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RIEMANN SUMS FOR SELF–ADJOINT OPERATORS

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Abstract. This paper focuses on Riemann sums for the functional calculus of bounded self-adjoint operaors. We first obtain some monotonicity properties of operator convex functions. Using these results we then refine an operator Hermite–Hadamard type inequality. Finally we extend the Alzer and Bennet inequalities to operators on Hilbert spaces.

1. Introduction and preliminaries

Alzer's inequality [3] states that

$$\frac{n}{n+1} \leqslant \left(\frac{(n+1)\sum_{i=1}^{n} i^r}{n\sum_{i=1}^{n+1} i^r} \right)^{\frac{1}{r}} \tag{1.1}$$

where r is a real number and n is a positive integer, in other words, the sequence of Riemann sums $\{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^r\}$ of the function x^r is decreasing for $r \geqslant 0$ and increasing for $r \leqslant 0$. This is equivalent with

$$\left(\frac{n}{n+1}\right)^r n\sum_{i=1}^{n+1} i^r \begin{cases} \leqslant (n+1)\sum_{i=1}^{n} i^r & r \geqslant 0 \\ \geqslant (n+1)\sum_{i=1}^{n+1} i^r & r < 0. \end{cases}$$

Actually, this inequality has been implicitly discovered in 1975 by Jan van de Lune [1]. The proof of Alzer in the case of $r > 0$ uses some techniques combined with Stirling's inequality and is rather complicated. Sandor [17] gave an easy proof of the Alzer inequality by using the Cauchy mean value theorem and mathematical induction. Several mathematicians have provided elementary proofs and extensions of this inequality; see [9, 6, 11, 2] and the references cited therein.

In 1992, Bennet [4] proved the inequality

$$\left(\frac{(n+1)\sum_{i=1}^{n} i^r}{n\sum_{i=1}^{n+1} i^r} \right)^{\frac{1}{r}} \begin{cases} \leqslant \frac{n+1}{n+2} & r \geqslant 1 \\ \geqslant \frac{n+1}{n+2} & r \leqslant 1\,, \end{cases} \tag{1.2}$$

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or equivalently,

$$(n+1)\sum_{i=1}^{n} i^r \begin{cases} \leqslant \left(\frac{n+1}{n+2}\right)^r n\sum_{i=1}^{n+1} i^r \ \ r\leqslant 0 \text{ or } r\geqslant 1 \\ \geqslant \left(\frac{n+1}{n+2}\right)^r n\sum_{i=1}^{n+1} i^r \quad 0\leqslant r\leqslant 1\,. \end{cases}$$

Clearly if $r\leqslant 1$, Bennet's inequality (1.2) is a refinement of Alzer's inequality (1.1), while if $r\geqslant 1$ Bennet's inequality (1.2) is a converse of Alzer's inequality (1.1); see also [16]. The aim of this paper is to extend the Alzer and Bennet inequalities to inequalities for operators.

Throughout the paper, let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators acting on a complex Hilbert space $(\mathscr{H},\langle\cdot,\cdot\rangle)$ and I be the identity operator. In the case where $\dim\mathscr{H}=n$, we identify $\mathbb{B}(\mathscr{H})$ with the full matrix algebra $\mathscr{M}_n(\mathbb{C})$ of all $n\times n$ matrices with entries in the complex field $\mathbb{C}$. An operator $A\in\mathbb{B}(\mathscr{H})$ is called positive if $\langle Ax,x\rangle\geqslant 0$ holds for every $x\in\mathscr{H}$ and then we write $A\geqslant 0$. If A is positive and invertible, we write $A>0$. For self-adjoint operators A,B, we say $A\leqslant B$ if $B-A\geqslant 0$. A continuous real valued function f defined on an interval J is called operator monotone if $A\leqslant B$ implies that $f(A)\leqslant f(B)$ for all self-adjoint operators A,B with spectra in J. A function f is said to be operator convex on J if

$$f(\lambda A+(1-\lambda)B)\leqslant \lambda f(A)+(1-\lambda)f(B)$$

for all $A,B\in\mathbb{B}(\mathscr{H})$ with spectra in J and all $\lambda\in[0,1]$. A function f is called operator concave if $-f$ is operator convex.

The Löwner–Heinz Inequality asserts that $f(x)=x^r$ $(0\leqslant r\leqslant 1)$ is operator monotone on $[0,\infty)$. For more information on operator inequalities see [10] and [5].

2. A refinement of an operator Hermite–Hadamard inequality

In this section we present some monotonicity properties of operator convex functions and refine the operator Hermite–Hadamard inequality.

THEOREM 2.1. *Let $f:J\to\mathbb{R}$ be an operator convex function and A,B be self-adjoint operators with spectra in J. Then for each $n=1,2,\cdots$,*

$$\frac{1}{n+2}\sum_{i=0}^{n+1} f\left(A+i\frac{B-A}{n+1}\right)\leqslant\frac{1}{n+1}\sum_{i=0}^{n} f\left(A+i\frac{B-A}{n}\right) \tag{2.1}$$

$$\frac{1}{n}\sum_{i=1}^{n} f\left(A+i\frac{B-A}{n+1}\right)\leqslant\frac{1}{n+1}\sum_{i=1}^{n+1} f\left(A+i\frac{B-A}{n+2}\right). \tag{2.2}$$

If f is operator concave, all inequalities are reversed.

Proof. Put

$$C_i^{(n)}=A+i\frac{B-A}{n}=\left(1-\frac{i}{n}\right)A+\frac{i}{n}B \qquad (i=0,1,\cdots,n;\ \ n=1,2,\cdots).$$

Due to J is a convex set, the spectrum of $C_i^{(n)}$ is contained in J. We observe that

$$C_i^{(n+1)} = \frac{i}{n+1} C_{i-1}^{(n)} + \frac{n+1-i}{n+1} C_i^{(n)} \quad (1 \leqslant i \leqslant n).$$

Since $f(t)$ is operator convex we obtain

$$f(C_i^{(n+1)}) \leqslant \frac{i}{n+1} f(C_{i-1}^{(n)}) + \frac{n+1-i}{n+1} f(C_i^{(n)}) \quad (1 \leqslant i \leqslant n). \tag{2.3}$$

By summing up (2.3) from $i=1$ to $i=n$ we deduce that

$$\begin{aligned}\sum_{i=1}^{n} f(C_i^{(n+1)}) &\leqslant \sum_{i=1}^{n} \frac{i}{n+1} f(C_{i-1}^{(n)}) + \sum_{i=1}^{n} \frac{n+1-i}{n+1} f(C_i^{(n)}) \\ &= \sum_{i=0}^{n-1} \frac{i+1}{n+1} f(C_i^{(n)}) + \sum_{i=1}^{n} \frac{n+1-i}{n+1} f(C_i^{(n)}) \\ &= \frac{1}{n+1} f(A) + \frac{1}{n+1} f(B) + \sum_{i=1}^{n-1} \frac{n+2}{n+1} f(C_i^{(n)}).\end{aligned}$$

Now by adding $f(A)$ and $f(B)$ to both sides of the above inequality we get

$$\sum_{i=1}^{n} f(C_i^{(n+1)}) + f(A) + f(B) \leqslant \frac{n+2}{n+1} f(A) + \frac{n+2}{n+1} f(B) + \frac{n+2}{n+1} \sum_{i=1}^{n-1} f(C_i^{(n)}).$$

Consequently

$$\sum_{i=0}^{n+1} f(C_i^{(n+1)}) \leqslant \frac{n+2}{n+1} \sum_{i=0}^{n} f(C_i^{(n)})$$

from which we derive (2.1). To prove (2.2), we consider the following convex combination

$$C_{i-1}^{(n)} = \frac{n+1-i}{n} C_{i-1}^{(n+1)} + \frac{i-1}{n} C_i^{(n+1)} \quad (1 \leqslant i \leqslant n+1).$$

Since $f(t)$ is operator convex function on J, we have

$$f(C_{i-1}^{(n)}) \leqslant \frac{n+1-i}{n} f(C_{i-1}^{(n+1)}) + \frac{i-1}{n} f(C_i^{(n+1)}) \quad (1 \leqslant i \leqslant n+1). \tag{2.4}$$

Again by summing up (2.4) from $i=1$ to $i=n+1$ we obtain

$$\begin{aligned}\sum_{i=1}^{n+1} f(C_{i-1}^{(n)}) &\leqslant \sum_{i=1}^{n+1} \frac{n+1-i}{n} f(C_{i-1}^{(n+1)}) + \sum_{i=1}^{n+1} \frac{i-1}{n} f(C_i^{(n+1)}) \\ &= \sum_{i=0}^{n} \frac{n-i}{n} f(C_i^{(n+1)}) + \sum_{i=1}^{n+1} \frac{i-1}{n} f(C_i^{(n+1)}) \\ &= f(A) + f(B) + \sum_{i=1}^{n} \left(\frac{n-i}{n} + \frac{i-1}{n} \right) f(C_i^{(n+1)}).\end{aligned}$$

Therefore we have

$$\sum_{i=2}^{n} f(C_{i-1}^{(n)}) \leqslant \sum_{i=1}^{n} \frac{n-1}{n} f(C_i^{(n+1)}),$$

or equivalently

$$\sum_{i=1}^{n-1} f(C_i^{(n)}) \leqslant \frac{n-1}{n} \sum_{i=1}^{n} f(C_i^{(n+1)}).$$

Now by changing n by $n+1$ we get (2.2).

If f is operator concave, apply (2.1) and (2.2) for operator convex function $-f$.

For a convex function f on $[a,b]$, the double inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a} \int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}$$

is known as the Hermite–Hadamard inequality. It gives us an estimation of the mean value of the convex function f. There is an extensive amount of literature devoted to this inequality, which has many applications. Interestingly, each of two sides of the Hermite–Hadamard inequality characterizes convex functions.

Let f be an operator convex function on an interval J of the real line and A,B be self-adjoint operators with spectra in J. An operator version of the Hermite–Hadamard inequality reads as follows

$$f\Big(\frac{A+B}{2}\Big) \leqslant \int_0^1 f((1-t)A+tB)dt \leqslant \frac{f(A)+f(B)}{2}. \tag{2.5}$$

Recently, Dragomir [8] presented the following generalization of the above operator Hermite–Hadamard inequality:

$$\begin{aligned} f\Big(\frac{A+B}{2}\Big) &\leqslant \frac{1}{2}\Big[f\Big(\frac{3A+B}{4}\Big) + f\Big(\frac{A+3B}{4}\Big)\Big] \\ &\leqslant \int_0^1 f((1-t)A+tB)dt \\ &\leqslant \frac{1}{2}\Big[f\Big(\frac{A+B}{2}\Big) + \frac{f(A)+f(B)}{2}\Big] \leqslant \frac{f(A)+f(B)}{2}. \end{aligned}$$

For other recent refinements of operator Hermite–Hadamard inequality see [13]. We should remark that there is a gap in the proof of Theorem 3.6 of [13]. In the proof, the unitary U should be different for each t. The following result gives a refinement of the operator Hermite–Hadamard inequality (2.5).

COROLLARY 2.2. *Let $f : J \to \mathbb{R}$ be an operator convex function on J and let A,B be self-adjoint operators with spectra in J. Then*

$$\begin{aligned} f\left(\frac{A+B}{2}\right) &\leqslant \frac{1}{m}\sum_{i=1}^{m} f\left(A+i\frac{B-A}{m+1}\right) \leqslant \int_0^1 f\big((1-t)A+tB\big)dt \\ &\leqslant \frac{1}{n+1}\sum_{i=0}^{n} f\left(A+i\frac{B-A}{n}\right) \leqslant \frac{f(A)+f(B)}{2} \qquad (m,n=1,2,\cdots). \end{aligned} \tag{2.6}$$

If f is operator concave, all inequalities are reversed.

Proof. Since f is continuous, we have

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n} f\left(A+i\frac{B-A}{n}\right)=\int_0^1 f\big((1-t)A+tB\big)dt.$$

It follows from inequality (2.1) that

$$\int_0^1 f\big((1-t)A+tB\big)dt \leqslant \frac{1}{n+1}\sum_{i=0}^{n} f\left(A+i\frac{B-A}{n}\right) \qquad (n=1,2,\cdots).$$

Also

$$\lim_{m\to\infty}\frac{1}{m}\sum_{i=1}^{m} f\left(A+i\frac{B-A}{m+1}\right)=\int_0^1 f\big((1-t)A+tB\big)dt\,.$$

Inequality (2.2) yields that

$$\frac{1}{m}\sum_{i=1}^{m} f\left(A+i\frac{B-A}{m+1}\right) \leqslant \int_0^1 f\big((1-t)A+tB\big)dt \qquad (m=1,2,\cdots).$$

In the sequel, suppose that $f:J\to\mathbb{R}$ is a real-valued function on J, and A,B are two self-adjoint operators with spectra contained in J. Let us set

$$S_n:=\frac{1}{n}\sum_{i=1}^{n} f\left(A+i\frac{B-A}{n}\right) \qquad \text{and} \qquad T_n:=\frac{1}{n}\sum_{i=0}^{n-1} f\left(A+i\frac{B-A}{n}\right).$$

PROPOSITION 2.3. *Let $f:J\to\mathbb{R}$ be an operator convex function on J and let A,B be self-adjoint operators with spectra in J. Then for each $n=1,2,\cdots$,*

$$S_{n+1}+\frac{1}{n(n+2)}\left(S_{n+1}-f(A)\right)\leqslant S_n\leqslant S_{n+1}+\frac{1}{n^2}\left(f(B)-S_{n+1}\right) \tag{2.7}$$

and

$$T_{n+1}+\frac{1}{n(n+2)}\left(T_{n+1}-f(B)\right)\leqslant T_n\leqslant T_{n+1}+\frac{1}{n^2}\left(f(A)-T_{n+1}\right). \tag{2.8}$$

In the case that f is operator concave, all inequalities in (2.7) and (2.8) are reversed.

Proof. Put $C_i^{(n)}=A+i\frac{B-A}{n}$. Then

$$nS_n+f(A)=\sum_{i=0}^{n} f(C_i^{(n)}) \tag{2.9}$$

and

$$(n+1)S_{n+1}+f(A)=\sum_{i=0}^{n+1} f(C_i^{(n+1)}). \tag{2.10}$$

Putting (2.9) and (2.10) in inequality (2.1) we get

$$\frac{1}{n+2}\left((n+1)S_{n+1}+f(A)\right)\leqslant\frac{1}{n+1}\left(nS_n+f(A)\right).$$

A straightforward computation yields the first inequality of (2.7). To prove the second inequality of (2.7) we observe that

$$(n+1)S_{n+1}-f(B)=\sum_{i=1}^{n}f(C_i^{(n+1)}) \tag{2.11}$$

and

$$(n+2)S_{n+2}-f(B)=\sum_{i=1}^{n+1}f(C_i^{(n+2)}). \tag{2.12}$$

Substituting (2.11) and (2.12) in inequality (2.2) we get

$$\frac{1}{n}\left((n+1)S_{n+1}-f(B)\right)\leqslant\frac{1}{n+1}\left((n+2)S_{n+2}-f(B)\right). \tag{2.13}$$

If $n=1$ equality holds in the right hand side of (2.7). If $n\geqslant 2$ by changing n by $n-1$ in (2.13) and again by simplifying the above inequality we reach the second inequality of (2.7). The double inequality (2.8) can be established in a similar manner.

In the next result we give an operator version of [2, Theorem 2].

COROLLARY 2.4. *Let $f:[\alpha,\infty)\to\mathbb{R}$ is bounded below and operator concave. Let A,B be self-adjoint operators with spectra in $[\alpha,\infty)$ such that $A\leqslant B$. Then for each $n=1,2,\cdots$,*

$$S_{n+1}\leqslant S_n,\qquad T_n\leqslant T_{n+1}. \tag{2.14}$$

Proof. It follows from [10, Theorem 1.15] that f is operator monotone. The monotonicity of f implies that

$$f\left(\left(1-\frac{i}{k}\right)A+\frac{i}{k}B\right)\leqslant f(B)\qquad (i=0,1,\cdots;k=1,2,\cdots).$$

Therefore for each $k=1,2,\cdots$,

$$S_k=\frac{1}{k}\sum_{i=1}^{k}f(C_i^{(k)})=\frac{1}{k}\sum_{i=1}^{k}f\left(\left(1-\frac{i}{k}\right)A+\frac{i}{k}B\right)\leqslant\frac{1}{k}\sum_{i=1}^{k}f(B)=f(B). \tag{2.15}$$

Considering (2.15) and the second reversed inequality of (2.7) we obtain

$$S_n-S_{n+1}\geqslant\frac{1}{n^2}\left(f(B)-S_{n+1}\right)\geqslant 0,$$

which yields the first inequality of (2.14). Similarly note that

$$T_k\leqslant f(B)\qquad (k=1,2,\cdots),$$

whence from the first reversed inequality of (2.8)

$$T_{n+1}-T_n\geqslant\frac{1}{n(n+2)}\left(f(B)-T_{n+1}\right)\geqslant 0,$$

which yield the second inequality of (2.14).

3. Operator extensions of Alzer and Bennet inequalities

In this section we obtain some operator versions of the Alzer and Bennet inequalities.

COROLLARY 3.1. *(i) If either $A,B \geqslant 0$ and $1 \leqslant r \leqslant 2$ or $A,B > 0$ and $-1 \leqslant r \leqslant 0$, then*

$$\frac{(n+1)^2}{n(n+2)}\left(\frac{n}{n+1}\right)^r n\sum_{i=0}^{n+1}((n+1-i)A+iB)^r \leqslant (n+1)\sum_{i=0}^{n}((n-i)A+iB)^r\,, \tag{3.1}$$

$$(n+1)\sum_{i=1}^{n}((n+1-i)A+iB)^r \leqslant \left(\frac{n+1}{n+2}\right)^r n\sum_{i=1}^{n+1}((n+2-i)A+iB)^r \tag{3.2}$$

and

$$\left(\frac{A+B}{2}\right)^r \leqslant \frac{1}{m}\sum_{i=1}^{m}\left(A+i\frac{B-A}{m+1}\right)^r \leqslant \int_0^1((1-t)A+tB)^r\,dt$$
$$\leqslant \frac{1}{n+1}\sum_{i=0}^{n}\left(A+i\frac{B-A}{n}\right)^r \leqslant \frac{A^r+B^r}{2} \quad (m,n=1,2,\cdots). \tag{3.3}$$

(ii) If $A,B \geqslant 0$ and $0 \leqslant r \leqslant 1$, then all above inequalities are reversed.

Proof. From [10, Corollary 1.16] the power function t^r on $[0,\infty)$ is operator convex if $1 \leqslant r \leqslant 2$ and operator concave if $0 \leqslant r \leqslant 1$. In addition, t^r is operator convex on $(0,\infty)$ if $-1 \leqslant r \leqslant 0$. Now the assertion follows from Theorem 2.1 and Corollary 2.2.

COROLLARY 3.2. *If $1 \leqslant r \leqslant 2$ and $A,B \geqslant 0$, then*

$$\frac{n}{n+1}\left(n\sum_{i=0}^{n+1}((n+1-i)A+iB)^r\right)^{\frac{1}{r}}$$
$$\leqslant \left(1+\frac{1}{n(n+2)}\right)^{\frac{1}{r}}\left(\frac{n}{n+1}\right)\left(n\sum_{i=0}^{n+1}((n+1-i)A+iB)^r\right)^{\frac{1}{r}}$$
$$\leqslant \left((n+1)\sum_{i=0}^{n}((n-i)A+iB)^r\right)^{\frac{1}{r}}$$

and

$$\left((n+1)\sum_{i=1}^{n}((n+1-i)A+iB)^r\right)^{\frac{1}{r}} \leqslant \frac{n+1}{n+2}\left(n\sum_{i=1}^{n+1}((n+2-i)A+iB)^r\right)^{\frac{1}{r}}.$$

Moreover

$$\left(\frac{1}{m}\sum_{i=1}^{m}\left(A+i\frac{B-A}{m+1}\right)^r\right)^{\frac{1}{r}} \leqslant \left(\int_0^1((1-t)A+tB)^r\,dt\right)^{\frac{1}{r}}$$
$$\leqslant \left(\frac{1}{n+1}\sum_{i=0}^{n}\left(A+i\frac{B-A}{n}\right)^r\right)^{\frac{1}{r}} \qquad (m,n=1,2,\cdots).$$

Proof. Since $0<\frac{1}{r}\leqslant 1$, by Löwner–Heinz Theorem (see also [14] and references therein) $t^{\frac{1}{r}}$ is operator monotone. Now the assertion follows by taking $(\frac{1}{r})-$th root from each sides of (3.1), (3.2) and (3.3).

REMARK 3.3. (i) If $1\leqslant r\leqslant 2$, put $A=0$ and $B=I$ in (3.1) and (3.2) to get

$$\left(\frac{n}{n+1}\leqslant\right)\frac{n}{n+1}\left(1+\frac{1}{n(n+2)}\right)^{\frac{1}{r}} \leqslant \left(\frac{(n+1)\sum_{i=1}^{n}i^r}{n\sum_{i=1}^{n+1}i^r}\right)^{\frac{1}{r}} \leqslant \frac{n+1}{n+2}. \tag{3.4}$$

The left inequality is a refinement of Alzer's inequality (1.1) and the right one is Bennet's inequality (1.2).

If $0\leqslant r\leqslant 1$, set $A=0$ and $B=I$ in the reversed of (3.1) and (3.2) to reach

$$\frac{n+1}{n+2}\leqslant\left(\frac{(n+1)\sum_{i=1}^{n}i^r}{n\sum_{i=1}^{n+1}i^r}\right)^{\frac{1}{r}}\leqslant\frac{n}{n+1}\left(1+\frac{1}{n(n+2)}\right)^{\frac{1}{r}}$$

which is Bennet's inequality (1.2) and a converse of Alzer's inequality (1.1).

Finally if $-1\leqslant r\leqslant 0$, by taking $B=I$ and by letting $A\to 0$ in (3.2) we get Bennet's inequality (1.2).

These show that indeed Corollaries 3.1 and 3.2 are some operator generalizations of numerical Alzer and Bennet inequalities (1.1) and (1.2) in the special case of $-1\leqslant r\leqslant 2$.

(ii) If $1\leqslant r\leqslant 2$ (respectively $0\leqslant r\leqslant 1$), by taking $A=0$ and $B=I$ in (the reversed form of) (3.3) and by changing m by n, we get (the reversed form of) the following estimations

$$\frac{n^r(n+1)}{r+1}\leqslant\sum_{i=1}^{n}i^r\leqslant\frac{n(n+1)^r}{r+1} \qquad (n=1,2,\cdots). \tag{3.5}$$

If $-1\leqslant r\leqslant 0$ taking $B=I$ and tending $A\to 0$, we get only the right hand of (3.5).

REMARK 3.4. Chen et al. [7, Corollary 1] used the convexity or concavity together with the monotonicity property of the power function t^r to deduce

$$\left(\frac{(n+1)\sum_{i=1}^{n-1}i^r}{n\sum_{i=1}^{n}i^r}\right)^{\frac{1}{r}}\leqslant\frac{n}{n+1}\leqslant\left(\frac{(n+1)\sum_{i=1}^{n}i^r}{n\sum_{i=1}^{n+1}i^r}\right)^{\frac{1}{r}} \tag{3.6}$$

for any real number $r > 0$. The right inequality is Alzer's inequality (1.1).

Changing n by $n-1$ in the right hand of (3.4) we can easily seen that if $1 \leqslant r \leqslant 2$, then (3.4) is equivalent to

$$\left(\frac{n\sum_{i=1}^{n-1} i^r}{(n-1)\sum_{i=1}^{n} i^r}\right)^{\frac{1}{r}} \leqslant \frac{n}{n+1} \leqslant \left(\frac{(n+2)\sum_{i=1}^{n} i^r}{(n+1)\sum_{i=1}^{n+1} i^r}\right)^{\frac{1}{r}}. \tag{3.7}$$

This shows that both inequalities in (3.7) are stronger than (3.6). In the case $0 \leqslant r \leqslant 1$, we get a similar conclusion.

COROLLARY 3.5. *If $A, B > 0$, then*

$$\frac{1}{n+2}\sum_{i=0}^{n+1} \log\left(A + i\frac{B-A}{n+1}\right) \geqslant \frac{1}{n+1}\sum_{i=0}^{n} \log\left(A + i\frac{B-A}{n}\right)$$
$$\frac{1}{n}\sum_{i=1}^{n} \log\left(A + i\frac{B-A}{n+1}\right) \geqslant \frac{1}{n+1}\sum_{i=1}^{n+1} \log\left(A + i\frac{B-A}{n+2}\right) \tag{3.8}$$

as well as

$$\log\left(\frac{A+B}{2}\right) \geqslant \frac{1}{m}\sum_{i=1}^{m} \log\left(A + i\frac{B-A}{m+1}\right) \geqslant \int_0^1 \log((1-t)A + tB)\,dt$$
$$\geqslant \frac{1}{n+1}\sum_{i=0}^{n} \log\left(A + i\frac{B-A}{n}\right) \geqslant \frac{\log A + \log B}{2} \qquad (m, n = 1, 2, \cdots). \tag{3.9}$$

Proof. The assertion follows from (2.1), (2.2) and (2.6) and operator concavity of $f(t) = \log t$ on $(0, \infty)$; see [10, Chapter 1].

COROLLARY 3.6. *For each $n = 1, 2, \cdots$ we have*

$$\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \geqslant \frac{n+1}{n+2}, \qquad \sqrt[n]{n!} \geqslant \frac{n+1}{e}. \tag{3.10}$$

Proof. Let $B = I$ and $A \to 0$ in (3.8) and in the second inequality of (3.9).

REMARK 3.7. The first inequality in (3.10) is a refinement of the Minc–Sathre inequality, $\sqrt[n]{n!}/\sqrt[n+1]{(n+1)!} \geqslant n/(n+1)$; see [12]. An extension of the Minc–Sathre inequality was given by Kuang [11] by showing that if f is a strictly increasing convex (or concave) function in $(0, 1]$, then

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) > \frac{1}{n+1}\sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right) > \int_0^1 f(x)\,dx.$$

Our results can be regarded as operator versions of the first inequality above. Some numerical extensions of the Minc–Sathre inequality may be found in [15].

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