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SOME INEQUALITIES FOR HAUSDORFF OPERATORS

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Abstract. In this paper, we give the sufficient and necessary conditions for the boundedness of Hausdorff operators on various function spaces. Moreover, we consider Lipschitz estimates for the commutator of Hausdorff operators. We extend some known results.

1. Introduction

Hausdorff means, including the Cesàro means, have a deep root in the study of some classical problems in analysis, for example, summability of the Fourier series [17] and Hausdorff summability of number series [16]. A brief history of the study of the Hausdorff summability can be found in [21].

In [30], Siskakis considered the Cesàro means for power series on the Hardy space H^1 in the unit circle. The Fourier transform setting of this problem was considered by Giang and Moricz in [10]. In fact, general Hausdorff means of a Fourier-Stieltjes transform were introduced even earlier, in [12], but only on $L^1(\mathbb{R})$. The one-dimensional Hausdorff operator H_φ generated by a function $\varphi \in L^1(\mathbb{R})$ is given by

$$H_\varphi f(x) = \int_{\mathbb{R}} \frac{\varphi(t)}{|t|} f(x/t)dt, \quad x \in \mathbb{R}.$$

If $f \in L^1(\mathbb{R})$, the Fubini's theorem gives the formula

$$(H_\varphi f)^\wedge(\xi) = \int_{\mathbb{R}} \widehat{f}(t\xi)\varphi(t)dt, \quad t \in \mathbb{R},$$

where $\widehat{f}$ is the Fourier transform of a function f. In particular, if we choose $\varphi(t) = \alpha(1-t)^{\alpha-1}\chi(0,1)(t)$ for $\alpha = 1,2,\ldots$, then $H_\varphi = C_\alpha$ is called the Cesàro operator of order α. For H_φ, Goldberg [13] investigated its properties on $L^p(\mathbb{R})$ with $1 < p \leqslant 2$. Georgakis [12] obtained its Fourier analytic properties on the space of complex bounded regular Borel measures on $\mathbb{R}$, and as a special case he showed if $\varphi \in L^1(\mathbb{R})$, then H_φ is a bounded operator on $L^1(\mathbb{R})$. In [22], by using a thoughtful method, Liflyand and Móricz proved H_φ is bounded on $H^1(\mathbb{R})$ and obtained an interchangeability relation between H_φ and the Hilbert transform, which contains partial result in [10]. For

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$0<p<1$, Kanjin [18] proved H_φ is bounded on the Hardy space $H^p(\mathbb{R})$ with the assumption on $\widehat{\varphi}$, and specially he showed that C_α is bounded on H^p provided $\alpha\in\mathbb{N}$ and $\frac{2}{2\alpha+1}<p<1$. Miyachi proved C_α is bounded on H^p for every $\alpha>0$ and every $0<p<1$ in [27]. Recently, Liflyand and Miyachi [24] obtained the boundedness for H_φ on the $H^p(\mathbb{R})(0<p<1)$ under some smoothness conditions on φ.

In the multidimensional case, the Cesàro means in [11] and Hausdorff means of a special form in [23] were considered in dimension 2 only for the so-called product (mixed) Hardy spaces. In the recent paper [32], a slight extension was made in the same direction of product Hardy spaces. In [4], Brown and Móricz defined the multivariate Hausdorff operator $H(\mu,c,A)$ acting on Borel measurable functions $f:\mathbb{R}^n\to\mathbb{C}$ by setting

$$H(\mu,c,A)f(x):=\int_{\mathbb{R}^n}c(s)f(A(s)x)d\mu(s),$$

where μ is a σ-finite complex measure defined on the Borel measurable subsets of $\mathbb{R}^n$, $c:\mathbb{R}^n\to\mathbb{C}$ is a Borel measurable function which is nonzero μ-a.e., and $A:=[a_{ij}]$ is a $n\times n$ matrix whose entries $a_{ij}:\mathbb{R}^n\to\mathbb{C}$ are Borel measurable functions and such that A is nonsingular μ-a.e., and they gave the $L^p(\mathbb{R}^n)$ boundedness for $H(\mu,c,A)$. For the $H^1(\mathbb{R}^n)$ boundedness, two commonly used methods can not be applied for $H(\mu,c,A)$, one is commuting $H(\mu,c,A)$ and Riesz transform, the other is using the duality of $H^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$. Despite all this, there are some special cases. In [19], Lerner and Liflyand obtained the $H^1(\mathbb{R}^n)$ boundedness when μ is absolutely continuous. And then, in [20], Liflyand proved the same boundedness under weaker condition based on atomic decomposition of $H^1(\mathbb{R}^n)$. In [28], the boundedness for such operators in $H^1(\mathbb{R}^n)$ was proved for diagonal matrices A with all entries on the diagonal equal to one another.

For the $H^1(\mathbb{R}^n)$ boundedness, Andersen [2] proved it for the following Hausdorff operator

$$H_\mu f(x)=\int_{\mathbb{R}}f(tx)d\mu(t),\qquad x\in\mathbb{R}^n,$$

and its formal adjoint operator

$$H_\mu^* f(x)=\int_{\mathbb{R}}|t|^{-n}f(x/t)d\mu(t),\qquad x\in\mathbb{R}^n,$$

where μ is a signed σ-finite Borel measure on $\mathbb{R}$. In addition to the $H^1(\mathbb{R}^n)$ boundedness, Andersen also obtained the $L^p(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$ boundedness for H_μ and H_μ^* in the same paper. Actually, before Andersen's results, Brown and Móricz obtained the L^p-boundedness for H_μ and H_μ^* for $n=1$ in [3]. We want to point out that H_μ is a generalization of H_φ indeed. As two special cases of Hausdorff operators H_μ and H_μ^*, the weighted Hardy-Littlewood average and weighted Cesàro average are defined by

$$(U_\psi f)(x)=\int_0^1 f(tx)\psi(t)dt,\qquad (V_\psi f)(x)=\int_0^1 f(x/t)t^{-n}\psi(t)dt,$$

where $\psi:[0,1]\to[0,\infty)$. Xiao [33] gave the operator norms of U_ψ and V_ψ on both $L^p(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$. Recently, Tang and Zhai [31] extended Xiao's results to $Q_p^{\alpha,q}(\mathbb{R}^n)$, and Fu [8] extended Xiao's results to much more function spaces, such as $\dot{B}^{q,\lambda}(\mathbb{R}^n)$, $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$ (for their definitions, see Section 2).

Inspired by the above results, in this paper, we consider and obtain some sufficient and necessary conditions of the boundedness for H_μ and H_μ^* on some more function spaces, such as $L^p(\mathbb{R}^n)$, $BMO(\mathbb{R}^n)$ and Herz spaces. Some sufficient conditions have been given, for example, see the following Theorem A and Theorem B, but we also obtain some necessary conditions. Unfortunately, we can not extend our necessity results to the general measure and the Hausdorff type operator $H(\mu,c,A)$, since we do not know how to choose suitable functions. So we only study H_μ and H_μ^* on some function spaces.

It is worth pointing out that one can learn much more backgrounds and developments of the Hausdorff operators by reading a comprehensive survey written by Liflyand [21]. Also, we no longer list much more results on some other spaces, such as local Hardy space and Herz type Hardy space, see [5, 6, 7]. For the sake of convenience, we describe Andersen's results.

THEOREM A. ([2]) *Let $1 \leqslant p \leqslant \infty$, $1/p+1/p'=1$, and suppose μ is a signed σ-finite Borel measure on $\mathbb{R}$ with $\mu(\{0\})=0$.*

(a) If $\|\mu\|_{n/p'} < \infty$, then H_μ^* is defined on $L^p(\mathbb{R}^n)$ and satisfies

$$\|H_\mu^* f\|_{L^p(\mathbb{R}^n)} \leqslant \|\mu\|_{n/p'} \|f\|_{L^p(\mathbb{R}^n)}.$$

(b) If $\|\mu\|_{n/p} < \infty$, then H_μ is defined on $L^p(\mathbb{R}^n)$ and satisfies

$$\|H_\mu f\|_{L^p(\mathbb{R}^n)} \leqslant \|\mu\|_{n/p} \|f\|_{L^p(\mathbb{R}^n)}.$$

(c) If $\|\mu\|_{n/p'} < \infty$, then H_μ and H_μ^* satisfy

$$\int_{\mathbb{R}^n} H_\mu^* f(x) g(x) dx = \int_{\mathbb{R}^n} f(x) H_\mu g(x) dx$$

for $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$.

THEOREM B. ([2]) *Let μ be a signed σ-finite Borel measure on $\mathbb{R}$ with $\mu(\{0\}) = 0$.*

(a) If $\|\mu\|_n < \infty$, then

$$\|H_\mu^* f\|_{BMO(\mathbb{R}^n)} \leqslant \|\mu\|_n \|f\|_{BMO(\mathbb{R}^n)}, \quad \forall f \in BMO(\mathbb{R}^n).$$

(b) If $\|\mu\| < \infty$, then

$$\|H_\mu f\|_{BMO(\mathbb{R}^n)} \leqslant \|\mu\| \|f\|_{BMO(\mathbb{R}^n)}, \quad \forall f \in BMO(\mathbb{R}^n).$$

The plan of this paper is as follows. In Section 2, we give some notations and definitions. In Section 3, we give the sufficient and necessary conditions of the boundedness for H_μ and H_μ^* on $L^p(\mathbb{R}^n)$, $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ and Herz type spaces when μ is positive on $\mathbb{R}$. In Section 4, we characterize μ for which H_μ and H_μ^* are bounded on $BMO(\mathbb{R}^n)$, $Q_p^{\alpha,q}(\mathbb{R}^n)$, and $CBMO^{q,\lambda}(\mathbb{R}^n)$ when μ is positive on $\mathbb{R}^+$. In Section 5, we consider Lipschitz estimates for the commutator of H_μ^*.

2. Notations and definitions

A signed Borel measure μ on $\mathbb{R}$ may be decomposed as $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are positive Borel measures on $\mathbb{R}$, at least one of which is finite. Let $\|\mu\|_\alpha := \int_{\mathbb{R}} |t|^{-\alpha} d|\mu|$, where $|\mu| = \mu^+ + \mu^-$. For $\alpha = 0$, $\|\mu\|_0 = \|\mu\|$ is the total variation of μ.

Let I denote the cube with sides parallel to the axes, and $|I|$ be the Lebesgue measure of I; $f_I := \frac{1}{|I|}\int_I f(x)dx$ is the average of f on I. For any $t > 0$, $tB(x,r)$ stands for $B(x,tr)$, where $B(x,r)$ denotes the ball centered at x with radius r. For $k \in \mathbb{Z}$, let $B_k := \{x \in \mathbb{R}^n : |x| \leqslant 2^k\}$, $\Delta_k := B_k \backslash B_{k-1}$, and $\chi_k (k \in \mathbb{Z})$ is the characteristic function of the set Δ_k. The notation $L^p(\mathbb{R}^n)(1 \leqslant p \leqslant \infty)$ is for the ordinary Lebesgue space ([14]).

DEFINITION 2.1. ([14]) Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The space $BMO(\mathbb{R}^n)$ consists of those measurable functions f with bounded mean oscillation

$$\|f\|_{BMO(\mathbb{R}^n)} = \sup_{I \subset \mathbb{R}^n} \frac{1}{|I|}\int_I |f(x) - f_I| dx < \infty.$$

DEFINITION 2.2. ([1]) Let $q \in (1,\infty)$ and $\lambda \in (-1/q, 0)$. A function $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to the λ-central Morrey space $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ if

$$\|f\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)} = \sup_{R>0}\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)} |f(x)|^q\right)^{1/q} < \infty.$$

DEFINITION 2.3. ([25]) Let $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty)$ and $\lambda \in [0,\infty)$. A function $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\})$ belongs to the homogeneous Morrey-Herz space $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$ if

$$\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\|f\chi_k\|^p_{L^q(\mathbb{R}^n)}\right)^{1/p} < \infty.$$

Obviously, $M\dot{K}^{\alpha,0}_{p,q}(\mathbb{R}^n) = \dot{K}^{\alpha,p}_q(\mathbb{R}^n)$ is the homogeneous Herz space (see [26]), and the classical Morrey space $M^\lambda_q(\mathbb{R}^n)$ is a subspace of $M\dot{K}^{0,\lambda}_{q,q}(\mathbb{R}^n)$.

DEFINITION 2.4. ([1]) Let $q \in (1,\infty)$ and $\lambda \in (-1/q, 1/n)$. A function $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to the λ-central bounded mean oscillation space $CBMO^{q,\lambda}(\mathbb{R}^n)$ if

$$\|f\|_{CBMO^{q,\lambda}(\mathbb{R}^n)} = \sup_{R>0}\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)} |f(x) - f_{B(0,R)}|^q\right)^{1/q} < \infty.$$

DEFINITION 2.5. ([31]) Let $\alpha \in (0,1)$, $p \in (0,\infty]$ and $q \in [1,\infty]$. $Q^{\alpha,q}_p(\mathbb{R}^n)$ is the space of all $f \in S'_\infty(\mathbb{R}^n)$ such that $f(x) - f(y)$ is measurable functions on $\mathbb{R}^n \times \mathbb{R}^n$ and

$$\|f\|_{Q^{\alpha,q}_p(\mathbb{R}^n)} = \sup_I |I|^{1/p-1/q}\left(\int_I\int_I \frac{|f(x)-f(y)|^q}{|x-y|^{n+q\alpha}} dxdy\right)^{1/q} < \infty.$$

Throughout this paper the constant C is not necessarily the same at each occurrence. The notation $A \approx B$ means that there are positive constants C_1, C_2 such that $C_1 B \leqslant A \leqslant C_2 B$.

3. Best estimates for H_μ^* and H_μ

In this section, we suppose μ is positive on $\mathbb{R}$ and give the sufficient and necessary conditions for the boundedness of H_μ and H_μ^* on $L^p(\mathbb{R}^n)$, $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ and Herz type spaces.

THEOREM 3.1. *Let $p \in [1,\infty]$ and $f \in L^p(\mathbb{R}^n)$. Then*
(i)

$$\|H_\mu^* f\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)}$$

if and only if $\|\mu\|_{n/p'} < \infty$; moreover, $\|H_\mu^*\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \|\mu\|_{n/p'}$;
(ii)

$$\|H_\mu f\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)}$$

if and only if $\|\mu\|_{n/p} < \infty$; moreover, $\|H_\mu\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \|\mu\|_{n/p}$.

Proof. We only prove (i), and the proof of (ii) is similar. The sufficiency is proved by Andersen [2]. Suppose $\|H_\mu^* f\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)}$. For any $k \in \mathbb{Z}$, we take $f_k(x) = |x|^{-(n+1/k)/p}\chi_{|x|\geqslant 1}(x)$, then

$$H_\mu^* f_k(x) = |x|^{-(n+1/k)/p}\int_{-|x|}^{|x|} |t|^{-n}|t|^{(n+1/k)/p} d\mu(t).$$

By a simple calculation, we obtain $\|f_k\|_{L^p(\mathbb{R}^n)} = (k|S^{n-1}|)^{1/p}$, and

$$\begin{aligned}
\|H_\mu^* f_k\|_{L^p(\mathbb{R}^n)} &\geqslant \left\{\int_{|x|>k} |x|^{-(n+1/k)/p}\left(\int_{-|x|}^{|x|} |t|^{-n}|t|^{(n+1/k)/p} d\mu(t)\right)^p dx\right\}^{1/p} \\
&= \left(\int_{|x|>k} |x|^{-n-1/k} dx\right)^{1/p}\int_{-k}^{k} |t|^{-n}|t|^{(n+1/k)/p} d\mu(t) \\
&= k^{-\frac{1}{k}}\|f_k\|_{L^p(\mathbb{R}^n)}\int_{-k}^{k} |t|^{-n}|t|^{(n+1/k)/p} d\mu(t).
\end{aligned}$$

Then

$$k^{-1/k}\int_{-k}^{k} |t|^{-n}|t|^{(n+1/k)/p} d\mu(t) \leqslant C.$$

Let $k \to \infty$, we have $\|\mu\|_{n/p'} < \infty$. And then follow the argument in [33], we obtain

$$\|H_\mu^*\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \|\mu\|_{n/p'}. \quad \square$$

THEOREM 3.2. *Let $q \in (1,\infty)$ and $\lambda \in [-1/q, 0]$. If $f \in \dot{B}^{q,\lambda}(\mathbb{R}^n)$, then*
(i)

$$\|H_\mu^* f\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)} \leqslant C\|f\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)}$$

if and only if $\|\mu\|_{n(1+\lambda)} < \infty$; moreover, $\|H_\mu^*\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)\to \dot{B}^{q,\lambda}(\mathbb{R}^n)} = \|\mu\|_{n(1+\lambda)}$;
(ii)

$$\|H_\mu f\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)} \leqslant C\|f\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)}$$

if and only if $\|\mu\|_{-n\lambda} < \infty$; *moreover,* $\|H_\mu\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)\to\dot{B}^{q,\lambda}(\mathbb{R}^n)} = \|\mu\|_{-n\lambda}$.

Proof. It suffices to prove (i). By the Minkowski's inequality, we get

$$\begin{aligned}
&\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)}|H_\mu^* f(x)|^q dx\right)^{1/q}\\
&\leqslant \int_{\mathbb{R}}\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)}|f(x/t)|^q dx\right)^{1/q}|t|^{-n}d\mu(t)\\
&=\int_{\mathbb{R}}\left(\frac{1}{|B(0,R/|t|)|^{1+\lambda q}}\int_{B(0,R/|t|)}|f(x)|^q dx\right)^{1/q}|t|^{-n(1+\lambda)}d\mu(t)\\
&\leqslant \|\mu\|_{n(1+\lambda)}\|f\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)}.
\end{aligned}$$

Conversely, the case $\lambda = -1/q$ is obvious because of $\dot{B}^{q,-1/q}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$ and Theorem 3.1. While $\lambda \in (-1/q, 0]$, take $f_0(x) = |x|^{n\lambda}$, $x \in \mathbb{R}^n$, then $\|f_0\|_{\dot{B}^{q,\lambda}(\mathbb{R}^n)} = v_n^{-q\lambda}/(q\lambda+1)$, and

$$H_\mu^* f_0(x) = f_0(x)\int_{\mathbb{R}}|t|^{-n(1+\lambda)}d\mu(t) < \infty.$$

Therefore $\|\mu\|_{n(1+\lambda)} < \infty$. □

THEOREM 3.3. *Let* $\alpha \in \mathbb{R}$ *and* $p,\ q \in (1,\infty)$. *If* $f \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, *then*

(i)

$$\|H_\mu^* f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \leqslant C\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}$$

if and only if $\|\mu\|_{n/q'-\alpha} < \infty$, *moreover,* $\|H_\mu^*\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)\to\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \approx \|\mu\|_{n/q'-\alpha}$.

(ii)

$$\|H_\mu f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \leqslant C\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}$$

if and only if $\|\mu\|_{n/q-\alpha} < \infty$, *moreover,* $\|H_\mu\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)\to\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \approx \|\mu\|_{n/q-\alpha}$.

Proof. It suffices to prove (i). By the definition of Herz space and the Minkowski's inequality, we have

$$\begin{aligned}
\|H_\mu^* f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}^p &= \sum_{k\in\mathbb{Z}} 2^{k\alpha p}\left(\int_{\Delta_k}\left|\sum_{k\in\mathbb{Z}}\int_{\Delta_j}|t|^{-n}f(x/t)d\mu(t)\right|^q dx\right)^{p/q}\\
&\leqslant \sum_{k\in\mathbb{Z}} 2^{k\alpha p}\left(\sum_{j\in\mathbb{Z}}\left(\int_{\Delta_k}\left|\int_{\Delta_j}|t|^{-n}f(x/t)d\mu(t)\right|^q dx\right)^{1/q}\right)^p\\
&\leqslant \sum_{k\in\mathbb{Z}} 2^{k\alpha p}\left(\sum_{j\in\mathbb{Z}}\int_{\Delta_j}|t|^{-n}\|f(x/t)\|_{L^q(\Delta_k)}d\mu(t)\right)^p.
\end{aligned}$$

For $t \in \Delta_j$, note that

$$\begin{aligned}
\|f(x/t)\|_{L^q(\Delta_k)} &\leqslant |t|^{n/q}\left(\int_{\Delta_{k-j}}|f(x)|^q dx + \int_{\Delta_{k-j+1}}|f(x)|^q dx\right)^{1/q}\\
&= C|t|^{n/q}(\|f\chi_{\Delta_{k-j}}\|_{L^q(\mathbb{R}^n)} + \|f\chi_{\Delta_{k-j+1}}\|_{L^q(\mathbb{R}^n)}).
\end{aligned}$$

Thus,

$$\|H^*_\mu f\|_{\dot K^{\alpha,p}_q(\mathbb{R}^n)} \leqslant C\Big(\sum_{k\in\mathbb{Z}} 2^{k\alpha p}\Big(\sum_{j\in\mathbb{Z}}\Big(\int_{\Delta_j}|t|^{-n/q'}\sum_{i=0}^{1}\|f\chi_{\Delta_{k-j+i}}\|_{L^q(\mathbb{R}^n)}d\mu(t)\Big)^p\Big)^{1/p}$$
$$\leqslant C\sum_{j\in\mathbb{Z}}\Big(\int_{\Delta_j}|t|^{-n/q'}\Big(\sum_{k\in\mathbb{Z}}2^{k\alpha p}\sum_{i=0}^{1}\|f\chi_{\Delta_{k-j+i}}\|^p_{L^q(\mathbb{R}^n)}\Big)^{1/p}d\mu(t)\Big)$$
$$= C\|\mu\|_{n/q'-\alpha}\|f\|_{\dot K^{\alpha,p}_q(\mathbb{R}^n)}.$$

Conversely, take $f_\varepsilon(x) = |x|^{-(n+\varepsilon)/q-\alpha}\chi_{\{|x|>1\}}(x)$, where $\varepsilon\in(0,1)$. By a simple calculation, we have $\|f_\varepsilon\chi_k\|^q_{L^q(\mathbb{R}^n)} = |S^{n-1}|2^{-k(\varepsilon+\alpha q)}(2^{\varepsilon+\alpha q}-1)/(\varepsilon+\alpha q)$, and

$$\|f_\varepsilon\|_{\dot K^{\alpha,p}_q(\mathbb{R}^n)} = |S^{n-1}|^{\frac{1}{q}}\Big(\frac{2^{\frac{-\varepsilon p}{q}}}{1-2^{\frac{-\varepsilon p}{q}}}\Big)^{1/p}\Big(\frac{2^{\varepsilon+\alpha q}-1}{\varepsilon+\alpha q}\Big)^{1/q}.$$

On the other hand,

$$H^*_\mu f_\varepsilon(x) = \int_{-|x|}^{|x|}|t|^{-n}|t|^{(n+\varepsilon)/q+\alpha}|x|^{-(n+\varepsilon)/q-\alpha}d\mu(t).$$

Thus,

$$\|H^*_\mu f_\varepsilon\|^p_{\dot K^{\alpha,p}_q(\mathbb{R}^n)} \geqslant \sum_{k=1}^{\infty}2^{k\alpha p}\Big(\int_{|x|>1}|H^*_\mu f_\varepsilon(x)|^q\chi_k(x)dx\Big)^{p/q}.$$

Let $\delta := 1/\varepsilon > 1$, then there exists $m\in\mathbb{Z}^+$ such that $\delta\in[2^{m-1},2^m)$, and we have

$$\|H^*_\mu f_\varepsilon\|^p_{\dot K^{\alpha,p}_q(\mathbb{R}^n)}$$
$$\geqslant \sum_{k=1}^{\infty}2^{k\alpha p}\Big(\int_{|x|>\delta}\Big|\int_{-|x|}^{|x|}|t|^{-n}|t|^{(n+\varepsilon)/q+\alpha}|x|^{-(n+\varepsilon)/q-\alpha}d\mu(t)\Big|^q\chi_k(x)dx\Big)^{p/q}$$
$$\geqslant \sum_{k=1}^{\infty}2^{k\alpha p}\Big(\int_{|x|>\delta}|x|^{-n-\varepsilon-\alpha q}\chi_k(x)dx\Big)^{p/q}\Big(\int_{-1/\varepsilon}^{1/\varepsilon}|t|^{-n}|t|^{(n+\varepsilon)/q+\alpha}d\mu(t)\Big)^p$$
$$= |S^{n-1}|^{p/q}\Big(\frac{2^{\frac{-(m+1)\varepsilon p}{q}}}{1-2^{\frac{-\varepsilon p}{q}}}\Big)^{1/p}\Big(\frac{2^{\varepsilon+\alpha q}-1}{\varepsilon+\alpha q}\Big)^{p/q}\Big(\int_{-1/\varepsilon}^{1/\varepsilon}|t|^{-n}|t|^{(n+\varepsilon)/q+\alpha}d\mu(t)\Big)^p$$
$$\geqslant \varepsilon^{\varepsilon p/q}\|f_\varepsilon\|^p_{\dot K^{\alpha,p}_q(\mathbb{R}^n)}\Big(\int_{-1/\varepsilon}^{1/\varepsilon}|t|^{-n}|t|^{(n+\varepsilon)/q+\alpha}d\mu(t)\Big)^p.$$

Thus,

$$\|H^*_\mu f_\varepsilon\|_{\dot K^{\alpha,p}_q(\mathbb{R}^n)} \geqslant \varepsilon^{\varepsilon/q}\|f_\varepsilon\|_{\dot K^{\alpha,p}_q(\mathbb{R}^n)}\int_{-1/\varepsilon}^{1/\varepsilon}|t|^{-n}|t|^{(n+\varepsilon)/q+\alpha}d\mu(t).$$

Let $\varepsilon\to 0$, we reach

$$\|H^*_\mu\|_{\dot K^{\alpha,p}_q(\mathbb{R}^n)\to\dot K^{\alpha,p}_q(\mathbb{R}^n)} \geqslant \int_{-\infty}^{\infty}|t|^{-n}|t|^{n/q+\alpha}d\mu(t). \quad \square$$

THEOREM 3.4. *Let* $\alpha \in \mathbb{R}$, $p,\ q \in (1,\infty)$ *and* $\lambda \in (0,\infty)$. *If* $f \in M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$, *then*

(i)

$$\|H^*_\mu f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} \leqslant C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}$$

if and only if $\|\mu\|_{n/q'-\alpha+\lambda} < \infty$, *moreover,* $\|H^*_\mu\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)\to M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} \approx \|\mu\|_{n/q'-\alpha+\lambda}$;

(ii)

$$\|H_\mu f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} \leqslant C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}$$

if and only if $\|\mu\|_{n/q-\alpha+\lambda} < \infty$, *moreover,* $\|H_\mu\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)\to M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} \approx \|\mu\|_{n/q-\alpha+\lambda}$.

Proof. It suffices to prove (i). Use the Minkowski's inequality, we have

$$\begin{aligned}\|(H^*_\mu f)\chi_k\|_{L^q} &\leqslant \int_{\mathbb{R}} \left(\int_{\Delta_k} |f(x/t)|^q dx\right)^{1/q} |t|^{-n} d\mu(t)\\ &= \int_{\mathbb{R}} \left(\int_{2^{k-1}/|t|<|x|\leqslant 2^k/|t|} |f(x)|^q dx\right)^{1/q} |t|^{-n/q'} d\mu(t).\end{aligned}$$

For any $|t| \in (0,\infty)$, there exists $m \in \mathbb{Z}$ such that $|t| \in (2^{m-1},2^m]$ and then

$$\begin{aligned}\|(H^*_\mu f)\chi_k\|_{L^q} &\leqslant \int_{\mathbb{R}} \left(\int_{2^{k-m-1}/|t|<|x|\leqslant 2^{k-m+1}/|t|} |f(x)|^q dx\right)^{1/q} |t|^{-n/q'} d\mu(t)\\ &\leqslant \int_{\mathbb{R}} \left(\sum_{i=0}^{1} \|f\chi_{k-m-i}\|_{L^q(\mathbb{R}^n)}\right) |t|^{-n/q'} d\mu(t).\end{aligned}$$

Therefore,

$$\begin{aligned}&\|H^*_\mu f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}\\ &\leqslant \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left(\int_{\mathbb{R}} \left(\sum_{i=0}^{1} \|f\chi_{k-m-i}\|_{L^q(\mathbb{R}^n)}\right) |t|^{-n/q'} d\mu(t)\right)^p\right\}^{1/p}\\ &\leqslant \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \int_{\mathbb{R}} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|f\chi_{k-m}\|^p_{L^q(\mathbb{R}^n)} |t|^{-np/q'}\right)^{1/p} d\mu(t)\\ &\quad + \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \int_{\mathbb{R}} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|f\chi_{k-m+1}\|^p_{L^q(\mathbb{R}^n)} |t|^{-np/q'}\right)^{1/p} d\mu(t)\\ &\leqslant \|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} \int_{\mathbb{R}} (2^{m\alpha} + 2^{(m-1)\alpha}) |t|^{-n/q'} |t|^{-\lambda} d\mu(t)\\ &= C\|\mu\|_{n/q'-\alpha+\lambda} \|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}.\end{aligned}$$

Conversely, take $g_0(x) := |x|^{-(\alpha-\lambda+n/q)} \in M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$, then

$$H^*_\mu g_0(x) = g_0(x) \int_{\mathbb{R}} |t|^{\alpha-\lambda-n/q'} d\mu(t).$$

For $\alpha \neq \lambda$, it is easy to show that

$$\|g_0\chi_k\|_{L^q}^q = 2^{kq(\lambda-\alpha)}|S^{n-1}|(1-2^{(\alpha-\lambda)p})(\lambda q - \alpha q),$$

and

$$\|g_0\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} = \left(\frac{1}{1-2^{-\lambda p}}\right)^{1/p} |S^{n-1}|^{\frac{1}{q}} \left(\frac{1-2^{(\alpha-\lambda)p}}{(\lambda-\alpha)q}\right)^{1/q}.$$

For $\alpha = \lambda$, we have $\|g_0\chi_k\|_{L^q}^q = \ln 2 \cdot |S^{n-1}|$, and

$$\|g_0\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} = \left(\frac{1}{1-2^{-\lambda p}}\right)^{1/p} (\ln 2 \cdot |S^{n-1}|)^{1/q}.$$

Thus,

$$\|H_\mu^* g_0\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} = C\|g_0\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} \int_{\mathbb{R}} |t|^{\alpha-\lambda-n/q'} d\mu(t).$$

We obtain $\|\mu\|_{n/q'-\alpha+\lambda} < \infty$. □

REMARK 3.1. If $d\mu(t) = \psi(t)\chi_{(0,1)}(t)dt$ for $\psi : [0,1] \to [0,\infty)$, we can get the corresponding result in [8] and [33].

4. Some estimates for H_μ^* and H_μ

In this section, we suppose μ is positive on $\mathbb{R}^+$ and we characterize μ for which H_μ and H_μ^* are bounded on $BMO(\mathbb{R}^n)$, $Q_p^{\alpha,q}(\mathbb{R}^n)$, and $CBMO^{q,\lambda}(\mathbb{R}^n)$. Note that $H_\mu f(x) = \int_0^\infty f(tx)d\mu(t)$, $H_\mu^* f(x) = \int_0^\infty t^{-n} f(x/t)d\mu(t)$ and $\|\mu\|_\alpha = \int_0^\infty t^{-\alpha} d\mu(t)$.

THEOREM 4.1. *If $f \in BMO(\mathbb{R}^n)$, then*
(i)

$$\|H_\mu^* f\|_{BMO(\mathbb{R}^n)} \leqslant C\|f\|_{BMO(\mathbb{R}^n)}$$

if and only if $\|\mu\|_n < \infty$, moreover, $\|H_\mu^*\|_{BMO(\mathbb{R}^n)\to BMO(\mathbb{R}^n)} = \|\mu\|_n$;
(ii)

$$\|H_\mu f\|_{BMO(\mathbb{R}^n)} \leqslant C\|f\|_{BMO(\mathbb{R}^n)}$$

if and only if $\|\mu\| < \infty$, moreover, $\|H_\mu\|_{BMO(\mathbb{R}^n)\to BMO(\mathbb{R}^n)} = \|\mu\|$.

Proof. It suffices to prove (i). See [2] for the sufficiency part. Conversely, take

$$f_0(x) = \begin{cases} 1, & x \in \mathbb{R}^n_r, \\ -1, & x \in \mathbb{R}^n_l. \end{cases}$$

where $\mathbb{R}^n_r$ and $\mathbb{R}^n_l$ denote the left and right halves of $\mathbb{R}^n$, separated by the hyperplane $x_1 = 0$ (x_1 is the first coordinate of $x \in \mathbb{R}^n$). Obviously, $f_0 \in L^\infty(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$ and $\|f_0\|_{BMO(\mathbb{R}^n)} \neq 0$. On the other hand,

$$(H_\mu^* f_0)(x) = \begin{cases} \int_0^\infty t^n d\mu(t), & x \in \mathbb{R}^n_r, \\ -\int_0^\infty t^n d\mu(t), & x \in \mathbb{R}^n_l, \end{cases} = f_0(x)\int_0^\infty t^n d\mu(t).$$

So we obtain $\|\mu\|_n < \infty$. □

THEOREM 4.2. *Let* $p \in (0,\infty]$, $q \in [1,\infty)$ *and* $\alpha \in (0,\min(1,n/p))$. *If* $f \in Q_p^{\alpha,q}(\mathbb{R}^n)$, *then*

(i)

$$\|H_\mu^* f\|_{Q_p^{\alpha,q}(\mathbb{R}^n)} \leqslant C\|f\|_{Q_p^{\alpha,q}(\mathbb{R}^n)}$$

if and only if $\|\mu\|_{n/p'+\alpha} < \infty$, *moreover, when* $\alpha > n(1/p-1/q)$, *we have*

$$\|H_\mu^*\|_{Q_p^{\alpha,q}(\mathbb{R}^n)\to Q_p^{\alpha,q}(\mathbb{R}^n)} = \|\mu\|_{n/p'+\alpha};$$

(ii)

$$\|H_\mu f\|_{Q_p^{\alpha,q}(\mathbb{R}^n)} \leqslant C\|f\|_{Q_p^{\alpha,q}(\mathbb{R}^n)}$$

if and only if $\|\mu\|_{n/p-\alpha} < \infty$, *moreover, when* $\alpha > n(1/p-1/q)$, *we have*

$$\|H_\mu\|_{Q_p^{\alpha,q}(\mathbb{R}^n)\to Q_p^{\alpha,q}(\mathbb{R}^n)} = \|\mu\|_{n/p-\alpha}.$$

Proof. We only prove (i). For any cube I, use the Minkowski's inequality, we have

$$\begin{aligned}
&\left(\int_I\int_I \frac{|H_\mu^* f(x)-H_\mu^* f(y)|^q}{|x-y|^{n+q\alpha}}dxdy\right)^{1/q}\\
&\leqslant \int_0^\infty |t|^{-n}\left(\int_I\int_I \frac{|f(x/t)-f(y/t)|^q}{|x-y|^{n+q\alpha}}dxdy\right)^{1/q} d\mu(t)\\
&= \int_0^\infty |t|^{-n}|t|^{n/q-\alpha}\left(\int_{I/|t|}\int_{I/|t|} \frac{|f(x)-f(y)|^q}{|x-y|^{n+q\alpha}}dxdy\right)^{1/q} d\mu(t).
\end{aligned}$$

Thus,

$$\begin{aligned}
&|I|^{1/p-1/q}\left(\int_I\int_I \frac{|H_\mu^* f(x)-H_\mu^* f(y)|^q}{|x-y|^{n+q\alpha}}dxdy\right)^{1/q}\\
&\leqslant \int_0^\infty |t|^{-n}|t|^{n/q-\alpha}|t|^{n/p-n/q}|I/|t||^{1/p-1/q}\left(\int_{I/|t|}\int_{I/|t|} \frac{|f(x)-f(y)|^q}{|x-y|^{n+q\alpha}}dxdy\right)^{1/q} d\mu(t)\\
&\leqslant \|\mu\|_{n/p'+\alpha}\|f\|_{Q_p^{\alpha,q}(\mathbb{R}^n)}.
\end{aligned}$$

Conversely, take

$$f_0(x) = \begin{cases} |x|^{-n/p+\alpha}, & x \in \mathbb{R}_r^n,\\ -|x|^{-n/p+\alpha}, & x \in \mathbb{R}_l^n. \end{cases}$$

Then $f_0 \in Q_p^{\alpha,q}(\mathbb{R}^n)$ and $\|f_0\|_{Q_p^{\alpha,q}(\mathbb{R}^n)} \neq 0$ (see [31]). On the other hand,

$$(H_\mu^* f_0)(x) = f_0(x)\int_0^\infty t^{-n+n/p-\alpha}d\mu(t).$$

Therefore $\|\mu\|_{n/p'+\alpha} < \infty$. Moreover, we get $\|H_\mu^*\|_{Q_p^{\alpha,q}(\mathbb{R}^n)\to Q_p^{\alpha,q}(\mathbb{R}^n)} = \|\mu\|_{n/p'+\alpha}$. □

THEOREM 4.3. *Let* $q \in (1,\infty)$ *and* $\lambda \in (-1/q, 1/n)$. *Then*
(i)

$$\|H_\mu^* f\|_{CBMO^{q,\lambda}(\mathbb{R}^n)} \leqslant C\|f\|_{CBMO^{q,\lambda}(\mathbb{R}^n)}$$

if and only if $\|\mu\|_{n(1+\lambda)} < \infty$, *moreover,* $\|H_\mu^*\|_{CBMO^{q,\lambda}(\mathbb{R}^n)\to CBMO^{q,\lambda}(\mathbb{R}^n)} = \|\mu\|_{n(1+\lambda)}$;
(ii)

$$\|H_\mu f\|_{CBMO^{q,\lambda}(\mathbb{R}^n)} \leqslant C\|f\|_{CBMO^{q,\lambda}(\mathbb{R}^n)}$$

if and only if $\|\mu\|_{-n\lambda} < \infty$, *moreover,* $\|H_\mu\|_{CBMO^{q,\lambda}(\mathbb{R}^n)\to CBMO^{q,\lambda}(\mathbb{R}^n)} = \|\mu\|_{-n\lambda}$.

Proof. We only prove (i). Use the Fubini's theorem, we have

$$(H_\mu^* f)_{B(0,R)} = \int_0^\infty f_{B(0,R/|t|)}|t|^{-n} d\mu(t).$$

Apply the Minkowski's inequality, we get

$$\begin{aligned}
&\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)} |H_\mu^* f(x) - (H_\mu^* f)_{B(0,R)}|^q dx\right)^{1/q} \\
&= \left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)} \left|\int_0^\infty \left(f(x/t) - f_{B(0,R/|t|)}\right)|t|^{-n} d\mu(t)\right|^q dx\right)^{1/q} \\
&\leqslant \int_0^\infty \left(\frac{1}{|B(0,R/|t|)|^{1+\lambda q}}\int_{B(0,R/|t|)} |f(x) - f_{B(0,R/|t|)}|^q dx\right)^{1/q} |t|^{-n\lambda}|t|^{-n} d\mu(t) \\
&\leqslant \|\mu\|_{n(1+\lambda)}\|f\|_{CBMO^{q,\lambda}(\mathbb{R}^n)}.
\end{aligned}$$

Conversely, when $\lambda \in (-1/q, 1/n)$, take

$$f_0(x) = \begin{cases} |x|^{n\lambda}, & x \in \mathbb{R}^n_r, \\ -|x|^{n\lambda}, & x \in \mathbb{R}^n_l. \end{cases},$$

then $(f_0)_{B(0,R)} = 0$ and $\|f_0\|_{CBMO^{q,\lambda}(\mathbb{R}^n)} = v_n^{-q\lambda}/(q\lambda+1)$. On the other hand,

$$H_\mu^* f_0(x) = f_0(x)\int_0^\infty t^{-n(1+\lambda)} d\mu(t) < \infty,$$

so we obtain $\|\mu\|_{n(1+\lambda)} < \infty$. $\square$

REMARK 4.1. Similar to Remark 3.1, we can get some corresponding results in [8], [31] and [33].

REMARK 4.2. In fact, the sufficiency parts of our main results in Section 3 and this section also hold if we remove the non-negativity of μ, for example, Theorem A and Theorem B. But for the necessary conditions under the general μ, we can not find some suitable functions so far.

5. Lipschitz estimates for commutator of H_μ^*

In this section, we will show Lipschitz estimates for the commutator of H_μ^*. Some related results about the commutator of another high dimensional Hausdorff operator can be found in [9]. But here our idea and method come from [8], which are different from [9]. First we define commutator $H_\mu^{*,b}$, and then state our results. Note that for H_μ, we have similar results and omit them here.

DEFINITION 5.1. ([29]) Let $\beta \in (0,1)$. The Besov-Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ is defined by

$$\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} = \sup_{x,h\in\mathbb{R}^n} \frac{|b(x+h)-b(x)|}{|h|^\beta} < \infty.$$

DEFINITION 5.2. Let $b \in L_{\mathrm{loc}}(\mathbb{R}^n)$. The commutator of Hausdorff operator $H_\mu^{*,b}$ is defined by

$$(H_\mu^{*,b} f)(x) = b(x)(H_\mu^* f)(x) - H_\mu^*(bf)(x).$$

THEOREM 5.1. *Let* $\beta \in (0,1)$, $p_1 \in (1,n/\beta)$, $p_2 \in (1,\infty]$, $1/p_1 - 1/p_2 = \beta/n$ *and* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.

If $\max\{\|\mu\|_{n/p_2'}, \|\mu\|_{n/p_1'}\} < \infty$, *then*

$$\|H_\mu^{*,b} f\|_{L^{p_2}(\mathbb{R}^n)} \leqslant C\max\{\|\mu\|_{n/p_2'}, \|\mu\|_{n/p_1'}\}\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\|f\|_{L^{p_1}(\mathbb{R}^n)}.$$

THEOREM 5.2. *Let* $p_1 \in (1,\infty]$, $p_2 \in [p_1,\infty]$, $\beta \in (0,1)$, $q_1 \in (1,n/\beta)$, $1/q_1 - 1/q_2 = \beta/n$, $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *and* $-n/q_1 + \beta < \alpha < n(1-1/q_1)$.

If $\max\{\|\mu\|_{n/q_2'-\alpha}, \|\mu\|_{n/q_1'-\alpha}\} < \infty$, *then*

$$\|H_\mu^{*,b} f\|_{\dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n)} \leqslant C\max\{\|\mu\|_{n/q_2'-\alpha}, \|\mu\|_{n/q_1'-\alpha}\}\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\|f\|_{\dot{K}_{q_1}^{\alpha,p_1}(\mathbb{R}^n)}.$$

THEOREM 5.3. *Let* $p_1 \in (1,\infty]$, $p_2 \in [p_1,\infty]$, $\beta \in (0,1)$, $q_1 \in (1,n/\beta)$, $1/q_1 - 1/q_2 = \beta/n$, $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, $-n/q_1 + \beta < \alpha < n(1-1/q_1)$ *and* $\lambda > 0$.

If $\max\{\|\mu\|_{n/q_2'-\alpha+\lambda}, \|\mu\|_{n/q_1'-\alpha+\lambda}\} < \infty$, *then*

$$\|H_\mu^{*,b} f\|_{M\dot{K}_{p_2,q_2}^{\alpha,\lambda}(\mathbb{R}^n)} \leqslant C\max\{\|\mu\|_{n/q_2'-\alpha+\lambda}, \|\mu\|_{n/q_1'-\alpha+\lambda}\}\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\|f\|_{M\dot{K}_{p_1,q_1}^{\alpha,\lambda}(\mathbb{R}^n)}.$$

THEOREM 5.4. *Let* $\beta \in (0,1)$, $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *and* $p \in (1,\infty)$.

If $\max\{\|\mu\|_{n/p'}, \|\mu\|_{n/p'+\beta}\} < \infty$, *then*

$$\|H_\mu^{*,b} f\|_{\dot{F}_p^{\beta,\infty}(\mathbb{R}^n)} \leqslant C\max\{\|\mu\|_{n/p'}, \|\mu\|_{n/p'+\beta}\}\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}.$$

To prove our results, we need some lemmas.

LEMMA 5.1. ([29]) *If* $\beta \in (0,1)$ *and* $f \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, *then for any cube* Q, *we have*

$$\sup_{x\in Q} |f(x) - f_Q| \leqslant C|Q|^{\beta/n}\|f\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}.$$

LEMMA 5.2. *If* $\beta \in (0,1)$ *and* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, *then*

$$M(H_\mu^{*,b} f)(x) \leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}} |t|^{-n}(1+|t|^\beta) M_\beta f(x/t) d|\mu|(t),$$

where M *is the Hardy-Littlewood maximal operator, and* M_β *is the fractional maximal operator (*[14]*).*

Proof. For a fixed Q and any $x \in Q$, then

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |H_\mu^{*,b} f(y)| dy &\leqslant \frac{1}{|Q|}\int_Q \int_{\mathbb{R}} |(b(y)-b(y/t)) f(y/t) t^{-n}| d|\mu|(t) dy \\
&\leqslant \frac{1}{|Q|}\int_{\mathbb{R}} \int_Q |(b(y)-b_Q) f(y/t)| dy |t|^{-n} d|\mu|(t) \\
&\quad + \frac{1}{|Q|}\int_{\mathbb{R}} \int_Q |(b_Q - b_{Q/|t|}) f(y/t)| dy |t|^{-n} d|\mu|(t) \\
&\quad + \frac{1}{|Q|}\int_{\mathbb{R}} \int_Q |(b_{Q/|t|} - b(y/t)) f(y/t)| dy |t|^{-n} d|\mu|(t) \\
&:= I_1 + I_2 + I_3.
\end{aligned}$$

After applying Lemma 5.1, we get

$$\begin{aligned}
I_1 &\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \frac{1}{|Q|}\int_{\mathbb{R}} \left(\int_Q |f(y/t)| dy\right) |t|^{-n} d|\mu|(t) \\
&\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}} \left(\frac{1}{|Q/|t||^{1-\beta/n}} \int_{Q/|t|} |f(y/t)| dy\right) |t|^\beta |t|^{-n} d|\mu|(t) \\
&\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}} |t|^{\beta-n} M_\beta f(x/t) d|\mu|(t).
\end{aligned}$$

Similarly,

$$I_3 \leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}} |t|^{-n} M_\beta f(x/t) d|\mu|(t).$$

Note that $\beta \in (0,1)$, then

$$\begin{aligned}
|b_Q - b_{Q/|t|}| &\leqslant \frac{1}{|Q|}\int_Q |b(y) - b_{Q/|t|}| dy \\
&\leqslant \frac{1}{|Q|}\frac{1}{|Q/t|}\int_Q \int_{Q/|t|} |b(y)-b(z)| dz dy \\
&\leqslant \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left(\frac{1}{|Q|}\int_Q |y|^\beta dy + \frac{1}{|Q/t|}\int_{Q/|t|} |z|^\beta dz\right) \\
&\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} (1+|t|^{-\beta}) |Q|^{\beta/n}.
\end{aligned}$$

Thus,

$$
\begin{aligned}
I_2 &\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \frac{1}{|Q|^{1-\beta/n}} \int_{\mathbb{R}} |t|^{-n}(1+|t|^{-\beta}) \left(\int_Q |f(y/t)|dy \right) d|\mu|(t) \\
&\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}} |t|^{-n}(1+|t|^{-\beta}) \left(\frac{1}{|Q/|t||^{1-\frac{\beta}{n}}} \int_{Q/|t|} |f(y)|dy \right) |t|^{\beta} d|\mu|(t) \\
&\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}} |t|^{-n}(1+|t|^{\beta}) M_\beta f(x/t) d|\mu|(t).
\end{aligned}
$$

Combining the estimates of I_1, I_2 and I_3, we obtain

$$
\frac{1}{|Q|} \int_Q |H_\mu^{*,b} f(y)|dy \leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}} |t|^{-n}(1+|t|^{\beta}) M_\beta f(x/t) d|\mu|(t).
$$

Then

$$
M(H_\mu^{*,b} f)(x) \leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}} |t|^{-n}(1+|t|^{\beta}) M_\beta f(x/t) d|\mu|(t). \quad \square
$$

LEMMA 5.3. ([26]) *Let* $p_1 \in (0,\infty]$, $p_2 \in [p_1,\infty]$, $\beta \in (0,1)$, $q_1 \in (1, n/\beta)$ *and* $1/q_1 - 1/q_2 = \beta/n$. *If* $-n/q_1 + \beta < \alpha < n(1-1/q_1)$, *then*

$$
\|M_\beta f\|_{\dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n)} \leqslant C\|f\|_{\dot{K}_{q_1}^{\alpha,p_1}(\mathbb{R}^n)}.
$$

LEMMA 5.4. ([15]) *Let* $p_1 \in (0,\infty]$, $p_2 \in [p_1,\infty]$, $\beta \in (0,1)$, $q_1 \in (1,n/\beta)$, $1/q_1 - 1/q_2 = \beta/n$ *and* $\lambda > 0$. *If* $-n/q_1 + \beta + \lambda < \alpha < n(1-1/q_1) + \lambda$, *then*

$$
\|M_\beta f\|_{M\dot{K}_{p_2,q_2}^{\alpha,\lambda}(\mathbb{R}^n)} \leqslant C\|f\|_{M\dot{K}_{p_1,q_1}^{\alpha,\lambda}(\mathbb{R}^n)}. \quad \square
$$

LEMMA 5.5. ([29]) *If* $\beta \in (0,1)$ *and* $p \in (1,\infty)$, *then*

$$
\|f\|_{\dot{F}_p^{\beta,\infty}(\mathbb{R}^n)} \approx \left\| \sup_{Q \ni x} \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |f(y) - f_Q| dy \right\|_{L^p(\mathbb{R}^n)}.
$$

Proof of Theorem 5.2. By Lemma 5.2 and the Minkowski's inequality, while $1/q_2 = 1/q_1 - \beta/n$, we have

$$
\begin{aligned}
&\|(M(H_\mu^{*,b} f))\chi_k\|_{L^{q_2}(\mathbb{R}^n)} \\
&\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left(\int_{\Delta_k} \left| \int_{\mathbb{R}} (|t|^{-n} + |t|^{\beta-n}) M_\beta f(x/t) d|\mu|(t) \right|^{q_2} \right)^{1/q_2} \\
&\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}} \left(\int_{2^{k-1}/|t| < |x| \leqslant 2^k/|t|} (M_\beta f(x))^{q_2} dx \right)^{1/q_2} (|t|^{-n/q_1'} + |t|^{-n/q_2'}) d|\mu|(t).
\end{aligned}
$$

For any $|t| \in (0,\infty)$, there exists $m \in \mathbb{Z}$ such that $|t| \in (2^{m-1}, 2^m]$. So

$$\begin{aligned}
&\|(M(H_\mu^{*,b} f))\chi_k\|_{L^{q_2}(\mathbb{R}^n)} \\
&\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}} \left(\int_{2^{k-m-1}/|t|<|x|\leqslant 2^{k-m+1}/|t|} (M_\beta f(x))^{q_2} dx \right)^{1/q_2} \\
&\quad \times (|t|^{-n/q_1'} + |t|^{-n/q_2'}) d|\mu|(t) \\
&\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}} \sum_{i=0}^{1} \|(M_\beta f)\chi_{k-m+i}\|_{L^{q_2}(\mathbb{R}^n)} (|t|^{-n/q_1'} + |t|^{-n/q_2'}) d|\mu|(t).
\end{aligned}$$

Then

$$\begin{aligned}
&\|M(H_\mu^{*,b} f)\|_{\dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n)} \\
&\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left\{ \sum_{k\in\mathbb{Z}} 2^{k\alpha p_2} \left(\int_{\mathbb{R}} \sum_{i=0}^{1} \|(M_\beta f)\chi_{k-m+i}\|_{L^{q_2}(\mathbb{R}^n)} \right.\right. \\
&\quad \left.\left. \times (|t|^{-n/q_1'} + |t|^{-n/q_2'}) d|\mu|(t) \right)^{p_2} \right\}^{1/p_2} \\
&\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}} \sum_{i=0}^{i=1} \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha p_2} \|(M_\beta f)\chi_{k-m+i}\|_{L^{q_2}(\mathbb{R}^n)}^{p_2} \right)^{1/p_2} (|t|^{-n/q_1'} + |t|^{-n/q_2'}) d|\mu|(t) \\
&\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|M_\beta f\|_{\dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n)} \int_{\mathbb{R}} (|t|^{-n/q_1'+\alpha} + |t|^{-n/q_2'+\alpha}) d|\mu|(t).
\end{aligned}$$

By Lemma 5.3, we reach

$$\|H_\mu^{*,b} f\|_{\dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n)} \leqslant C \max\{\|\mu\|_{n/q_2'-\alpha}, \|\mu\|_{n/q_1'-\alpha}\} \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|f\|_{\dot{K}_{q_1}^{\alpha,p_1}(\mathbb{R}^n)}. \qquad \square$$

Proof of Theorem 5.3. By the proof of Theorem 5.2, we have

$$\begin{aligned}
&\|M(H_\mu^{*,b} f)\|_{M\dot{K}_{p_2,q_2}^{\alpha,\lambda}(\mathbb{R}^n)} \\
&\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} \int_{\mathbb{R}} \left(\sum_{i=0}^{1} \|(M_\beta f)\chi_{k-m+i}\|_{L^{q_2}(\mathbb{R}^n)} \right.\right. \\
&\quad \left.\left. \times (|t|^{-n/q_1'} + |t|^{-n/q_2'}) d|\mu|(t) \right)^{p_2} \right\}^{1/p_2} \\
&\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \int_{\mathbb{R}} \sum_{i=0}^{i=1} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p_2} \|(M_\beta f)\chi_{k-m+i}\|_{L^{q_2}(\mathbb{R}^n)}^{p_2} \right)^{1/p_2} \\
&\quad \times (|t|^{-n/q_1'} + |t|^{-n/q_2'}) d|\mu|(t) \\
&\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|M_\beta f\|_{M\dot{K}_{p_2,q_2}^{\alpha,\lambda}(\mathbb{R}^n)} \int_{\mathbb{R}} (2^{m\alpha} + 2^{(m-1)\alpha}) (|t|^{-n/q_1'} + |t|^{-n/q_2'}) |t|^{-\lambda} d|\mu|(t) \\
&\leqslant C \max\{\|\mu\|_{n/q_2'-\alpha+\lambda}, \|\mu\|_{n/q_1'-\alpha+\lambda}\} \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|M_\beta f\|_{M\dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n)}.
\end{aligned}$$

Using Lemma 5.4, we obtain

$$\|M(H_\mu^{*,b}f)\|_{M\dot{K}^{\alpha,\lambda}_{p_2,q_2}(\mathbb{R}^n)} \leqslant C\max\{\|\mu\|_{n/q_2'-\alpha+\lambda}, \|\mu\|_{n/q_1'-\alpha+\lambda}\} \\ \times \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1}(\mathbb{R}^n)}. \quad \square$$

Proof of Theorem 5.4. Let $x \in Q$, we have

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q |H_\mu^{*,b}f(z) - (H_\mu^{*,b}f)_Q|dz \leqslant \frac{2}{|Q|^{1+\beta/n}}\int_Q |H_\mu^{*,b}f(z)|dz \\
&\leqslant \frac{1}{|Q|^{1+\beta/n}}\int_{\mathbb{R}}\int_Q |(b(z)-b_Q)f(z/t)|dz|t|^{-n}d\mu(t) \\
&\quad + \frac{1}{|Q|^{1+\beta/n}}\int_{\mathbb{R}}\int_Q |(b_Q - b_{Q/|t|})f(z/t)|dz|t|^{-n}d\mu(t) \\
&\quad + \frac{1}{|Q|^{1+\beta/n}}\int_{\mathbb{R}}\int_Q |(b_{Q/|t|} - b(1/t))f(z/t)|dz|t|^{-n}d\mu(t) \\
&=: J_1 + J_2 + J_3.
\end{aligned}$$

By the same procedure as in Lemma 5.2, we obtain

$$\begin{aligned}
J_1 &\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}} Mf(x/t)|t|^{-n}d|\mu|(t); \\
J_2 &\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}} Mf(x/t)|t|^{-n}(1+|t|^{-\beta})d|\mu|(t); \\
J_3 &\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}} Mf(x/t)|t|^{-(n+\beta)}d|\mu|(t).
\end{aligned}$$

By Lemma 5.5, we get

$$\begin{aligned}
\|H_\mu^{*,b}f\|_{\dot{F}_p^{\beta,\infty}(\mathbb{R}^n)} &\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\left\|\int_{\mathbb{R}} Mf(x/t)|t|^{-n}(1+|t|^{-\beta})d|\mu|(t)\right\|_{L^p(\mathbb{R}^n)} \\
&\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\|Mf\|_{L^p(\mathbb{R}^n)}\int_{\mathbb{R}} |t|^{-n/p'}(1+|t|^{-\beta})d|\mu|(t) \\
&\leqslant C\max\{\|\mu\|_{n/p'}, \|\mu\|_{n/p'+\beta}\}\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)},
\end{aligned}$$

the last step by the boundedness of M on $L^p(\mathbb{R}^n)$ ([14]). $\square$

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