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THE ORLICZ AFFINE ISOPERIMETRIC INEQUALITY

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Abstract. In this paper, the Orlicz-affine surface area is introduced. Isoperimetric inequalities for this new afine surface area are established.

1. Introduction

The affine surface area is one of the most important concepts in affine differential geometry. The initial study of the classical affine surface area went back to Blaschke (cf. [4]) in the last century. That is the $SL(n)$ and translation invariant surface area in $\mathbb{R}^3$. It was extended by Leichtweiss [10] to convex bodies in $\mathbb{R}^n$ with sufficiently smooth boundary (cf. [17, 29, 33, 37]). In recently, the affine surface area has been to establish notions and theorems for arbitrary convex bodies. It provides a tool to measure the boundary structure of a convex body and be key ingredients in many applications. For instance, in the theory of valuations (cf. [1, 12, 13, 31]), approximation of convex bodies by polytopes (cf. [7, 14, 32, 34]) and information theory (cf. [25, 39]). It also has an impact in the study of the affine PDE (cf. [20, 35, 36]). More important is that it relates to various isoperimetric inequalities (e.g., the curvature image inequalities, the Blaschke-Santaló inequality and the Mahler volume product inequality) (cf. [15, 28]). During the recent years, the notion of affine surface area has attracted increasing interest (cf. [2, 3, 12, 17, 19, 24, 37, 38, 40, 41]).

During the past decades it has come to be seen that the classical Brunn-Minkowski theory of convex bodies is a part of a more general L_p-Brunn-Minkowski theory introduced by Lutwak (cf. [18, 19]). The L_p extensions of the affine surface area have been found within the L_p-Brunn-Minkowski theory. For $1 < p < \infty$, the L_p-affine surface area is defined by Lutwak (cf. [17, 19]) for all convex bodies K in $\mathbb{R}^n$. He finds that the L_p-affine surface area attains it's maximum among all convex bodies in $\mathbb{R}^n$ with fixed volume if and only if the body is an ellipsoid. This is very important in studying the L_p-affine surface area. Specially, if $p = 1$, the L_p-affine surface area is just the classical affine surface area defined by Petty (cf. [27]). In [9] Hug gave a new definition of the L_p-affine surface area for $p > 0$ and proves that these new definitions give

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the same L_p-affine surface area as that defined by Lutwak. Moreover, he shows that the L_p-affine surface area is the well known centro-affine surface area for the case $p = n$. Thus the notion of the L_p-affine surface area connects two important affine geometric functionals.

Recently, much effort has been made to extend the L_p-Brunn-Minkowski theory to the Orlicz-Brunn-Minkowski thoery (cf. [5, 8, 21, 22]). The extension of the L_p-affine surface area, namely, the general affine surface area, were investigated by Ludwig, Reitzner and Ye (cf. [11, 13, 39, 42]). In this paper, inspired by the Orlicz-Brunn-Minkowski theory, we consider the Orlicz extension of the affine surface area. We obtain the Orlicz-affine isoperimetric inequality and the Orlicz-analogue Blaschke-Santaló inequality.

Let $\phi : (0,\infty) \to (0,\infty)$ be a function such that ϕ is increasing and concave on $(0,\infty)$, $\lim_{t\to 0}\phi(t) = 0$ and $\lim_{t\to\infty}\phi(t)/t = 0$. In this case, we set $\phi(0) = 0$. The class of such ϕ is denoted by $\mathscr{C}$. Let $\mathscr{K}_o^n$ denote the set of convex bodies whose centroids are at the origin. For $K \in \mathscr{K}_o^n$, the Orlicz-affine surface area of K is defined as

$$\Omega_\phi(K) = \inf\left\{\lambda > 0 : \int_{\partial K} \phi\left(\frac{\kappa_0(K,x)}{\lambda}\right) d\mu_K(x) \leqslant n|K|\right\}, \tag{1.1}$$

where $d\mu_K(x) = x\cdot v_(x) d\mathscr{H}^{n-1}(x)$ is the cone measure on ∂K, $x\cdot v$ is the standard inner product of x and v in $\mathbb{R}^n$, $v_K(x)$ is the exterior unit normal vector to K at $x \in \partial K$, $\mathscr{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. Let $|K|$ denote the volume of K, and

$$\kappa_0(K,x) = \frac{\kappa(K,x)}{(x\cdot v_K(x))^{n+1}}$$

where $\kappa(K,x)$ is the Gaussian curvature of K at $x \in \partial K$.

If one takes $\phi(t) = t^{\frac{p}{n+p}}$ and $p > 0$ in (1.1), then $\phi(t) \in \mathscr{C}$ and it turns out that

$$[\Omega_\phi(K)]^{\frac{p}{n+p}} = \frac{\Omega_p(K)}{n|K|} \tag{1.2}$$

where $\Omega_p(K)$ is the L_p-affine surface area defined as (cf. [9, 17, 19])

$$\Omega_p(K) = \int_{\partial K} \kappa_0(K,x)^{\frac{p}{n+p}} d\mu_K(x).$$

More specially, if $p = 1$ in $\phi(t) = t^{\frac{p}{n+p}}$, then

$$[\Omega_\phi(K)]^{\frac{1}{n+1}} = \frac{\Omega(K)}{n|K|}, \tag{1.3}$$

where $\Omega(K)$ is the classical affine surface area defined by (cf. [26, 27])

$$\Omega(K) = \int_{\partial K} \kappa_0(K,x)^{\frac{1}{n+1}}.$$

In this paper, we will mainly prove the following Orlicz-affine isoperimetric inequality.

Orlicz-affine isoperimetric inequality: *Let $K \subset \mathscr{K}_o^n$ and B_K be the origin symmetric Euclidean ball with $|K| = |B_K|$. If $\phi \in \mathscr{C}$ be a concave function, then the following inequality holds*

$$\Omega_\phi(K) \leqslant \Omega_\phi(B_K). \tag{1.4}$$

For strictly increasing ϕ, the equality holds if and only if K is an ellipsoid.

This paper is organized as follows. In section 2, we establish some notations and list some basic facts regarding convex bodies and convex functions. In section 3, some of the properties of Orlicz-affine surface area are established. Sections 4 contains some results of the Orlicz-affine inequality.

2. Basic definition and notation

The setting will be n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathscr{K}^n$ denote the set of convex bodies in $\mathbb{R}^n$. For $K \in \mathscr{K}^n$, let $h(K;\cdot) = h_K : \mathbb{R}^n \to \mathbb{R}$ denote the support function of K

$$h(K;x) = h_K(x) = \max\{x \cdot y : y \in K\}.$$

Thus if $y \in \partial K$, then

$$h_K(v_K(y)) = \sigma(K,y) \cdot y,$$

were $\sigma(K,y)$ denotes an outer unit normal to ∂K at y. Obviously, when $c > 0$, for the support function of the convex body $cK = \{cx : x \in K\}$ we have

$$h_{cK} = ch_K.$$

By the definition of the support function, it follows immediately that the support function of the image AK $(= \{Ay : y \in K\})$ is given by

$$h_{AK}(x) = h_K(A^t x) \tag{2.1}$$

for $A \in SL(n)$. Here A^t denotes the transpose of A.

Let $K_i \in \mathscr{K}^n$, then $K_i \to K_0 \in \mathscr{K}^n$ provided

$$|h_{K_i} - h_{K_0}|_\infty = \max_{u \in S^{n-1}} |h_{K_i}(u) - h_{K_0}(u)| \to 0.$$

For $K \in \mathscr{K}_o^n$, the polar body K^* of K is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leqslant 1 \ \textit{for all}\ y \in K\}.$$

It is easy to see that for $c \geqslant 0$

$$(cK)^* = \frac{1}{c} K^*. \tag{2.2}$$

More generally for $A \in GL(n)$

$$(AK)^* = A^{-t}K^*$$

where A^{-t} denotes the inverse of the transpose of A. It is easy to verity that if $K \in \mathscr{K}_o^n$, then

$$K^{**} = K. \tag{2.3}$$

An inequality involving the volume of K and K^* is the Blashke-Santaló inequality (cf. [23, 27, 30])

$$|K||K^*| \leqslant \omega_n^2 \tag{2.4}$$

where K is the convex body centered at the origin and ω_n is the volume of the unit ball B^n in $\mathbb{R}^n$.

The classical Aleksandrov-Fenchel -Jessen surface area measure, S_K of the convex body K, can be defined as the unique Borel measure on S^{n-1} such that

$$\int_{S^{n-1}} f(u)dS_K(u) = \int_{\partial K} f(\nu_K(y))d\mathscr{H}^{n-1}(y) \tag{2.5}$$

for each continuous $f : S^{n-1} \to \mathbb{R}$. Here $\nu_K(y)$ is the outer unit normal to ∂K at y and $dS_K(u)$ is the surface area measure of ∂K. Moreover, the surface area measure of cK satisfies

$$S_{cK} = c^{n-1}S_K. \tag{2.6}$$

It is known that the measure S_K cannot be concentrated on a hemisphere of S^{n-1} for $K \in \mathscr{K}_o^n$, and it is weakly continuous in K, i.e., if $K_i \in \mathscr{K}_o^n$, then

$$K_i \to K_0 \Rightarrow S_{K_i} \to S_{K_0} \ \textit{weakly}. \tag{2.7}$$

A convex body $K \in \mathscr{K}^n$ is said to have a curvature function $f(K,\cdot) : S^{n-1} \to \mathbb{R}$, if its surface area measure $S(K,\cdot)$ is absolutely continuous with respect to the spherical Lebesgue measure S, and

$$\frac{dS(K,\cdot)}{dS} = f(K,\cdot) = f_K(\cdot) \tag{2.8}$$

almost everywhere with respect to S.

For $A \in SL(n)$, it's shown in [16] the curvature function satisfies

$$f(AK,u) = f(K,A^t u) \tag{2.9}$$

for $u \in S^{n-1}$.

For $K \in \mathscr{K}_o^n$, it will be convenient to use the volume normalized conical measure V_K defined by (cf. [21, 22])

$$|K|dV_K = \frac{1}{n}h_K dS_K. \tag{2.10}$$

Observe that V_K is probability measure on S^{n-1}.

We say the sequence $\{f_i\} \to f_0$ provided

$$\max_{t\in I} |f_i(t) - f_0(t)| \to 0$$

for every compact interval $I \subset \mathbb{R}$. For each f, we define c_f by

$$c_f = \min\{c > 0 : \max\{f(c), f(-c)\} \leqslant 1\}. \tag{2.11}$$

For convex body $K \in \mathscr{K}^n$, we say that ∂K is line free in direction $u \in S^{n-1}$ if $\partial K \cap (x + Ru)$ consists of no more than two points for each $x \in \partial K$. The Ewald-Larman-Rogers theorem (cf. [6]) guarantees that for each convex body K

$$\mathscr{H}^{n-1}(\{u \in S^{n-1} : \partial K \text{ is not line free in direction } u\}) = 0. \tag{2.12}$$

3. The Orlicz-affine surface area

Let $\phi \in \mathscr{C}$, by the definition of (1.1) and (2.10), the Orlicz-affine surface area can be rewritten as

$$\Omega_\phi(K) = \inf\left\{\lambda > 0 : \int_{S^{n-1}} \phi\left(\frac{\kappa_0(K,u)}{\lambda}\right) dV_K(u) \leqslant 1\right\} \tag{3.1}$$

where $\kappa_0(K,u) = \frac{\kappa(K,u)}{(x\cdot\sigma(K,u))^{n+1}}$. Since ϕ is strictly increasing on $[0,\infty)$, it follows that the function

$$\lambda \to \int_{S^{n-1}} \phi\left(\frac{\kappa_0(K,u)}{\lambda}\right) dV_K(u) \tag{3.2}$$

is strictly decreasing in $(0,\infty)$. Thus, we have

LEMMA 3.1. *Suppose $\phi \in \mathscr{C}$ and $K \in \mathscr{K}_o^n$, then we have*

(1) $\int_{S^{n-1}} \phi\left(\frac{\kappa_0(K,u)}{\lambda_0}\right) dV_K(u) > 1$ if and only if $\Omega_\phi(K) > \lambda_0$.

(2) $\int_{S^{n-1}} \phi\left(\frac{\kappa_0(K,u)}{\lambda_0}\right) dV_K(u) = 1$ if and only if $\Omega_\phi(K) = \lambda_0$.

(3) $\int_{S^{n-1}} \phi\left(\frac{\kappa_0(K,u)}{\lambda_0}\right) dV_K(u) < 1$ if and only if $\Omega_\phi(K) < \lambda_0$.

Moreover, if ∂K is line free, the Orlicz-affine surface area can be rewritten as

$$\Omega_\phi(K) = \inf\left\{\lambda > 0 : \int_{S^{n-1}} \phi\left(\frac{f_K(u)^{-1}h_K(u)^{-(n+1)}}{\lambda}\right) dV_K(u) \leqslant 1\right\}. \tag{3.3}$$

The following lemma shows that $\Omega_\phi(K)$ is $SL(n)$ invariant.

LEMMA 3.2. *Let $K \in \mathscr{K}_o^n$, $\phi \in \mathscr{C}$ and $A \in SL(n)$, then*

$$\Omega_\phi(AK) = \Omega_\phi(K). \tag{3.4}$$

Proof. Suppose $\Omega_\phi(AK) = \lambda_0$. By Lemma 3.1 we obtain

$$\int_{S^{n-1}} \phi\left(\frac{f_{AK}(u)^{-1}h_{AK}(u)^{-(n+1)}}{\lambda_0}\right) dV_{AK}(u) = 1.$$

Combine with

$$h(AK,u) = h(K,A^t u) \text{ and } f(AK,u) = f(K,A^t u)$$

we have

$$\int_{S^{n-1}} \phi\left(\frac{f_K(A^t u)^{-1}h_K(A^t u)^{-(n+1)}}{\lambda_0}\right) dV_{AK}(u) = 1.$$

Since $A \in SL(n)$, we obtain

$$\int_{S^{n-1}} \phi\left(\frac{f_K(u)^{-1}h_K(u)^{-(n+1)}}{\lambda_0}\right) dV_K(u) = 1.$$

By Lemma 3.1 again we complete the proof. □

The following lemma will be useful for our proof of the main result (cf. [11]).

LEMMA 3.3. *Let $K \in \mathscr{K}_0^n$, $|K^*|$ denote the volume of polar body K and $\kappa_0(K,x)$ be defined as above. Then the following inequality holds*

$$\int_{\partial K} \kappa_0(K,x)d\mu_K(x) \leqslant n|K^*| \tag{3.5}$$

with equality holds if and only if $\kappa_0(K,x)$ is constant.

4. Proof of the Orlicz-affine isoperimetric inequality

Now, we are ready to prove the main result of the present paper.

THEOREM 4.1. *Let $K \subset \mathscr{K}_o^n$ be a convex body with the centroid at the origin, and B_K be the origin symmetric Euclidean ball with $|K| = |B_K|$. Then for all $\phi \in \mathscr{C}$, the following inequality holds*

$$\Omega_\phi(K) \leqslant \Omega_\phi(B_K). \tag{4.1}$$

For strictly increasing ϕ, the equality holds if and only if K is an ellipsoid.

Proof. Let $\phi \in \mathscr{C}$ and $K \in \mathscr{K}_o^n$. Suppose $\Omega_\phi(B_K) = \lambda_0$. By Lemma 3.1 we obtain

$$\int_{\partial B_K} \phi\left(\frac{\kappa_0(B_K,x)}{\lambda_0}\right) d\mu_{B_K}(x) = n|B_K|. \tag{4.2}$$

In order to prove (4.1) we only need to prove

$$\int_{\partial K} \phi\left(\frac{\kappa_0(K,x)}{\lambda_0}\right) d\mu_K(x) \leqslant n|K|. \tag{4.3}$$

Since $|K| = |B_K|$, we have

$$B_K = \left(\frac{|K|}{\omega_n}\right)^{\frac{1}{n}} B^n$$

where B^n is the unit ball in $\mathbb{R}^n$ and ω_n is the volume of B^n. By (4.2) we have

$$\begin{aligned} n|B_K| &= \int_{S^{n-1}} \phi\left(\frac{f_{B_K}(u)^{-1} h_{B_K}(u)^{-(n+1)}}{\lambda_0}\right) h_{B_K}(u) f_{B_K}(u) dS(u) \qquad (4.4)\\ &= \int_{S^{n-1}} \phi\left(\frac{(\frac{|K|}{\omega_n})^{-2}}{\lambda_0}\right) \frac{|K|}{\omega_n} dS(u) \\ &= n|K|\phi\left(\frac{\omega_n^2}{\lambda_0|K|^2}\right). \end{aligned}$$

On the other hand since ϕ is concave and increasing, by Blaschke-Santaló inequality, Lemma 3.3 and Jensen's inequality we obtain

$$\begin{aligned} n|K| = n|B_K| = n|K|\phi\left(\frac{\omega_n^2}{\lambda_0|K|^2}\right) &\geqslant n|K|\phi\left(\frac{|K^*|}{\lambda_0|K|}\right) \qquad (4.5)\\ &\geqslant n|K|\phi\left(\frac{1}{n|K|}\int_{\partial K}\left(\frac{\kappa_0(K,x)}{\lambda_0}\right) d\mu_K(x)\right) \\ &\geqslant \int_{\partial K} \phi\left(\frac{\kappa_0(K,x)}{\lambda_0}\right) d\mu_K(x). \end{aligned}$$

It implies

$$\Omega_\phi(K) \leqslant \lambda_0.$$

For strict increasing function ϕ, the equality in the first inequality of (4.5) holds if and only if there is equality in the Blaschke-Santaló inequality, that is exactly for ellipsoids. By Lemma 3.3 the equality in the second inequality of (4.5) holds for origin centered ellipsoids. It also implies that the equality in the third inequality of (4.5) holds. That means only when K is an ellipsoid. □

In fact, if $p > 0$ and $\phi(t) = t^{\frac{p}{n+p}}$, then ϕ is a concave function. In this case the Orlicz-affine isoperimetric inequality is exactly the L_p-affine isoperimetric inequality obtained by Lutwak (cf. [19]).

COROLLARY 4.2. *Let $p>0$, $K\in\mathscr{K}_o^n$ and $\Omega_p(K)$ denote the L_p-affine surface area. Then the following L_p-affine isoperimetric inequality holds*

$$\Omega_p(K)^{n+p}\leqslant n^{n+p}\omega_n^{2p}|K|^{n-p} \tag{4.6}$$

with equality if and only if K is an ellipsoid.

Proof. Let $K\in\mathscr{K}_o^n$, $|K|=|B_K|$ and $\phi(t)=t^{\frac{p}{n+p}}$. Suppose $\Omega_\phi(K)=\lambda_0$ and $\Omega_\phi(B_K)=\lambda_1$. By the definition of the Orlicz-affine surface and L_p-affine surface area we obtain

$$\lambda_0^{\frac{p}{n+p}}=\frac{\Omega_p(K)}{n|K|}\quad and\quad \lambda_1^{\frac{p}{n+p}}=\frac{\Omega_p(B_K)}{n|B_K|}. \tag{4.7}$$

By Theorem 4.1 we know

$$\Omega_p(K)\leqslant\Omega_p(B_K).$$

In fact

$$\Omega_p(B_K)=\Omega_p\left(\left(\frac{|K|}{\omega_n}\right)^{\frac{1}{n}}B^n\right).$$

Note that $\Omega_p(\cdot)$ is homogeneous of degree $q=\frac{n(n-p)}{n+p}$ (cf. [11, 19]). We have

$$\Omega_p(B_K)=\Omega_p\left(\left(\frac{|K|}{\omega_n}\right)^{\frac{1}{n}}B^n\right)=\left(\frac{|K|}{\omega_n}\right)^{\frac{n-p}{n+p}}\Omega_p(B^n). \tag{4.8}$$

It is easy to show that

$$\Omega_p(B^n)=O_{n-1}$$

where

$$O_{n-1}=n\omega_n\ is\ the\ surface\ area\ of\ S^{n-1}.$$

By simple computation, we obtain

$$\Omega_p(K)^{n+p}\leqslant n^{n+p}\omega_n^{2p}|K|^{n-p}.\quad\square$$

Moreover, if one takes $p=1$ in (4.6), it is the classical affine isoperimetric inequality established by Petty (cf. [27]).

COROLLARY 4.3. *Let $K\in\mathscr{K}_o^n$, $\Omega(K)$ be the affine surface area. Then the following affine isoperimetric inequality holds*

$$\Omega(K)^{n+1}\leqslant n^{n+1}\omega_n^{2}|K|^{n-1} \tag{4.9}$$

with equality if and only if K is an ellipsoid.

The following result is an analogue of the the classical Blaschke-Santaló inequality for the Orlicz-affine surface area of K^* and K.

THEOREM 4.4. *Let $\phi \in \mathscr{C}$ and $K \in \mathscr{K}_o^n$, then*

$$\Omega_\phi(K)\Omega_\phi(K^*) \leqslant \left(\frac{\omega_n^2}{c_\phi |K||K^*|}\right)^2. \tag{4.10}$$

If ϕ is strictly increasing, then with equality hold if and only if K is an ellipsoid.

Proof. By Theorem 4.1 we have

$$\Omega_\phi(K) \leqslant \Omega_\phi(B_K) \ \ and \ \ \Omega_\phi(K^*) \leqslant \Omega_\phi(B_{K^*}). \tag{4.11}$$

By simple computation we obtain

$$\Omega_\phi(B_K) = \frac{\omega_n^2}{c_\phi |K|^2} \ \ and \ \ \Omega_\phi(B_{K^*}) = \frac{\omega_n^2}{c_\phi |K^*|^2}. \tag{4.12}$$

Hence, we have

$$\Omega_\phi(K)\Omega_\phi(K^*) \leqslant \left(\frac{\omega_n^2}{c_\phi |K||K^*|}\right)^2.$$

Here equality holds if and only if $\Omega_\phi(K) = \Omega_\phi(B_K)$ *and* $\Omega_\phi(K^*) = \Omega_\phi(B_{K^*})$. This means K must be an ellipsoid. □

Letting $p > 0$ and $\phi(t) = t^{\frac{p}{n+p}}$ in Theorem 4.4 will lead to the following corollary which is obtained by E. Lutwak (cf. [19]).

COROLLARY 4.5. *Let $K \in \mathscr{K}_o^n$ and $p > 0$, then*

$$\Omega_p(K)\Omega_p(K^*) \leqslant (n\omega_n)^2 \tag{4.13}$$

with equality if and only if K is an ellipsoid.

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