

which yields $p \geqslant p_0 = \frac{\ln 2}{2(\ln \pi - \ln 2)}$.

Sufficiency. We prove the condition $p \geqslant p_0 = \frac{\ln 2}{2(\ln \pi - \ln 2)}$ is also sufficient. In view of Lemma 1 it is enough to show that the reverse inequality of (1.4) holds if $p = p_0$. We shall prove now that there exits a x_{02} in $(0,1)$ such that $g_2(x) > 0$ for $x \in (0, x_{02})$ and $g_2(x) < 0$ for $x \in (x_{02}, 1)$ if $p \in (3/4, 4/5)$. For this end, we need to prove the following lemmas. □

LEMMA 3. *For $p \in (3/4, 4/5)$, let g_3 be defined on $(0,1)$ by (3.9). Then there is a $x_3 \in (0,1)$ such that $g_3'''(x) < 0$ for $x \in (0, x_3)$ and $g_3'''(x) > 0$ for $x \in (x_3, 1)$.*

Proof. Differentiation yields

$$\begin{aligned} g_3'(x) = {} & 16\left(-p^2 - p + 3\right) + 16\left(p^2 + 4p - 1\right)x \\ & + 4p(p+2)x^{p+1} + 2(p+1)(7-p)x^p \\ & + (2p+1)x^{2p} - 64(2-p)x^{1-p} - p(2p-1)x^{p-1}, \end{aligned} \tag{4.1}$$

$$\begin{aligned} g_3''(x) = {} & 16\left(p^2 + 4p - 1\right) + 4p(p+2)(p+1)x^p + 2p(p+1)(7-p)x^{p-1} \\ & + 2p(2p+1)x^{2p-1} - 64(1-p)(2-p)x^{-p} + p(1-p)(2p-1)x^{p-2}, \end{aligned}$$

$$\begin{aligned} x^{3-p}g_3'''(x) = {} & -p(2p-1)(1-p)(2-p) - 2p(1-p)(p+1)(7-p)x \\ & + 4p^2(p+1)(p+2)x^2 + 64p(1-p)(2-p)x^{2-2p} \\ & + 2p(2p-1)(2p+1)x^{p+1} \\ : = {} & g_4(x), \end{aligned}$$

$$\begin{aligned} g_4'(x) = {} & -2p(1-p)(p+1)(7-p) + 8p^2(p+1)(p+2)x \\ & + 128p(1-p)^2(2-p)x^{1-2p} + 2p(2p-1)(p+1)(2p+1)x^p. \end{aligned}$$

For $x \in (0,1)$ and $p_0 \in (3/4, 4/5)$, since both the second and fourth terms of $g_4'(x)$ are positive and the last member of the third term is greater than 1 due to $1 - 2p < 0$, we easily get

$$\begin{aligned} g_4'(x) > {} & -2p(1-p)(p+1)(7-p) + 128p(p-1)^2(2-p) \\ = {} & 130p(1-p)\left(\tfrac{11}{13} - p\right)\left(\tfrac{11}{5} - p\right) > 0, \end{aligned}$$

which implies that g_4 is increasing on $(0,1)$. Note that

$$\begin{aligned} \operatorname{sgn} g_4\left(0^+\right) &= \operatorname{sgn}\left(-p(2p-1)(p-1)(p-2)\right) < 0 \\ g_4\left(1^-\right) &= 3p\left(35p^2 - 63p + 38\right) > 0, \end{aligned}$$

it is seen that there is a unique $x_3 \in (0,1)$ such that $g_4(x) < 0$, then $g_3'''(x) < 0$ for $x \in (0, x_3)$ and $g_4(x) > 0$, then $g_3'''(x) > 0$ for $x \in (x_3, 1)$. This indicates that g_3'' is decreasing on $(0, x_3)$ and increasing on $(x_3, 1)$. □

REMARK 1. For $p \in (3/4, 4/5)$, let g_3 be defied on $(0,1)$ by (3.9). It is easy to check that the equation $g_3'(x) = 0$ has three roots on $(0,1)$ at least. To this end, it suffices to verify that

$$g_3'(0^+) < 0,\ g_3'(e^{-8}) > 0,\ g_3'(e^{-2}) < 0,\ \ g_3'(1^-) > 0.$$

In fact, from (4.1) it is derived that

$$\begin{aligned} \operatorname{sgn} g_3'(0^+) &= \operatorname{sgn}(-p(2p-1)) < 0, \\ g_3'(1^-) &= 27(5p-3) > 0. \end{aligned}$$

We now show $g_3'(e^{-8}) > 0$. Clearly, $g_3'(x)$ consists of seven terms. For convenience, we denote in sequential order these terms by u_0 and $u_i(x)$, $i = 1,2,3,4,5,6$. Then for $p \in (3/4, 4/5)$, we have

$$\begin{aligned} u_0 &= 16(-p^2 - p + 3) > 16\left(-\left(\tfrac{4}{5}\right)^2 - \tfrac{4}{5} + 3\right) = \tfrac{624}{25}, \\ u_i(e^{-8}) &> 0,\ \ i = 1,2,3,4, \\ u_5(e^{-8}) &= -64(2-p)e^{8p-8} > -64\left(2 - \left(\tfrac{3}{4}\right)\right)e^{8(4/5)-8} = -80e^{-8/5}, \\ u_6(e^{-8}) &= -p(2p-1)e^{8-8p_0} > -\left(\tfrac{4}{5}\right)\left(2\left(\tfrac{4}{5}\right) - 1\right)e^{8-8(3/4)} = -\tfrac{12}{25}e^2. \end{aligned}$$

Hence,

$$g_3'(e^{-8}) = u_0 + \sum_{i=1}^{6} u_i(e^{-8}) > \tfrac{624}{25} - 80e^{-8/5} - \frac{12}{25}e^2 \approx 5.2615 > 0.$$

At last, we show that $g_3'(e^{-2}) < 0$. It is obtained that

$$\begin{aligned} u_0 &= 16(-p^2 - p + 3) < 16\left(-\left(\tfrac{3}{4}\right)^2 - \left(\tfrac{3}{4}\right) + 3\right) = 27, \\ u_1(e^{-2}) &= 16(p^2 + 4p - 1)e^{-2} < 16\left(\left(\tfrac{4}{5}\right)^2 + 4\left(\tfrac{4}{5}\right) - 1\right)e^{-2} = \tfrac{1136}{25}e^{-2}, \\ u_2(e^{-2}) &= (2p+1)e^{-4p} < \left(2\left(\tfrac{4}{5}\right) + 1\right)e^{-4(3/4)} = \tfrac{13}{5}e^{-3}, \\ u_3(e^{-2}) &= 4p(p+2)e^{-2p-2} < 4\left(\tfrac{4}{5}\right)\left(\left(\tfrac{4}{5}\right) + 2\right)e^{-2(3/4)-2} = \tfrac{224}{25}e^{-7/2}, \\ u_4(e^{-2}) &= 2(p+1)(7-p)e^{-2p} < 2\left(\left(\tfrac{4}{5}\right) + 1\right)\left(7 - \left(\tfrac{3}{4}\right)\right)e^{-2(3/4)} = \tfrac{45}{2}e^{-3/2}, \\ u_5(e^{-2}) &= -64(2-p)e^{2p-2} < -64\left(2 - \left(\tfrac{4}{5}\right)\right)e^{2(3/4)-2} = -\tfrac{384}{5}e^{-1/2}, \\ u_6(e^{-2}) &= -p(2p-1)e^{2-2p} < -\left(\tfrac{3}{4}\right)\left(2\left(\tfrac{3}{4}\right) - 1\right)e^{2-2(4/5)} = -\tfrac{3}{8}e^{2/5}, \end{aligned}$$

and therefore,

$$\begin{aligned} g_3'(e^{-2}) &= u_0 + \sum_{i=1}^{6} u_i(e^{-2}) \\ &< 27 + \tfrac{1136}{25}e^{-2} + \frac{13}{5}e^{-3} + \frac{224}{25}e^{-7/2} + \frac{45}{2}e^{-3/2} - \frac{384}{5}e^{-1/2} - \frac{3}{8}e^{2/5} \\ &\approx -8.5709 < 0. \end{aligned}$$

LEMMA 4. *For* $p\in(3/4,4/5)$, *let* g_3 *be defined on* $(0,1)$ *by (3.9). Then* $g_3(x)>0$ *for* $x\in(0,e^{-8})$.

Proof. From (3.9) we see that $g_3(x)$ is a sum of eight terms and denote in sequential order them by v_0 and $v_i(x)$, $i=1,2,3,4,5,6,7$, respectively. Obviously, we have

$$\begin{aligned} v_0 &= 8(p-1)^2 > 8\left(1-\tfrac{4}{5}\right)^2 = \tfrac{8}{25} \\ v_i(x) &> 0,\ \ i=2,3,4,5,6, \\ v_7(x) &= -64e^{8p-16} > -64e^{8(4/5)-16} = -64e^{-48/5}, \\ v_8(x) &= -(2p-1)e^{-8p} > -\left(2\left(\frac{4}{5}\right)-1\right)e^{-8(3/4)} = -\tfrac{3}{5}e^{-6}, \end{aligned}$$

and so

$$\begin{aligned} g_3(x) &= v_0+\sum_{i=1}^{7}u_i(x) > v_0+v_7+v_8 \\ &> \tfrac{8}{25}-64e^{-48/5}-\tfrac{3}{5}e^{-6} \approx 0.31418 > 0, \end{aligned}$$

which proves the lemma. □

LEMMA 5. *For* $p\in(3/4,4/5)$, *let* g_3 *be defined on* $(0,1)$ *by (3.9). Then there is a* $x_{02}\in(0,1)$ *so that* $g_3(x)>0$ *for* $x\in(0,x_{02})$ *and* $g_3(x)<0$ *for* $x\in(x_{02},1)$.

Proof. Lemma 3 tell us there is a $x_3\in(0,1)$ such that $g_3'''(x)<0$ for $x\in(0,x_3)$ and $g_3'''(x)>0$ for $x\in(x_3,1)$, which implies that $g_3''(x)$ is decreasing on $(0,x_3)$ and increasing on $(x_3,1)$.

By Remark 1, $g_3'(x)$ has three zero on $(0,1)$ at least. Consequently, we claim that $g_3''(x_3)<0$. If not, that is, $g_3''(x_3)\geqslant 0$, which in combination with

$$\begin{aligned} \operatorname{sgn} g_3''(0^+) &= \operatorname{sgn}(p(1-p)(2p-1)) > 0, \\ g_3''(1^-) &= -17p^2+279p-144 > 0 \end{aligned}$$

leads to $g_3''(x)>0$ for $x\in(0,1)$, that is, g_3' is increasing on $(0,1)$. This together with the facts

$$\begin{aligned} \operatorname{sgn} g_3'(0^+) &= \operatorname{sgn}(-p(2p-1)) < 0, \\ g_3'(1^-) &= 27(5p-3) > 0 \end{aligned}$$

implies that the equation $g_3'(x)=0$ has a unique solution on $(0,1)$, which yields a contradiction.

Thus, there are two numbers $x_{21}\in(0,x_3)$ and $x_{22}\in(x_3,1)$ such that $g_3''(x)>0$ for $x\in(0,x_{21})\cup(x_{22},1)$ and $g_3''(x)<0$ for $x\in(x_{21},x_{22})$, which shows that $g_3'(x)$ is increasing on $(0,x_{21})\cup(x_{22},1)$ and decreasing on (x_{21},x_{22}). Also, application of

Remark 1 again yields that $g_3'(x_{21}) > 0$ and $g_3'(x_{22}) < 0$. Because if $g_3'(x_{22}) > 0$, then $g_3'(x) > g_3'(x_{22}) > 0$ for $x \in (x_{21}, x_{22}) \cup (x_{22}, 1)$, which together with $g_3'(0^+) < 0$ yields that the equation $g_3'(x) = 0$ has a unique root on $(0,1)$ which is in $(0, x_{21})$. It is clearly impossible. Likewise, if $g_3'(x_{21}) < 0$, then $g_3'(x) < g_3'(x_{21}) < 0$ for $x \in (0, x_{21}) \cup (x_{21}, x_{22})$, which in conjunction with $g_3'(1^-) > 0$ leads to that the equation $g_3'(x) = 0$ also has a unique solution on $(0,1)$ which is in $(x_{22}, 1)$. Clearly, it is in contradiction with Remark 1.

Hence, the equation $g_3'(x) = 0$ has three solutions which are $x_{11} \in (0, x_{21})$, $x_{12} \in (x_{21}, x_{22})$ and $x_{13} \in (x_{22}, 1)$ such that $g_3'(x) < 0$ for $x \in (0, x_{11}) \cup (x_{12}, x_{13})$ and $g_3'(x) > 0$ for $x \in (x_{11}, x_{12}) \cup (x_{13}, 1)$.

Clearly, $g_3(0^+) = 8(p-1)^2 > 0$, $g_3(x_{13}) < g_3(1^-) = 0$. On the other hand, it is easy to see that $x_{11} \in \left(0, e^{-8}\right)$ by virtue of Remark 1, and application of Lemma 4 leads to $g_3(x_{11}) > 0$. Then $g_3(x_{12}) > g_3(x_{11}) > 0$. Thus the equation $g_3(x) = 0$ has a unique solution x_{02} in (x_{12}, x_{13}) such that $g_3(x) > 0$ for $x \in (0, x_{02})$ and $g_3(x) < 0$ for $x \in (x_{02}, 1)$.

This completes the proof of this lemma. □

LEMMA 6. *For $p \in (3/4, 4/5)$, let g_2 be defined on $(0,1)$ by (3.4). Then there is a $x_0 \in (0,1)$ such that $g_2(x) > 0$ for $x \in (0, x_0)$ and $g_2(x) < 0$ for $x \in (x_0, 1)$.*

Proof. As mentioned by the Case 1 of proof of Theorem 1, if $p \in (0,1)$ then there is a $x_{01} \in (0,1)$ such that $g_{22}(x) < 0$ for $x \in (0, x_{01})$ and $g_{22}(x) > 0$ for $x \in (x_{01}, 1)$, and $g_2(x) > 0$ for $x \in (0, x_{01})$. We take $x_0 = \max(x_{01}, x_{02})$ and consider two cases: $x_{01} \geqslant x_{02}$ and $x_{01} < x_{02}$.

Case 1: $x_{01} \geqslant x_{02}$. Then $x_0 = x_{01}$. When $x \in (0, x_{01})$ we have shown that $g_2(x) > 0$. While $x \in (x_{01}, 1) \subset (x_{02}, 1)$, since $g_{21}(x), g_{22}(x) > 0$, we have $\operatorname{sgn} g_2(x) = \operatorname{sgn} g_3(x)$ due to (3.7). Application of Lemma 5 yields $g_2(x) < 0$.

Case 2: $x_{01} < x_{02}$. Then $x_0 = x_{02}$. If $x \in (0, x_{01})$ then $g_2(x) > 0$. If $x \in (x_{01}, x_{02}) \subset (0, x_{02})$, then it is obtained by Lemma 5 that $g_2(x) > 0$. While $x \in (x_{02}, 1)$, Lemma 5 gives $g_2(x) < 0$.

This completes the proof. □

Now we continue proving Theorem 2.

Continuation of the proof of Theorem 2. For $p \in (3/4, 4/5)$, applying Lemma 6 and noting that $x = \cos^2 t$, we see that there is a unique $t_0 = \arccos\sqrt{x_0} \in (0, \pi/2)$ such that $g_2\left(\cos^2 t\right) > 0$ for $t \in (t_0, \pi/2)$ and $g_2\left(\cos^2 t\right) < 0$ for $t \in (0, t_0)$, which implies that $g_1'(t) < 0$ for $t \in (t_0, \pi/2)$ and $g_1'(t) > 0$ for $t \in (0, t_0)$. Thus it can be seen that $g_1(t) > g_1(0^-) = 0$ for $t \in (0, t_0)$ and $g_1(t)$ decreases from $g_1(t_0)$ to $g_1(\pi/2^-) = -\pi/2 < 0$ with t increases from t_0 to $\pi/2^-$, which means that the equation $g_1(t) = 0$ has a unique solution t_1 on $(0, \pi/2)$ which is in $(t_0, \pi/2)$ such that $g_1(t) > 0$ for $t \in (0, t_1)$ and $g_1(t) < 0$ for $t \in (t_1, \pi/2)$. By the relation between $g_1(t)$ and $G_p'(t)$ given by (3.1), this yields that G_p is increasing on $(0, t_1)$ and decreasing on $(t_1, \pi/2)$.

When $p = p_0 = \frac{\ln 2}{2(\ln\pi - \ln 2)} \in (3/4, 4/5)$, $G_{p_0}(\pi/2^-) = 0$, we conclude that

$$0 = G_p\left(0^+\right) < G_{p_0}(t) < G_{p_0}(t_1) \text{ if } t \in (0, t_1),$$
$$0 = G_p\left(\pi/2^-\right) < G_{p_0}(t) < G_{p_0}(t_1) \text{ if } t \in (t_1, \pi/2),$$

that is, $0 < G_{p_0}(t) < G_{p_0}(t_1)$ if $t \in (0, \pi/2)$.

Solving the equation $g_1(t) = 0$ by mathematical computation software we find that $t_1 \in (1.4427, 1.4428)$, and $\exp\left(-G_{p_0}(t_1)\right) \approx 0.98213$, which proves the sufficiency and the proof of Theorem 2 is completed. □

5. Remarks

REMARK 2. The following sharp inequality is contained in [21, Theorem 1] (see also [17]):

$$\frac{\sin t}{t} > \left(\frac{2}{3} + \frac{1}{3}\cos^{4/5} t\right)^{5/4} \tag{5.1}$$

for $t \in (0, \pi/2)$. However, our sharp lower bound for $(\sin t)/t$ given by (1.5), denote by Y, is superior to the Zhu's, denote by Z. For proving $Y > Z$, we set $\cos^{1/5} t = x$, then $x \in (0, 1)$. Thus it suffices to show that for $x \in (0, 1)$

$$D(x) := Y^{12/5}(x) - Z^{12/5}(x) = \left(\frac{\sqrt{x^6+8} + x^3}{4}\right)^4 - \left(\tfrac{1+x^4}{2}\right)^3 > 0.$$

Rearranging yield

$$32D(x) = x^3\left(x^6+4\right)\sqrt{x^6+8} - l(x),$$

where

$$l(x) = 3x^{12} + 12x^8 - 8x^6 + 12x^4 - 4.$$

Differentiation leads to

$$l'(x) = 12x^3\left(3x^8 + 8x^4 + 4\left(1 - x^2\right)\right) > 0,$$

which together with $l(3/4) < 0$ and $l(1) > 0$ yields that there is a unique $z_0 \in (3/4, 1)$ such that $l(x) < 0$ for $x \in (0, z_0)$ and $l(x) > 0$ for $x \in (z_0, 1)$.

In the case of $x \in (0, z_0)$, it is clear that $D(x) > 0$.

In the case of $x \in (z_0, 1)$, we define

$$\begin{aligned} u(x) &:= 32D(x) \times \left(x^3\left(x^6+4\right)\sqrt{x^6+8} + l(x)\right) \\ &:= 8\left(1 - x^2\right)v(x), \end{aligned}$$

where

$$\begin{aligned} v(x) &= w(x) + 10x^4 - 2x^2 - 2, \\ w(x) &= x^{22} + x^{20} + 10x^{18} + 2x^{16} + 29x^{14} + 5x^{12} + 36x^{10} + 12x^8 + 18x^6. \end{aligned}$$

Since $z_0 > 3/4$, so $w(x) > w(3/4) = 7.1887 > 7$, then

$$v(x) = w(x) + 10x^4 - 2x^2 - 2 > 10x^4 - 2x^2 + 5 > 0.$$

Thus it can be seen that $D(x) > 0$ for $x \in (0,1)$, that is, $Y(x) > Z(x)$.

It is easy to verify that

$$\left(\left(\frac{\sin t}{t}\right)^p - \frac{\sqrt{\cos^{2p} t + 8} + \cos^p t}{4}\right)\left(\left(\frac{\sin t}{t}\right)^p + \frac{\sqrt{\cos^{2p} t + 8} - \cos^p t}{4}\right)$$
$$= \left(\frac{\sin t}{t}\right)^{2p} - \frac{1}{2}(\cos^p t)\left(\frac{\sin t}{t}\right)^p - \frac{1}{2},$$

and therefore our main results can be restated as a equivalent assertion.

PROPOSITION 1. *For $t \in (0, \pi/2)$ the inequality*

$$\left(\frac{\sin t}{t}\right)^{2p} > \frac{1}{2}\left(\left(\frac{\sin 2t}{2t}\right)^p + 1\right) \tag{5.2}$$

holds if and only if $0 < p \leqslant 3/5$. While its reverse is valid if and only if $p \geqslant p_0 = \frac{\ln 2}{2(\ln \pi - \ln 2)}$ or $p < 0$.

REMARK 3. The inequality 5.2 seems to be a new type of inequality for trigonometric functions.

We easily check that the identity

$$\left(\frac{\sin t}{t}\right)^{2p} \frac{\sqrt{\cos^{2p} t + 8} + \cos^p t}{4}\left(\left(\frac{t}{\sin t}\right)^{2p} + \left(\frac{t}{\tan t}\right)^p - 2\right)$$
$$= \left(\frac{\sqrt{\cos^{2p} t + 8} + \cos^p t}{4} - \left(\frac{\sin t}{t}\right)^p\right)\left(1 + \frac{\sqrt{\cos^{2p} t + 8} + \cos^p t}{2}\left(\frac{\sin t}{t}\right)^p\right)$$

is true, and so our Theorems 1 and 2 are in fact equivalent to the following assertion:

PROPOSITION 2. *For $t \in (0, \pi/2)$ the inequality*

$$\left(\frac{t}{\sin t}\right)^{2p} + \left(\frac{t}{\tan t}\right)^p > 2$$

holds true if and only if $p \geqslant p_0 = \frac{\ln 2}{2(\ln \pi - \ln 2)}$ or $p < 0$. Its reverse holds if and only if $0 < p \leqslant 3/5$.

REMARK 4. Clearly, the proposition gives an answer for a problem posed by Zhu in [20].

REFERENCES

[1] A. BARICZ AND J. SÁNDOR, *Extensions of generalized Wilker inequality to Bessel functions*, J. Math. Inequal. **2** (2008), 397–406.
[2] C.-P. CHEN AND W.-S. CHEUNG, *Wilker- and Huygens-type inequalities and solution to Oppenheim's problem*, Integral Transforms Spec. Funct. **23**, 5 (2012), 325–336.
[3] C.-P. CHEN, W.-S. CHEUNG, *Sharpness of Wilker and Huygens type inequalities*, J. Inequal. Appl. **2012** (2012): 72.
[4] J.-C. KUANG, *Applied Inequalities*, Shangdong Science and Technology Press, Jinan, China, 3rd edition, 2004 (in Chinese).
[5] I. LAZAREVIĆ, *Neke nejednakosti sa hiperbolickim funkcijama*, Univerzitet u Beogradu, Publikacije Elektrotehničkog Fakulteta, Serija Matematika i Fizika **170** (1966), 41–48.
[6] D. S. MITRINOVIČ, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.
[7] C. MORTICI, *The natural approach of Wilker-Cusa-Huygens inequalities*, Math. Inequal. Appl. **14**, 3 (2011), 535–541.
[8] E. NEUMAN AND J. SÁNDOR, *On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker, and Huygens inequalities*, Math. Inequal. Appl. **13**, 4 (2010), 715–723.
[9] E. NEUMAN, *On Wilker and Huygens type inequalities*, Math. Inequal. Appl. **15**, 2 (2012), 271–279.
[10] J. SÁNDOR AND M. BENCZE, *On Huygens's trigonometric inequality*, RGMIA Research Report Collection **8**, 3 (2005); available online at `http://www.ajmaa.org/RGMIA/papers/v8n3/Huygens.pdf`.
[11] J. S. SUMNER, A. A. JAGERS, M. VOWE AND J. ANGLESIO, *Inequalities involving trigonometric functions*, Amer. Math. Monthly **98** (1991), 264–267.
[12] J. B. WILKER, *Problem E3306*, Amer. Math. Monthly **96**, 1 (1989), 55.
[13] S.-H. WU AND H. M. SRIVASTAVA, *A weighted and exponential generalization of Wilker's inequality and its applications*, Integral Transforms and Spec. Funct. **18** (2007), 529–535.
[14] S.-H. WU AND H. M. SRIVASTAVA, *A further refinement of Wilker's inequality*, Integral Transforms and Spec. Funct. **19** (2008), 757–765.
[15] S.-H. WU, *On extension and refinement of Wilker's inequality*, Rocky Mountain J. Math. **39** (2009), 683–687.
[16] S.-H. WU AND L. DEBNATH, *Wilker-type inequalities for hyperbolic functions*, Appl. Math. Lett. **25**, 5 (2012), 837–842.
[17] ZH.-H. YANG, *Sharp bounds for Seiffert mean in terms of weighted power means of arithmetic mean and geometric mean*, Math. Inequal. Appl. in press.
[18] L. ZHANG AND L. ZHU, *A new elementary proof of Wilker's inequalities*, Math. Inequal. Appl. **11**, 1 (2008), 149–151.
[19] L. ZHU, *On Wilker-type inequalities*, Math. Inequal. Appl. **10**, 4 (2007), 727–731.
[20] L. ZHU, *Some new Wilker-type inequalities for circular and hyperbolic functions*, Abstr. Appl. Anal. **2009** (2009), Art. ID 485842.
[21] L. ZHU, *A source of inequalities for circular functions*, Comput. Math. Appl. **58** (2009), 1998–2004.
[22] L. ZHU, *Inequalities for Hyperbolic functions and their Applications*, J. Inequal. Appl. **2010** (2010), Art. ID 130821.

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