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THE TRIANGLE INEQUALITY FOR GRADED REAL VECTOR SPACES OF LENGTH 3 AND 4

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Abstract. In [3] the classical theorem of Minkowski on lattice points and convex bodies in $\mathbb{R}^n$ was generalized to simply connected nilpotent Lie groups with a grading of length 2. In doing so it was necessary to prove the triangle inequality for a certain natural homogeneous norm (with respect to automorphisms) of the Lie algebra associated with the grading (the case of a grading of length 1 being the Schwarz inequality). Here we shall extend the homogeneous norms for which the triangle inequality holds to gradings of length 3 and 4. The results hold for any graded real vector space of those lengths.

In [3] the classical theorem of Minkowski on lattice points and convex bodies in $\mathbb{R}^n$ was extended to simply connected nilpotent Lie groups with a $\mathbb{Q}$-structure whose Lie algebra $\mathfrak{g}$ admits a grading of length 2 (and in particular to 2 step nilpotent groups). In doing so it was necessary to prove the triangle inequality for a certain natural homogeneous norm (with respect to automorphisms) of the Lie algebra associated with the grading (the case of a grading of length 1 being the Schwarz inequality). Here we shall extend the homogeneous norms for which the triangle inequality holds to gradings of length 3 and 4. As the reader will see at the end of this note, further extension by this method fails when r is 5 and by general principles must break down at some point. The author does not know if the triangle inequality holds for $r = 5$.

A Lie algebra $\mathfrak{g}$ is said to admit a *grading* (see [1]) if there is a finite family of subspaces $V_1, \dots, V_r$ with $\mathfrak{g} = V_1 \oplus \dots \oplus V_r$ satisfying $[V_i, V_j] \subseteq V_{i+j}$ for all i, j. If $\mathfrak{g}$ is a graded Lie algebra, define for $t \in \mathbb{R}^\times$,

$$\alpha_t(v_1, \dots, v_r) = (tv_1, t^2 v_2 \dots, t^r v_r).$$

An easy check shows α_t is a Lie algebra automorphism of $\mathfrak{g}$. Because of these automorphisms the theorem of [2] implies that if a Lie algebra, $\mathfrak{g}$, admits a grading it must be nilpotent.

As in [3] a *homogeneous norm* on a graded Lie algebra $\mathfrak{g}$ is a function, $\|\cdot\| : \mathfrak{g} \to \mathbb{R}$, satisfying the following conditions.

1. $\|\cdot\| \geqslant 0$ and is 0 only at 0.
2. $\| X \| = \| -X \|$ for all $X \in \mathfrak{g}$.

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3. $\| \alpha_t(X) \| = |t| \, \| X \|$, for all $t \in \mathbb{R}$ and $X \in \mathfrak{g}$.

4. $\| X+Y \| \leqslant \| X \| + \| Y \|$ for all X and $Y \in \mathfrak{g}$.

A graded Lie algebra, $\mathfrak{g}$, always possesses a natural candidate for homogeneous norm, where $\| \cdot \|_i$ is the Euclidean norm on each V_i. (Henceforth we shall suppress the subscript). For $X = (v_1, \ldots, v_r)$ let

$$\| X \| = (\| v_1 \|_1^{2r} + \| v_2 \|_2^{2r-2} + \ldots + \| v_r \|_r^2)^{\frac{1}{2r}},$$

However, the subadditivity 4) and homogeneity 3) properties may not be valid. For example, the reader can check that homogeneity is only valid when $r \leqslant 2$.

Our result here is the following: Its proof will proceed by successive reduction to simpler and simpler inequalities.

THEOREM 1. $\| X+Y \| \leqslant \| X \| + \| Y \|$ *whenever* $r \leqslant 4$.

Proof. Let $X = (v_1, \ldots, v_r)$ and $Y = (w_1, \ldots, w_r)$. Then

$$\| X \|^{2r} = \| v_1 \|^{2r} + \cdots + \| v_r \|^2$$

and

$$\| Y \|^{2r} = \| w_1 \|^{2r} + \cdots + \| w_r \|^2 .$$

That is,

$$\| X \|^{2r} = \| v_1 \|^{2r} + \| X' \|^{2r-2} . \tag{0}$$

where $X' = (v_2, \ldots, v_r)$ and similarly for Y and Y'.

Our objective is to prove

$$\| v_1 + w_1 \|^{2r} + \cdots + \| v_r + w_r \|^2 \leqslant (\| X \| + \| Y \|)^{2r}. \tag{1}$$

By the Schwarz inequality it is sufficient (and necessary) to show

$$(\| v_1 \| + \| w_1 \|)^{2r} + \cdots + (\| v_r \| + \| w_r \|)^2 \leqslant (\| X \| + \| Y \|)^{2r}. \tag{2}$$

Now we will argue by induction (the initial case being proved for $r = 2$ in [3]) and assume

$$(\| v_2 \| + \| w_2 \|)^{2r-2} + \cdots + (\| v_r \| + \| w_r \|)^2 \leqslant (\| X' \| + \| Y' \|)^{2r-2}, \tag{3}$$

Hence it's sufficient to prove

$$(\| v_1 \| + \| w_1 \|)^{2r} + (\| X' \| + \| Y' \|)^{2r-2} \leqslant (\| X \| + \| Y \|)^{2r}. \tag{4}$$

We first show (4) is correct when either X' or $Y' = 0$. For suppose $Y' = 0$. Then we have to verify

$$(\| v_1 \| + \| w_1 \|)^{2r} + (\| X' \|)^{2r-2} \leqslant (\| X \| + \| w_1 \|)^{2r}. \tag{5}$$

The right side of (5) is $\geqslant \| X \|^{2r} + 2 \| X \|^{r} \| w_1 \|^{r} + \| w_1 \|^{2r}$. Using (0), expanding $(\| v_1 \| + \| w_1 \|)^{2r}$ in a similar manner and cancelling appropriate terms, the verification reduces to $\| X \| \geqslant \| v_1 \|$. Similarly (5) is true if $X' = 0$.

Returning to (4) we may now assume both X' and $Y' \neq 0$. Expand the terms being raised to $2r$ power on both the left and right and take $v_2 = \ldots v_r = 0 = w_2 = \ldots w_r$ in the expansion on the right side. This yields each term of $(\| v_1 \| + \| w_1 \|)^{2r}$ being dominated by the corresponding term of $(\| X \| + \| Y \|)^{2r}$. Taking account of this for all but the middle term gives

$$\frac{(2r)!}{r!^2}(\| v_1 \|^{r} \| w_1 \|^{r}) + (\| X' \| + \| Y' \|)^{2r-2} \leqslant \frac{(2r)!}{r!^2}(\| X \|^{r} \| Y \|^{r}). \tag{6}$$

But $\| X \|^{2r} = \| v_1 \|^{2r} + \| X' \|^{2r-2}$ and $\| Y \|^{2r} = \| w_1 \|^{2r} + \| Y' \|^{2r-2}$.
Hence

$$\frac{(2r)!}{r!^2} \| v_1 \|^{r} \| w_1 \|^{r} + (\| X' \| + \| Y' \|)^{2r-2} \leqslant \frac{(2r)!}{r!^2} \sqrt{\| v_1 \|^{2r} + \| X' \|^{2r-2}} \sqrt{\| w_1 \|^{2r} + \| Y' \|^{2r-2}}. \tag{7}$$

Now square both sides of (7)

$$\left(\frac{(2r)!}{r!^2}\right)^2 \| v_1 \|^{2r} \| w_1 \|^{2r} + (\| X' \| + \| Y' \|)^{4r-4} + 2\frac{(2r)!}{r!^2} \| v_1 \|^{r} \| w_1 \|^{r} (\| X' \| + \| Y' \|)^{2r-2} \leqslant \left(\frac{(2r)!}{r!^2}\right)^2 (\| v_1 \|^{2r} + \| X' \|^{2r-2})(\| w_1 \|^{2r} + \| Y' \|^{2r-2}). \tag{8}$$

Multiplying out the right side of (8) and cancelling yields

$$(\| X' \| + \| Y' \|)^{4r-4} + 2\frac{(2r)!}{r!^2} \| v_1 \|^{r} \| w_1 \|^{r} (\| X' \| + \| Y' \|)^{2r-2} \leqslant \left(\frac{(2r)!}{r!^2}\right)^2 (\| v_1 \|^{2r} \| Y' \|^{2r-2} + \| w_1 \|^{2r} \| X' \|^{2r-2} + (\| X' \| \| Y' \|)^{2r-2}). \tag{9}$$

Hence it is sufficient to prove the following two inequalities

$$(\| X' \| + \| Y' \|)^{4r-4} \leqslant \left(\frac{(2r)!}{r!^2}\right)^2 (\| X' \| \| Y' \|)^{2r-2}) \tag{10}$$

and

$$2\frac{(2r)!}{r!^2} \| v_1 \|^{r} \| w_1 \|^{r} (\| X' \| + \| Y' \|)^{2r-2} \leqslant \left(\frac{(2r)!}{r!^2}\right)^2 (\| v_1 \|^{2r} \| Y' \|^{2r-2} + \| w_1 \|^{2r} \| X' \|^{2r-2}). \tag{11}$$

Since (11) is evidently true when $w_1 = 0$, here we may assume $w_1 \neq 0$.

Now (10), or rather its square root, implies (11) since by cancelling and dividing we get $2 \leqslant t + \frac{1}{t}$, where

$$t = \frac{\| v_1 \|^r \| Y' \|^{r-1}}{\| w_1 \|^r \| X' \|^{r-1}}.$$

Finally, we turn to the square root of (10) itself,

$$(\| X' \| + \| Y' \|)^{2r-2} \leqslant \frac{(2r)!}{r!^2} (\| X' \| \| Y' \|)^{r-1}. \tag{12}$$

We show the left side (12) is maximal when $\| X' \| = \| Y' \|$. Because we can ignore the exponents, this is a question of maximizing the function $\phi(x,y) = x+y$ where x and $y > 0$. Now consider the $\frac{1}{4}$ circle, $x^2+y^2 = r^2$. An application of Lagrange multipliers shows that constrained to this, ϕ is maximized where $\operatorname{grad}\phi = (1,1) = \lambda(2x,2y)$ so $x = y$. Since these $\frac{1}{4}$ circles fill out the first quadrant, this proves the claim. Similarly the right side is minimized when $\| X' \| = \| Y' \|$. Here again we can ignore exponents. The conclusion follows from a similar application of Lagrange multipliers considering the function $\psi(x,y) = xy$, where x and $y > 0$, (viz. $\operatorname{grad}\psi = (y,x) = \lambda(2x,2y)$ so that $\frac{y}{x} = \frac{x}{y}$, and since x and y are positive, $x = y$). The net effect of this is to reduce the proof of (10) to the following special case:

$$2^{2r-2} \leqslant \frac{(2r)!}{r!^2}, \tag{13}$$

which can be checked by hand. Evidently it is true for all $r \leqslant 4$ and false for $r = 5$. □

We remark that by Stirling's formula $n!$ is asymptotic to $n^{n+\frac{1}{2}}\sqrt{2\pi}e^{-n}$ as the positive integer $n \to \infty$. Thus $2^{2r-2}/\frac{(2r)!}{r!^2}$ is asymptotic to $\frac{\sqrt{r}\sqrt{\pi}}{4}$ which tends to infinity as r does. Hence even if the inequality (13) were valid for $r = 5$, there would be no possibility of extending Theorem 1 to high r by this method.

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