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SOME NEW SCALES OF REFINED JENSEN AND HARDY TYPE INEQUALITIES

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Abstract. Some scales of refined Jensen and Hardy type inequalities are derived and discussed. The key object in our technique is γ-quasiconvex functions $K(x)$ defined by $K(x)x^{-\gamma} = \varphi(x)$, where φ is convex on $[0,b)$, $0 < b \leqslant \infty$ and $\gamma \geqslant 0$.

1. Introduction

Hardy's famous inequality reads: If f is non-negative and p-integrable over $(0,\infty)$, then:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\,dx, \quad p > 1. \tag{1.1}$$

This inequality was stated by G. H. Hardy in 1920 (see [3]) and finally proved by him in 1925 (see [4]). The first weighted version of (1.1) was proved in 1928 also by G. H. Hardy (see [5]) and it reads: If f is nonnegative and measurable on $(0,\infty)$, then

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^\alpha dx \leqslant \left(\frac{p}{p-1-\alpha}\right)^p \int_0^\infty f^p(x)\,x^\alpha dx, \tag{1.2}$$

whenever $p > 1$ and $\alpha < p-1$. The inequality is sharp. Moreover, we also have the following version of (1.2) where $(0,\infty)$ is replaced by an interval $(0,b)$, $0 < b \leqslant \infty$, still with sharp constant:

$$\int_0^b \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^\alpha dx \leqslant \left(\frac{p}{p-1-\alpha}\right)^p \int_0^b f^p(x)\,x^\alpha \left(1-\left(\frac{x}{b}\right)^{\frac{p-\alpha-1}{p}}\right) dx, \tag{1.3}$$

where $p \geqslant 1$ and $\alpha < p-1$.

For this and more information of this type see [13]. In particular, let us mention the remarkable fact that (1.1), (1.2) and (1.3) are equivalent to the same Jensen inequality. Concerning Hardy type inequalities with general weights we refer to [7], [8] and [9].

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Even if all constants above are sharp we can improve all inequalities above by making so called “refinements”, i.e., inserting some additional strictly positive terms on the left hand-side of the inequalities.

Here we will mention some of these results.

An early result of this type is the following one by C. O. Imoru from 1977 [6]:

$$\int_0^b \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^\alpha dx + \frac{p}{p-1-\alpha} b^{1-p-\alpha}\left(\int_0^b f(y)\,dy\right)^p \leqslant \left(\frac{p}{p-1-\alpha}\right)^p \int_0^b f^p(x)\,x^\alpha dx,$$

where $p \geqslant 1,\ \alpha < p-1$ and $0 < b < \infty$. This result was further generalized (and also previous results by D. T. Shum) in the paper [11]. In the paper [10] (cf. also [12]) the same authors made a refinement of a completely different type, namely the following: Let $p \geqslant 1,\ \alpha < p-1$ and $0 < b \leqslant \infty$. If $p \geqslant 2$, then

$$\begin{aligned} &\int_0^b \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^\alpha dx \\ &\quad + \frac{p-1-\alpha}{p}\int_0^b \int_t^b \left|\frac{p}{p-\alpha-1}\left(\frac{t}{x}\right)^{1-\frac{p-\alpha-1}{p}} f(t) \right. \\ &\quad \left. - \frac{1}{x}\int_0^x f(\tau)\,d\tau\right|^p x^{\alpha-\frac{p-\alpha-1}{p}} dx \cdot t^{\frac{p-\alpha-1}{p}-1} dt \\ &\leqslant \left(\frac{p}{p-\alpha-1}\right)^p \int_0^b f^p(x)\,x^\alpha \left(1-\left(\frac{x}{b}\right)^{\frac{p-\alpha-1}{p}}\right) dx. \end{aligned} \tag{1.4}$$

If $1 < p \leqslant 2$, then (1.4) holds in the reverse direction. In particular, for $p = 2$ we have equality in (1.4). This means that the natural “breaking point” is $p = 2$ in this refined Hardy inequality. In all other Hardy type inequalities discussed above and elsewhere (see e.g. the books [7, 8, 9] and the references there) the corresponding natural breaking point is $p = 1$.

Recently S. Abramovich and L. E. Persson [2] proved that (1.4) is not unique and can be replaced by another inequality with breaking point $p = 2$. In the same paper another refined Hardy inequality with breaking point $p = 3$, was also proved. Moreover, in the paper [1] the same authors derived a scale $\{H_p\}$ of refined Hardy inequalities with natural corresponding breaking points $p,\ p \geqslant 2$. This results were obtained by using a similar technique as those introduced in [10] (see also [13]) with superquadratic or superterzatic functions and the corresponding refined Jensen type inequalities involved.

In this paper we continue this development by deriving some new scales of refined Hardy type inequalities for all $p \geqslant 1$ (see Theorem 2). The key argument is to work with functions $K(x)$ that satisfy $K(x)x^{-\gamma} = \varphi(x),\ x \in [0,b),\ 0 < b \leqslant \infty$ where $\gamma \geqslant 0$ and $\varphi(x)$ is convex. We call such functions γ-*quasiconvex*. One important step is to first prove some corresponding scales of Jensen type inequalities for such functions of independent interest (see e.g. Theorem 1).

The paper is organized as follows: The new Jensen type inequalities are presented, proved and discussed in Section 2 and the new Hardy type inequalities are presented, proved and discussed in Section 3.

2. Refinements of Jensen inequality for γ-quasiconvex functions

A convex function $\varphi = \varphi(x)$ on $[0,b)$, $0 < b \leqslant \infty$ is characterized by the following inequality:

$$\varphi(y) - \varphi(x) \geqslant C_\varphi(x)(y-x), \forall x,y \in [0,b). \tag{2.1}$$

In this section we derive some refined Jensen type inequalities that hold when the given function $K(x)$ satisfies $K(x)x^{-\gamma} = \varphi(x)$, $\gamma \in \mathbb{R}_+$, where φ is a convex function. We call the function $K(x)$ a γ-quasiconvex function. These inequalities include and generalize the results related to the convex function $\varphi : [0,b) \to R$. The announced scales of refined Jensen type inequalities is stated in Theorem 1, but first we need to state and prove some lemmas of independent interest.

For the γ-quasiconvex functions $K(x)$ we prove:

LEMMA 1. *Let $K(x) = x^\gamma \varphi(x)$, $\gamma \in \mathbb{R}_+$, where $\varphi(x)$ is convex on $[0,b)$. Then*

$$K(y) - K(x) \geqslant \varphi(x)(y^\gamma - x^\gamma) + C_\varphi(x) y^\gamma (y-x), \tag{2.2}$$

holds for $x \in [0,b)$, $y \in [0,b)$, where $C_\varphi(x)$ is defined by (2.1). Moreover,

$$\begin{aligned}&\int_\Omega K(f(s))\,d\mu(s) - K\left(\int_\Omega f(s)\,d\mu(s)\right) \\ &\geqslant \int_\Omega \left[\varphi(x)(f^\gamma(s) - x^\gamma) + C_\varphi(x) f^\gamma(s)(f(s)-x)\right] d\mu(s)\end{aligned} \tag{2.3}$$

holds, where f is any nonnegative function, f and $K(f(s))$ are μ-integrable functions on the probability measure space (Ω,μ) and $x = \int_\Omega f(s)\,d\mu(s) > 0$.

If φ is concave, then the reverse inequalities of (2.1), (2.2), and (2.3) hold, in particular

$$\begin{aligned}&\int_\Omega K(f(s))\,d\mu(s) \; - K\left(\int_\Omega f(s)\,d\mu(s)\right) \\ &\leqslant \int_\Omega \left[\varphi(x)(f^\gamma(s) - x^\gamma) + C_\varphi(x) f^\gamma(s)(f(s)-x)\right] d\mu(s).\end{aligned} \tag{2.4}$$

EXAMPLE 1. Inequalities (2.1), (2.2), and (2.3) are satisfied especially by $K(x) = x^p$, $p \geqslant \gamma+1$. For $\gamma < p \leqslant \gamma+1$ the reverse inequalities hold. They reduce to equalities for $p = \gamma+1$ e.g. it yields that

$$\begin{aligned}&\int_\Omega f^{1+\gamma}(s)\,d\mu(s) - \left(\int_\Omega f(\sigma)\,d\mu(\sigma)\right)^{1+\gamma} \\ &= x\int_\Omega (f^\gamma(s) - x^\gamma)\,d\mu(s) + \int_\Omega f^\gamma(s)(f(s)-x)\,d\mu(s).\end{aligned} \tag{2.5}$$

Proof. (of Lemma 1) By multiplying (2.1) by y^{γ} we obtain that

$$y^{\gamma}\varphi(y) - y^{\gamma}\varphi(x) \geqslant C_{\varphi}(x)(y-x)y^{\gamma} \tag{2.6}$$

so that, by making some simple munipulations, we conclude that $K(x) = x^{\gamma}\varphi(x)$ satisfies (2.2) when φ is convex.

In (2.2) we put $y = f(s)$ and $x = \int_{\Omega} f(s)d\mu(s)$. The inequality (2.3) follows now by just integrating (2.2) over the measure space (Ω, μ).

Similarly, since $-\varphi$ is convex, inequality (2.4) and the reverse inequalities of (2.2), (2.3) and (2.19) are obtained for concave functions φ. The proof is complete. □

LEMMA 2. *Let $K(x) = x^{\gamma}\varphi(x) = x^{\gamma-1}\psi(x)$, $\gamma \geqslant 1$, where φ is a differentiable, nonnegative increasing and convex function on $[0,b)$, $0 < b \leqslant \infty$ (i.e. $\psi(x) = x\varphi(x)$). Then the bound obtained for $K(x) = x^{\gamma}\varphi(x)$ is "stronger" than the bound obtained for $K(x) = x^{\gamma-1}\psi(x)$, that is:*

$$K(y) - K(x) \geqslant \varphi(x)(y^{\gamma} - x^{\gamma}) + \varphi'(x)y^{\gamma}(y-x) \tag{2.7}$$

implies that

$$K(y) - K(x) \geqslant \psi(x)\left(y^{\gamma-1} - x^{\gamma-1}\right) + \psi'(x)y^{\gamma-1}(y-x). \tag{2.8}$$

Moreover, if $K(x) = x^{n}\varphi(x)$, $\psi_k(x) = x^{k}\varphi(x)$, n is an integer, $k = 1,2,\ldots,n$, and $\varphi(x)$ is a nonnegative, increasing and convex function, then the inequalities

$$\begin{aligned}
&\int_{\Omega} K(f(s))\,d\mu(s) - K\left(\int_{\Omega} f(s)\,d\mu(s)\right) \\
&\geqslant \int_{\Omega} \left[\varphi(x)(f^{n}(s) - x^{n}) + C_{\varphi}(x) f^{n}(s)(f(s)-x)\right] d\mu(s) \\
&\geqslant \int_{\Omega} \left[\psi_{k-1}(x)\left(f^{n-k+1}(s) - x^{n-k+1}\right) + C_{\psi_{k-1}}(x) f^{n-k+1}(s)(f(s)-x)\right] d\mu(s) \\
&\geqslant \int_{\Omega} \left[\psi_{k}(x)\left(f^{n-k}(s) - x^{n-k}\right) + C_{\psi_{k}}(x) f^{n-k}(s)(f(s)-x)\right] d\mu(s) \geqslant 0
\end{aligned} \tag{2.9}$$

hold for all probability measure spaces (Ω, μ) of μ-integrable nonnegative functions f, where $x = \int_{\Omega} f(s)\,d\mu(s) > 0$.

Proof. We first note that in order to prove that (2.7) $\Rightarrow$ (2.8) we just need to prove that if φ is a differentiable and nonnegative convex increasing function, where $\psi(x) = x\varphi(x)$, then

$$\begin{aligned}
&\varphi(x)(y^{\gamma} - x^{\gamma}) + \varphi'(x)y^{\gamma}(y-x) \\
&\geqslant \psi(x)\left(y^{\gamma-1} - x^{\gamma-1}\right) + \psi'(x)y^{\gamma-1}(y-x).
\end{aligned} \tag{2.10}$$

After some manipulations we see that to prove (2.10) is equivalent to prove that

$$y^{\gamma-1}\varphi'(x)(x-y)^{2} \geqslant 0, \tag{2.11}$$

which is satisfied because both y, and $\varphi^{'}$ are nonnegative.

If φ is nonnegative, increasing and convex so is $x^k\varphi(x) = \psi_k(x)$, k is a nonnegative integer.

Then by using (2.8) repeatedly and integrating over (Ω,μ), we obtain (2.9) in a similar way as (2.3) was derived from (2.2). The proof is complete. □

We are now ready to formulate and prove the announced refined Jensen type inequalities for γ-quasiconvex functions.

THEOREM 1. *Let $\gamma \in \mathbb{R}_+$, f be nonnegative and $f(s)$ and $\varphi(f(s))$ are μ-integrable functions on the probability measure space (Ω,μ) and $x = \int_\Omega f(s)\,d\mu(s) > 0$. If φ is a differentiable, nonnegative, convex, increasing function on $[0,b)$, $0 < b \leqslant \infty$ and $\varphi(0) = \lim_{z\to 0^+} z\varphi'(z) = 0$, then the Jensen type inequalities*

$$\int_\Omega \varphi(f(s))\,f^\gamma(s)\,d\mu(s) - \varphi(x)\,x^\gamma \tag{2.12}$$
$$\geqslant \varphi(x)\int_\Omega (f^\gamma(s) - x^\gamma)\,d\mu(s) + \varphi'(x)\int_\Omega f^\gamma(s)\,(f(s)-x)\,d\mu(s) \geqslant 0$$

hold.

REMARK 1. Note especially that when $\gamma = 0$ the second inequality in (2.12) reduces to equality, so that then (2.12) coincides with the usual Jensen's inequality.

Proof of Theorem 1. First we prove that if $\gamma \geqslant 0$, then

$$\varphi(x)\int_\Omega (f^\gamma(s) - x^\gamma)\,d\mu(s) + \varphi'(x)\int_\Omega f^\gamma(s)\,(f(s)-x)\,d\mu(s) \geqslant 0, \tag{2.13}$$

Since the function $x^{1+\gamma}$, $x \geqslant 0$, is convex, by Jensen's inequality we get that

$$\int_\Omega f^{1+\gamma}(s)\,d\mu(s) - \left(\int_\Omega f(\sigma)\,d\mu(\sigma)\right)^{1+\gamma} \tag{2.14}$$
$$= x\int_\Omega (f^\gamma(s) - x^\gamma)\,d\mu(s) + \int_\Omega f^\gamma(s)\,(f(s)-x)\,d\mu(s) \geqslant 0.$$

We multiply now inequality (2.3) by $x = \int_\Omega f(\sigma)\,d\mu(\sigma)$ and find that

$$x\left(\int_\Omega \varphi(f(s)\,f^\gamma(s))\,d\mu(s) - \varphi(x)\,x^\gamma\right) \tag{2.15}$$
$$\geqslant \varphi(x)\,x\int_\Omega (f^\gamma(s) - x^\gamma)\,d\mu(\gamma) + \varphi'(x)\,x\left(\int_\Omega f^\gamma(s)\,(f(s)-x)\right)d\mu(s).$$

Since $\varphi(x)$ is a differentiable and convex function satisfying that $\varphi(0) = 0 = \lim_{z\to 0} z\varphi'(z)$, then

$$\varphi'(x)\,x \geqslant \varphi(x). \tag{2.16}$$

As $\gamma \geqslant 0,\ x = \int_\Omega f(s)\, d\mu(s) > 0$ and μ is a nonnegative probability measure, then because when $z > 0,\ z$ and z^γ are monotonic in the same direction, Chebyshev inequality yields that

$$\int_\Omega f^\gamma(s) f(s)\, d\mu(s) - \int_\Omega f(s)\, d\mu(s) \int_\Omega f^\gamma(s)\, d\mu(s) \geqslant 0, \tag{2.17}$$

i.e. that $\int\limits_\Omega f^\gamma(s)(f(s)-x)d\mu(x) \geqslant 0$. By using now (2.16)–(2.17) we get that

$$\begin{aligned} &\varphi(x)x\int_\Omega (f^\gamma(s) - x^\gamma)\, d\mu(\gamma) + \varphi'(x)x\left(\int_\Omega f^\gamma(s)(f(s)-x)\right) d\mu(s) \\ &\geqslant \varphi(x)x\int_\Omega (f^\gamma(s) - x^\gamma)\, d\mu(\gamma) + \varphi(x)\left(\int_\Omega f^\gamma(s)(f(s)-x)\right) d\mu(s). \end{aligned} \tag{2.18}$$

From (2.14) and (2.18) we get that for $\gamma \geqslant 0$ (2.13) holds. Hence, the second inequality in (2.12) is proved. The first inequality in (2.12) is just a special case of (2.3) in Lemma 1 so (2.12) is proved and the proof is complete. □

EXAMPLE 2. By applying (2.3) with $\mu(s) = \sum\limits_{i=1}^{N} a_i \delta_i$ with $\sum\limits_{i=1}^{N} a_i = 1$ and δ_i unit masses at $x = x_i,\ y_i = f(x_i),\quad i = 1,\ldots,N,\quad N \in \mathbb{Z}_+$, we obtain that the following special case of (2.3) yields the inequality

$$\begin{aligned} &\sum_{i=1}^{N} \alpha_i K(y_i) - K\left(\sum_{i=1}^{N} \alpha_i y_i\right) \\ &\geqslant \varphi\left(\sum_{j=1}^{N} \alpha_j y_j\right)\left(\sum_{i=1}^{N} \alpha_i y_i^\gamma - \left(\sum_{j=1}^{N} \alpha_j y_j\right)^\gamma\right) + C_\varphi\left(\sum_{j=1}^{N} \alpha_j y_j\right)\sum_{i=1}^{N} \alpha_i y_i^\gamma \left(y_i - \sum_{j=1}^{N} \alpha_j y_j\right) \end{aligned} \tag{2.19}$$

which holds for $x_i \in [0,b),\ y_i \in [0,b),\ 0 \leqslant \alpha_i \leqslant 1,\ i = 1,\ldots,n,$ and $\sum_{i=1}^{N} \alpha_i = 1$. Moreover, under the conditions on φ in Theorem 1, as φ is differentiable so that $C_\varphi = \varphi'$, then the right handside of (2.19) is nonnegative and therefore we get that (2.19) is a genuine scale of refined discrete Jensen type inequalities.

3. Some new scales of refined Hardy type inequalities

In this section we use the ideas and techniques of [10], and implement them on the γ-quasiconvex functions $K(x)$, that is $K(x)x^{-\gamma} = \varphi(x),\ \gamma \in \mathbb{R}_+$, where $\varphi(x)$ is convex. First we prove a proposition which is generalization of Proposition 2.1a in [10].

PROPOSITION 1. *Let $0 < b \leqslant \infty,\ u : (0,\infty) \to \mathbb{R}$, be a nonnegative weight function such that $\frac{u(x)}{x^2}$ is locally integrable on $(0,\infty)$ and let the weight function v be defined by*

$$v(t) = t\int_t^b \frac{u(x)}{x^2}dx, \quad t \in (0,b). \tag{3.1}$$

If the function φ *is integrable and convex on* $[0,b)$ *and* $K(x)=x^{\gamma}\varphi(x)$, $\gamma\in\mathbb{R}_+$, *then*

$$\int_0^b K(f(x))\frac{v(x)}{x}dx-\int_0^b K\left(\frac{1}{x}\int_0^x (f(t)dt)\right)\frac{u(x)}{x}dx \tag{3.2}$$
$$\geqslant \int_0^b\int_t^b\left(f^{\gamma}(t)-\left(\frac{1}{x}\int_0^x f(\tau)d\tau\right)^{\gamma}\right)\varphi\left(\frac{1}{x}\int_0^x f(\tau)d\tau\right)\frac{u(x)}{x^2}dxdt$$
$$+\int_0^b\int_t^b f^{\gamma}(t)\left(f(t)-\frac{1}{x}\int_0^x f(\tau)d\tau\right)C_{\varphi}\left(\frac{1}{x}\int_0^x f(\tau)d\tau\right)\frac{u(x)}{x^2}dxdt,$$

holds for all nonnegative locally integrable functions f. *If* φ *is concave, then the reverse of inequality (3.2) holds.*

Here C_{φ} *is the function in the definition (2.1) of convexity.*

COROLLARY 1. *For* $\gamma=0$ *Inequality (3.2) coincides with the statement in Proposition 2.1a in* [10], *that is*

$$\int_0^b \varphi(f(x))\frac{v(x)}{x}dx-\int_0^b \varphi\left(\frac{1}{x}\int_0^x f(t)dt\right)\frac{u(x)}{x}dx\geqslant 0. \tag{3.3}$$

holds whenever $u(x)$ *and* $v(x)$ *are related by the formula (3.1).*

Proof of Proposition 1. Let us apply Lemma 1 with the probability measure $d\mu(t) = \frac{1}{x}dt$, $0\leqslant t\leqslant x$, in (2.3). Then

$$\frac{1}{x}\int_0^x K(f(t))dt-K\left(\frac{1}{x}\int_0^x f(t)dt\right) \tag{3.4}$$
$$\geqslant \varphi\left(\frac{1}{x}\int_0^x f(t)dt\right)\frac{1}{x}\int_0^x\left(f^{\gamma}(t)-\left(\frac{1}{x}\int_0^x f(\tau)d\tau\right)^{\gamma}\right)dt$$
$$+C_{\varphi}\left(\frac{1}{x}\int_0^x f(t)dt\right)\frac{1}{x}\int_0^x f^{\gamma}(t)\left(f(t)-\frac{1}{x}\int_0^x f(\tau)d\tau\right)dt.$$

Multiplying (3.4) by $\frac{u(x)}{x}$ and integrating over the interval $0\leqslant x\leqslant b$, we get that

$$\int_0^b\int_0^x K(f(t))dt\frac{u(x)}{x^2}dx-\int_0^b K\left(\frac{1}{x}\int_0^x f(t)dt\right)\frac{u(x)}{x}dx \tag{3.5}$$
$$\geqslant \int_0^b\int_0^x\left(f^{\gamma}(t)-\left(\frac{1}{x}\int_0^x f(\tau)d\tau\right)^{\gamma}\right)\varphi\left(\frac{1}{x}\int_0^x f(\tau)d\tau\right)\frac{u(x)}{x^2}dtdx$$
$$+\int_0^b\int_0^x f^{\gamma}(t)\left(f(t)-\frac{1}{x}\int_0^x f(\tau)d\tau\right)C_{\varphi}\left(\frac{1}{x}\int_0^x f(\tau)d\tau\right)\frac{u(x)}{x^2}dtdx.$$

Now using (3.1) and Fubini's theorem we find that

$$\int_0^b\int_0^x K(f(t))\frac{u(x)}{x^2}dtdx=\int_0^b K(f(t))\int_t^b\frac{u(x)}{x^2}dx=\int_0^b K(f(t))v(t)\frac{dt}{t}. \tag{3.6}$$

(3.5) and (3.6) lead to (3.2). When φ is concave the proof is similar and therefore omitted. The proof is complete. □

EXAMPLE 3. From Proposition 1 for $\varphi(x) = x^p,\ p \geqslant 1$ (therefore $C_\varphi(x) = \varphi'(x) = px^{p-1}$), choosing $u(x) = 1$ and $\gamma \in \mathbb{R}_+$, so that $v(x) = 1 - \frac{x}{b}$ we find that (3.2) reads:

$$\begin{aligned}
&\int_0^b \left(1 - \frac{x}{b}\right) f^{p+\gamma}(x) \frac{dx}{x} - \int_0^b \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^{p+\gamma} \frac{dx}{x} \qquad (3.7)\\
\geqslant &\int_0^b \int_t^b \left(f^\gamma(t) - \left(\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)^\gamma\right)\left(\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)^p \frac{dx}{x^2}dt\\
&+\int_0^b \int_t^b f^\gamma(t)\left(f(t) - \frac{1}{x}\int_0^x f(\tau)\,d\tau\right) p\left(\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)^{p-1} \frac{dx}{x^2}dt.
\end{aligned}$$

The reverse inequality holds when $0 < p \leqslant 1$.

By using (3.7) we are now ready to derive our new scales of refined Hardy type inequalities.

THEOREM 2. *Let $p \geqslant 1,\ k > 1,\ 0 < b \leqslant \infty$, and $\gamma \in \mathbb{R}_+$, and let the function f be nonnegative and locally integrable on $(0,b)$. Then*

$$\begin{aligned}
&\left(\frac{p+\gamma}{k-1}\right)^{p+\gamma}\int_0^b \left(1-\left(\frac{x}{b}\right)^{\frac{k-1}{p+\gamma}}\right) x^{p+\gamma-k} f^{p+\gamma}(x)\,dx - \int_0^b x^{-k}\left(\int_0^x f(t)dt\right)^{p+\gamma} dx \qquad (3.8)\\
\geqslant &\left(\frac{k-1}{p+\gamma}\right)\int_0^b \int_t^b \left(\left(f(t)\frac{p+\gamma}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p+\gamma}}\right)^\gamma - \left(\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)^\gamma\right)\\
&\times \left(\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)^p x^{\left(1-\frac{k-1}{p+\gamma}\right)(p+\gamma-1)} t^{\frac{k-1}{p+\gamma}-1}\frac{dx}{x^2}dt\\
&+\left(\frac{k-1}{p+\gamma}\right)^{1-\gamma}\int_0^b \int_t^b \left(f(t)t^{1-\frac{k-1}{p+\gamma}}\right)^\gamma \left(f(t)\frac{p+\gamma}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p+\gamma}} - \frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)\\
&\times p\left(\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)^{p-1} x^{\left(1-\frac{k-1}{p+\gamma}\right)(p+1)} t^{\frac{k-1}{p+\gamma}-1}\frac{dx}{x^2}dt \geqslant 0.
\end{aligned}$$

Proof. We denote the right hand side of (3.7) by R and replace the parameter b by $b^{\frac{k-1}{p+\gamma}}$ and $f(x)$ by $f\left(x^{\frac{p+\gamma}{k-1}}\right)x^{\frac{p+\gamma}{k-1}-1}$. Then

$$\begin{aligned}
R = &\int_0^{b^{\frac{k-1}{p+\gamma}}} \int_t^{b^{\frac{k-1}{p+\gamma}}} \left(f^\gamma\left(t^{\frac{p+\gamma}{k-1}}\right) t^{\left(\frac{p+\gamma}{k-1}-1\right)\gamma} - \left(\frac{1}{x}\int_0^x f\left(\tau^{\frac{p+\gamma}{k-1}}\right)\tau^{\frac{p+\gamma}{k-1}-1}d\tau\right)^\gamma\right) \qquad (3.9)\\
&\times \left(\frac{1}{x}\int_0^x f\left(\tau^{\frac{p+\gamma}{k-1}}\right)\tau^{\frac{p+\gamma}{k-1}-1}d\tau\right)^p \frac{dx}{x^2}dt\\
&+\int_0^{b^{\frac{k-1}{p+\gamma}}} \int_t^{b^{\frac{k-1}{p+\gamma}}} \left(f\left(t^{\frac{p+\gamma}{k-1}}\right)t^{\frac{p+\gamma}{k-1}-1}\right)^\gamma \left(f\left(t^{\frac{p+\gamma}{k-1}}\right)t^{\frac{p+\gamma}{k-1}-1} - \frac{1}{x}\int_0^x f\left(\tau^{\frac{p+\gamma}{k-1}}\right)\tau^{\frac{p+\gamma}{k-1}-1}d\tau\right)\\
&\times p\left(\frac{1}{x}\int_0^x f\left(\tau^{\frac{p+\gamma}{k-1}}\right)\tau^{\frac{p+\gamma}{k-1}-1}d\tau\right)^{p-1}\frac{dx}{x^2}dt
\end{aligned}$$

We use now the substitutions

$$y = x^{\frac{p+\gamma}{k-1}} \qquad \text{and} \qquad s = t^{\frac{p+\gamma}{k-1}}$$

from which it follows that

$$\begin{aligned}
&t = b^{\frac{k-1}{p+\gamma}} \Rightarrow s = b, \qquad x = b^{\frac{k-1}{p+\gamma}} \Rightarrow y = b,\\
&dt = \frac{k-1}{p+\gamma} s^{\frac{k-1}{p+\gamma}-1} ds, \qquad \frac{k-1}{p+1} ds = t^{\frac{p+\gamma}{k-1}-1} dt, \qquad dx = y^{\frac{k-1}{p+\gamma}-1} \frac{k-1}{p+\gamma} dy,\\
&dy = \frac{p+\gamma}{k-1} x^{\frac{p+\gamma}{k-1}-1} dx, \quad \text{and} \quad t^{\frac{p+\gamma}{k-1}-1} = s^{1-\frac{k-1}{p+\gamma}}.
\end{aligned}$$

By using these substitutions we get from (3.9) that

$$\begin{aligned}
R = &\left(\frac{k-1}{p+\gamma}\right)^{p+\gamma+2} \int_0^b \int_s^b \left(\left(\frac{p+\gamma}{k-1} f(s) \left(\frac{s}{y}\right)^{1-\frac{k-1}{p+\gamma}} \right)^{\gamma} - \left(\frac{1}{y} \int_0^y f(\sigma)\, d\sigma \right)^{\gamma} \right) \qquad (3.10)\\
&\times \left(\frac{1}{y} \int_0^y f(\sigma)\, d\sigma \right)^{p} y^{\left(1-\frac{k-1}{p+\gamma}\right)(p+\gamma+1)} s^{\frac{k-1}{p+\gamma}-1} \frac{dy}{y^2} ds\\
&+ p \left(\frac{k-1}{p+\gamma}\right)^{p+2} \int_0^b \int_s^b \left(\frac{p+\gamma}{k-1} f(s) \left(\frac{s}{y}\right)^{\left(1-\frac{k-1}{p+\gamma}\right)} - \frac{1}{y} \int_0^y f(\sigma)\, d\sigma \right)\\
&\times \left(f(s)\, s^{\left(1-\frac{k-1}{p+\gamma}\right)} \right)^{\gamma} \left(\frac{1}{y} \int_0^y f(\sigma)\, d\sigma \right)^{p-1} y^{\left(1-\frac{k-1}{p+\gamma}\right)(p+1)} s^{\frac{k-1}{p+\gamma}-1} \frac{dy}{y^2} ds.
\end{aligned}$$

Now we make the same changes on the left hand side of (3.7), denoted by L, that is, we replace b by $b^{\frac{k-1}{p+\gamma}}$ and $f(x)$ by $f\left(x^{\frac{p+\gamma}{k-1}}\right) x^{\frac{p+\gamma}{k-1}-1}$ and by the substitution $y = x^{\frac{p+\gamma}{k-1}}$ we get that

$$\begin{aligned}
L = &\int_0^b \frac{k-1}{p+\gamma} \left(1 - \left(\frac{y}{b}\right)^{\frac{k-1}{p+\gamma}} \right) y^{p+\gamma-k} (f(y))^{p+\gamma} dy \qquad (3.11)\\
&- \left(\frac{k-1}{p+\gamma}\right)^{p+\gamma+1} \int_0^b y^{-k} \left(\int_0^y f(s)\, ds \right)^{p+\gamma} dy.
\end{aligned}$$

Therefore from (3.7)–(3.11), after dividing L and R by $\left(\frac{k-1}{p+\gamma}\right)^{p+\gamma+1}$, we get the first inequality in (3.8).

We obtain inequality (3.8) by starting with inequality (2.3) and chosing $\varphi(x) = x^p$, $p \geqslant 1,\ x \geqslant 0$. In Theorem 1 we proved that the right handside of (2.3) is non-negative when γ is nonnegative. Therefore, also the right hand-side of (3.8) is non-negative when γ is a non-negative integer. This completes the proof of the theorem. $\square$

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