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DETERMINATION OF ORDER OF MAGNITUDE OF MULTIPLE FOURIER COEFFICIENTS OF FUNCTIONS OF BOUNDED ϕ–VARIATION HAVING LACUNARY FOURIER SERIES USING JENSEN'S INEQUALITY

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Abstract. For a Lebesgue integrable complex-valued function f defined over the m-dimensional torus $\mathbb{T}^m := [0,2\pi)^m$, let $\hat{f}(\mathbf{n})$ denote the Fourier coefficient of f, where $\mathbf{n} = (n^{(1)},\ldots,n^{(m)}) \in \mathbb{Z}^m$. Recently, in one of our papers [to appear in Mathematical Inequalities & Applications], we have defined the notion of bounded ϕ-variation for a complex-valued function on a rectangle $[a_1,b_1]\times\ldots\times[a_m,b_m]$ and studied the order of magnitude of Fourier coefficients of such functions on $[0,2\pi]^m$. In this paper, the order of magnitude of Fourier coefficients of a function of bounded ϕ-variation from $[0,2\pi]^m$ to $\mathbb{C}$ and having lacunary Fourier series with certain gaps is studied and a generalization of our earlier result (Theorem in [Acta Sci. Math. (Szeged), 78, (2012), 97–109]) is proved. Interestingly, the Jensen's inequality for integrals is used to prove the main result.

1. Introduction

For a function of two variables several definitions of bounded variation are given and various properties are studied (see, for example, [8], [1]). In 2002 F. Móricz [9] studied the order of magnitude of double Fourier coefficients with the help of Riemann-Stieltjes integral of functions of two variables and in 2004 V. Fülöp and F. Móricz [4] studied the order of magnitude of multiple Fourier coefficients of functions of bounded variation in the sense of Vitali and Hardy (see [3]) in a straightforward way without using Riemann-Stieltjes integral. J. R. Patadia (see [11, Theorem 3]) studied the order of magnitude of Fourier coefficients of functions in $L^1(\mathbb{T}^m)$ having lacunary Fourier series with certain gaps and are satisfying Lipschitz condition locally (that is, on certain smaller subsets of $[-\pi,\pi]^m$). In [5], we have defined the notion of bounded p-variation ($p \geqslant 1$) for a function from a rectangle $[a_1,b_1]\times\ldots\times[a_m,b_m]$ to $\mathbb{C}$ and studied the order of magnitude of Fourier coefficients of such functions from $[0,2\pi]^m$ to $\mathbb{C}$. Later in [6] we have proved a lacunary analogue of the main result (Theorem 2) of [5] by considering lacunary condition similar to that considered by J. R. Patadia [11, Theorem 3]). Recently, in [7], we have defined the the notion of bounded ϕ-variation for a function from a rectangle $[a_1,b_1]\times\ldots\times[a_m,b_m]$ to $\mathbb{C}$ and studied the order of magnitude of Fourier coefficients of such functions from $[0,2\pi]^m$ to $\mathbb{C}$. Here we study the order of

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magnitude of Fourier coefficients of functions in $L^1(\mathbb{T}^m)$ having the same type of lacunary Fourier series and are of bounded ϕ-variation locally and prove result analogous to our earlier result (see [7, Theorem 2]). Interestingly, we use the Jensen's inequality for integrals to prove our main result.

2. Notation and Definitions

In [7] we have defined the notion of bounded ϕ-variation for functions of several variables that generalize our earlier definition of bounded p-variation and hence in turn generalizes the definitions of bounded variation for functions of several variables given by Vitali and by Hardy. For the sake of completeness, here we rewrite those definitions.

Let R be the rectangle $R = [a_1,b_1] \times \ldots \times [a_m,b_m]$. By a (finite) partition $\mathscr{P}$ of R we mean the set $\mathscr{P} = \{R_1,\ldots,R_n\}$, in which R_i's are pairwise disjoint (no two have common interior) subrectangles of R having their sides (faces) parallel to the standard coordinate hyperplanes and whose union is R. Let $f = f(x_1,\ldots,x_m)$ be a real or complex-valued function on R. For any subrectangle $R' = [\alpha_1,\beta_1] \times \ldots \times [\alpha_m,\beta_m]$ of R with $a_i \leqslant \alpha_i < \beta_i \leqslant b_i$ for all $i = 1,2,\ldots,m$, we define $\Delta f(R')$ as follows: When $m = 2$ we put

$$\begin{aligned}\Delta f(R') &:= \Delta f([\alpha_1,\beta_1] \times [\alpha_2,\beta_2]) \\ &= f(\beta_1,\beta_2) - f(\beta_1,\alpha_2) - f(\alpha_1,\beta_2) + f(\alpha_1,\alpha_2);\end{aligned}$$

for $m = 3$

$$\begin{aligned}\Delta f(R') &:= \Delta f([\alpha_1,\beta_1] \times [\alpha_2,\beta_2] \times [\alpha_3,\beta_3]) \\ &= [f(\beta_1,\beta_2,\beta_3) - f(\beta_1,\alpha_2,\beta_3) - f(\alpha_1,\beta_2,\beta_3) + f(\alpha_1,\alpha_2,\beta_3)] \\ &\quad - [f(\beta_1,\beta_2,\alpha_3) - f(\beta_1,\alpha_2,\alpha_3) - f(\alpha_1,\beta_2,\alpha_3) + f(\alpha_1,\alpha_2,\alpha_3)] \\ &= \Delta_{[\alpha_3,\beta_3]}\Delta f([\alpha_1,\beta_1] \times [\alpha_2,\beta_2]), \text{ say;}\end{aligned}$$

and successively for any $m \geqslant 3$

$$\begin{aligned}\Delta f(R') &:= \Delta f([\alpha_1,\beta_1] \times \ldots \times [\alpha_m,\beta_m]) \\ &= \Delta_{[\alpha_m,\beta_m]}\Delta f([\alpha_1,\beta_1] \times \ldots \times [\alpha_{m-1},\beta_{m-1}]).\end{aligned}$$

In what follows, we consider $\phi : [0,\infty) \to \mathbb{R}$ a convex function which increases strictly from 0 to ∞ and satisfy the conditions $\phi(0) = 0$, $\phi(1) = 1$. The function ϕ is said to be a Δ_2-function if there is a constant $d \geqslant 2$ such that $\phi(2x) \leqslant d\phi(x)$ for all $x \geqslant 0$.

DEFINITION 1. We say that f is of bounded ϕ-variation over R in the sense of Vitali (written as $f \in \phi\mathrm{BV_V}(R)$) if $V_\phi(f;R)$, the total ϕ-variation of f over R, is finite, where

$$V_\phi(f;R) := \sup\left\{\sum_{i=1}^{n} \phi\left(|\Delta f(R_i)|\right)\right\}, \tag{1}$$

in which the supremum is taken over all partitions $\{R_1,\ldots,R_n\}$ of R.

REMARK 1. Note that for $\phi(x) = x^p$ $(p \geqslant 1)$ above definition is same as the definition of a function of bounded p-variation (see [5, Definition V]) and hence for $\phi(x) = x$ above definition is equivalent to that of Vitali (see, for example, [3, 4] and [7, Remark 1]). Also, the class $\phi\mathrm{BV_V}(R)$ contains functions for which the m-dimensional Lebesgue integral over R fails to exist. The following notion of bounded ϕ-variation is motivated by this fact.

DEFINITION 2. In case $m = 2$, we say that a function $f = f(x_1, x_2)$ is of bounded ϕ-variation over $R := [a_1, b_1] \times [a_2, b_2]$ in the sense of Hardy, in symbol: $f \in \phi\mathrm{BV_H}(R)$, if it is in the class $\phi\mathrm{BV_V}(R)$ and if the marginal functions $f(x_1, a_2)$ and $f(a_1, x_2)$ are of bounded ϕ-variation on the intervals $I_1 := [a_1, b_1]$ and $I_2 := [a_2, b_2]$, respectively in the sense of Young [12].

In case $m \geqslant 3$, the notion of bounded ϕ-variation in the sense of Hardy over a rectangle R can be defined by the following recurrence: $f \in \phi\mathrm{BV_H}(R)$ if $f \in \phi\mathrm{BV_V}(R)$ and each of the marginal functions $f(x_1, \ldots, a_k, \ldots, x_m)$ is in the class $\phi\mathrm{BV_H}(R(a_k))$, where $k = 1, \ldots, m$ and

$$R(a_k) = \{(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m) \in \mathbb{R}^{m-1} : \, a_j \leqslant x_j \leqslant b_j \\ \text{for } j = 1, \ldots, k-1, k+1, \ldots, m\}.$$

This definition can be equivalently reformulated as follows: $f \in \phi\mathrm{BV_H}(R)$ if and only if $f \in \phi\mathrm{BV_V}(R)$ and for any choice of $(1 \leqslant)$ $j_1 < \ldots < j_n$ $(\leqslant m)$, $1 \leqslant n < m$, the function $f(x_1, \ldots, a_{j_1}, \ldots, a_{j_n}, \ldots, x_m)$ is in the class $\phi\mathrm{BV_V}(R(a_{j_1}, \ldots, a_{j_n}))$, where

$$R(a_{j_1}, \ldots, a_{j_n}) := \{(x_{\ell_1}, \ldots, x_{\ell_{m-n}}) \in \mathbb{R}^{m-n} : \, a_j \leqslant x_j \leqslant b_j \\ \text{for } j = \ell_1, \ldots, \ell_{m-n}\}$$

and $\{\ell_1, \ldots, \ell_{m-n}\}$ is the complementary set of $\{j_1, \ldots, j_n\}$ with respect to $\{1, \ldots, m\}$.

REMARK 2. When $\phi(x) = x^p$ $(p \geqslant 1)$ our Definition 2 is same as our earlier definition of a function of bounded p-variation (see, [5, Definition H]) and hence when $\phi(x) = x$ above definition is equivalent that given by Hardy and Krause (see, for example, [3, 4])(refer Lemma 2 below).

Next let $\mathbb{T}^m$ be the m-dimensional torus identified with $\mathbf{Q} = [-\pi, \pi]^m$ and let its dual be identified with $\mathbb{Z}^m$. The points $(x_1, \ldots, x_m)$ of $\mathbf{Q}$ and $(n^{(1)}, \ldots, n^{(m)})$ of $\mathbb{Z}^m$ are denoted by $\mathbf{x}$ and $\mathbf{n}$ respectively; $\mathbf{n} \cdot \mathbf{x}$ denotes the scalar product given by $\mathbf{n} \cdot \mathbf{x} = n^{(1)} \cdot x_1 + \ldots + n^{(m)} \cdot x_m$ and $|\mathbf{x}|$ denotes the number $\sqrt{|x_1|^2 + \ldots + |x_m|^2}$. For $f \in \mathrm{L}^1(\mathbb{T}^m)$ its formal Fourier series is given by

$$f(\mathbf{x}) \sim \sum_{\mathbf{n} \in \mathbb{Z}^m} \hat{f}(\mathbf{n}) e^{i(\mathbf{n} \cdot \mathbf{x})},$$

where $\hat{f}(\mathbf{n})$ denotes the $\mathbf{n}^{th}$ Fourier coefficient of $f(\mathbf{x})$ given by

$$\hat{f}(\mathbf{n}) = \frac{1}{(2\pi)^m} \int_{\mathbf{Q}} f(\mathbf{x}) e^{-i(\mathbf{n} \cdot \mathbf{x})} d\mathbf{x}.$$

Let $\mathbf{x_0} = (x_{01}, x_{02}, \ldots, x_{0m})$ denote an arbitrary point of $\mathbf{Q}$, let δ be any arbitrary real number such that $0 < \delta \leqslant \pi$, and let $I = I(\mathbf{x_0}, \delta)$ denote the m-dimensional subrectangle of $\mathbf{Q}$ given by

$$I(\mathbf{x_0}, \delta) = \{\mathbf{x} := (x_1, \ldots, x_m) \in \mathbf{Q} : |x_j - x_{0j}| \leqslant \delta \text{ for } j = 1, 2, \ldots, m\}.$$

Given a subset $E \subset \mathbb{Z}^m$, a function $f \in \mathrm{L}^1(\mathbb{T}^m)$ is said to be E-*spectral* (or, said to have spectrum E) if and only if $\hat{f}(\mathbf{n}) = 0$ for all $\mathbf{n}$ in $\mathbb{Z}^m \setminus E$. In what follows, we consider a set $E \subset \mathbb{Z}^m$ described in the following way: for each $j = 1, 2, \ldots, m$ consider sets $E^{(j)} = \{\ldots, n_{-2}^{(j)}, n_{-1}^{(j)}, n_0^{(j)}, n_1^{(j)}, n_2^{(j)}, \ldots\} \subset \mathbb{Z}$ with $n_{-k}^{(j)} = -n_k^{(j)}$ for $k = 0, 1, 2, \ldots$ and with $\{n_k^{(j)}\}_{k=1}^{\infty}$ strictly increasing such that

$$\liminf_{k \to \infty} \frac{N_k^{(j)}}{\ln n_k^{(j)}} = B^{(j)} > \frac{8e}{\delta}, \tag{2}$$

where $N_k^{(j)} = \min\left\{n_{k+1}^{(j)} - n_k^{(j)}, n_k^{(j)} - n_{k-1}^{(j)}\right\}$; and then put $E = \prod_{j=1}^m E^{(j)}$. Now $\mathbf{n_s} = (n_{s_1}^{(1)}, n_{s_2}^{(2)}, \ldots, n_{s_m}^{(m)})$ denotes the typical element of E. When $m = 1$, E will be taken to be $E^{(1)}$ with upper suffix in $n_k^{(1)}$'s and $N_k^{(1)}$'s omitted.

3. Results

We need the following lemmas. Lemmas 1 to 4 are proved in [7], Lemma 5 is due to Noble ([10], or [2, p. 270]) and Lemma 6 is its m-dimensional analogue by Patadia [11].

LEMMA 1. *If $f \in \phi\mathrm{BV_H}(R)$, then f is bounded over R.*

LEMMA 2. *If ϕ is Δ_2 and $f \in \phi\mathrm{BV_H}(R)$, then for any arbitrary fixed values $c_{j_1} \in [a_{j_1}, b_{j_1}], \ldots, c_{j_n} \in [a_{j_n}, b_{j_n}]$, $(1 \leqslant)\ j_1 < \ldots < j_n\ (\leqslant m)$, and $1 \leqslant n < m$, the function $f(\cdot, \ldots, c_{j_1}, \ldots, c_{j_n}, \ldots, \cdot)$ is in the class $\phi\mathrm{BV_H}(R(a_{j_1}, \ldots, a_{j_n}))$ and that*

$$V_\phi(f(\cdot, \ldots, c_{j_1}, \ldots, c_{j_n}, \ldots, \cdot); R(a_{j_1}, \ldots, a_{j_n})) \\ \leqslant d^n \left\{ V_\phi(f; R) + \sum_{k=1}^{n} \sum_{\substack{s_1 < \ldots < s_k, \\ s_1, \ldots, s_k \in \{j_1, \ldots, j_n\}}} V_\phi\left(f(\cdot, \ldots, a_{s_1}, \ldots, a_{s_k}, \ldots, \cdot); R(a_{s_1}, \ldots, a_{s_k})\right) \right\}.$$

LEMMA 3. *Let $f \in \phi\mathrm{BV_V}(R)$, where $R = [a_1, b_1] \times \ldots \times [a_m, b_m]$. Let $\{R_1, \ldots, R_k\}$ be a partition of R. Then $f \in \phi\mathrm{BV_V}(R_i)$ for each $i = 1, \ldots, k$, and that*

$$\sum_{i=1}^{k} V_\phi(f; R_i) \leqslant V_\phi(f; R).$$

LEMMA 4. *Let $f \in \phi\mathrm{BV_H}(R)$, where $R = [a_1,b_1] \times \ldots \times [a_m,b_m]$. Then the discontinuities of f are located on a countable number of $(m-1)$-dimensional hyperplanes parallel to some of the coordinate hyperplanes.*

LEMMA 5. *Let $\delta > 0$. Then for sufficiently large n there exists a trigonometric polynomial $T_n(x)$ of degree at most n, with constant term 1, such that*

(i) $|T_n(x)| \leqslant A_1\delta^{-1}$ *for all* $x \in [-\pi,\pi]$,

(ii) $|T_n(x)| \leqslant A_2 \exp(-n\delta/8e)$ *for all x such that* $\delta \leqslant |x| \leqslant \pi$,

where A_1 and A_2 are absolute constants.

LEMMA 6. *Let $\delta > 0$. Then for $\mathbf{n} = (n^{(1)},\ldots,n^{(m)})$ such that each $n^{(j)}$ is sufficiently large, there exists a trigonometric polynomial*

$$T_{\mathbf{n}}(\mathbf{x}) = \sum_{\substack{|k^{(j)}|\leqslant n^{(j)}\\ j=1,\ldots,m}} c_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{x})},$$

with constant term 1, such that

(i) $|T_{\mathbf{n}}(\mathbf{x})| \leqslant A_1\delta^{-m}$ *for all* $\mathbf{x} \in \mathbf{Q}$,

(ii) $|T_{\mathbf{n}}(\mathbf{x})| \leqslant A_2 \exp(-\delta(\mathbf{1}\cdot\mathbf{n})/8e)$ *for all* $\mathbf{x} \in \mathbf{Q} \setminus I(\mathbf{0},\delta)$,

where $\mathbf{1} = (1,\ldots,1)$ and A_1, A_2 are constants depending only on m.

Here we prove the following theorem.

THEOREM 1. *Let $E \subset \mathbb{Z}^m$ be described as above and $f : \mathbb{R}^m \to \mathbb{C}$ be 2π-periodic in each variable. If $f \in \phi\mathrm{BV_H}(I)$, f is E-spectral and $\mathbf{n_k} = (n^{(1)}_{k_1},\ldots,n^{(m)}_{k_m}) \in \mathbb{Z}^m$ is such that $|n^{(j)}_{k_j}|$ is sufficiently large for each j, then*

$$\hat{f}(\mathbf{n_k}) = O\left(\phi^{-1}\left(\frac{1}{\left|\prod_{j=1}^m n^{(j)}_{k_j}\right|}\right)\right).$$

REMARK 3. This theorem gives a lacunary analogue of our earlier result [7, Theorem 2]. Further, by taking $\phi(x) = x^p$ $(p \geqslant 1)$, we get our earlier result [6, Theorem]. The proof of this theorem is similar to that of [6, Theorem] where the Hölder's inequality is used, and explains the technique (at least to new readers) how to go from bounded p-variation to bounded ϕ-variation and how to use the Jensen's inequality in place of Hölder's inequality.

4. Proof of Theorem 1

We may assume without loss of generality that $\mathbf{x_0} = \mathbf{0}$. For, suppose the theorem is true when $\mathbf{x_0} = \mathbf{0}$ and consider the function $g(\mathbf{x}) = f(\mathbf{x}+\mathbf{x_0}) = (T_{-\mathbf{x_0}}f)(\mathbf{x})$. Then

$$\mathbf{x} \in I(\mathbf{0},\delta) \Leftrightarrow |x_j| \leqslant \delta \ \ \forall j \Leftrightarrow |x_j + x_{0j} - x_{0j}| \leqslant \delta \ \ \forall j \Leftrightarrow \mathbf{x}+\mathbf{x_0} \in I(\mathbf{x_0},\delta).$$

Since $f \in \phi\mathrm{BV_H}(I(\mathbf{x_0},\delta))$, it follows that $g \in \phi\mathrm{BV_H}(I(\mathbf{0},\delta))$. Also,

$$g = T_{-\mathbf{x_0}} f \Rightarrow \hat{g}(\mathbf{n}) = e^{i(\mathbf{n}\cdot\mathbf{x_0})}\hat{f}(\mathbf{n}) \ \ \forall \mathbf{n} \in \mathbb{Z}^m.$$

Since f is E-spectral, so is g and as the theorem is true when $\mathbf{x_0} = \mathbf{0}$, $\hat{g}(\mathbf{n_k}) = O(\phi^{-1}(1/|\prod_{j=1}^m n_{k_j}^{(j)}|))$. It follows now that $\hat{f}(\mathbf{n_k}) = O(\phi^{-1}(1/|\prod_{j=1}^m n_{k_j}^{(j)}|))$ in view of $|e^{i(\mathbf{n}\cdot\mathbf{x_0})}| = 1$.

For the sake of simplicity in writing, now onwards, we carry out the proof for $m = 2$, and we write (x,y) in place of (x_1,x_2). Since $f \in \phi\mathrm{BV_H}([0,2\pi]^2)$, in view of Lemma 4 (for $m = 2$), the discontinuities of f lie on a countable number of parallels to the axes and hence f is measurable over $\mathbb{T}^2$ in the sense of Lebesgue. Further, by Lemma 1, f is bounded over $[0,2\pi]^2$ and hence $f \in \mathrm{L}^1(\mathbb{T}^2)$. As $\phi\mathrm{BV_H}([0,2\pi]^2) \subset \phi\mathrm{BV_V}([0,2\pi]^2)$, $f \in \mathrm{L}^1(\mathbb{T}^2) \cap \phi\mathrm{BV_V}([0,2\pi]^2)$.

For a given $\mathbf{n_k} = (n_{k_1}^{(1)}, n_{k_2}^{(2)})$, we take $\mathbf{M_k} = (M_{k_1}^{(1)}, M_{k_2}^{(2)})$, where for each $j = 1,2$, $M_{k_j}^{(j)} = \min\{N_{k_j}^{(j)} - 1, |n_{k_j}^{(j)}|^{1/2}\}$. In view of the symmetry of the set $E^{(j)}$ and (2) we have

$$\liminf_{|k_j|\to\infty} \frac{N_{k_j}^{(j)} - 1}{\ln|n_{k_j}^{(j)}|} = \liminf_{|k_j|\to\infty} \frac{N_{k_j}^{(j)}}{\ln|n_{k_j}^{(j)}|} = B^{(j)} > \frac{8e}{\delta},$$

for each $j = 1,2$. Thus there is a positive integer K_0 such that $(N_{k_j}^{(j)} - 1)/(\ln|n_{k_j}^{(j)}|) > (8e/\delta)$ for all $k_j \geqslant K_0$ and each $j = 1,2$. Since

$$\lim_{k_j\to\infty} \frac{|n_{k_j}^{(j)}|^{1/2}}{\ln|n_{k_j}^{(j)}|} = \infty$$

for each j, there is a $K_1 \in \mathbb{N}$ such that $(|n_{k_j}^{(j)}|^{1/2})/(\ln|n_{k_j}^{(j)}|) > (8e/\delta)$ for all $k_j \geqslant K_1$ and each $j = 1,2$. Taking $K_2 = \max\{K_0, K_1\}$ we see that

$$M_{k_j}^{(j)} > \left(\frac{8e}{\delta}\right) \ln|n_{k_j}^{(j)}| \tag{3}$$

for all $k_j \geqslant K_2$ and each $j = 1,2$. Thus for $\mathbf{n_k}$ such that each $|n_{k_j}^{(j)}|$ is sufficiently large (3) holds.

Now consider the trigonometric polynomial $T_{\mathbf{M_k}}(\mathbf{x})$ satisfying conditions of Lemma 6 corresponding to this $\mathbf{M_k}$ and δ. Since f is E-spectral, the choice of $\mathbf{M_k}$ and $T_{\mathbf{M_k}}(\mathbf{x})$ gives us

$$\begin{aligned} \hat{f}(\mathbf{n_k}) &= \frac{1}{(2\pi)^2} \int_{\mathbf{Q}} f(\mathbf{x}) T_{\mathbf{M_k}}(\mathbf{x}) e^{-i(\mathbf{n_k}\cdot\mathbf{x})} d\mathbf{x} \\ &= \frac{1}{(2\pi)^2} \left(\int_{I(\mathbf{0},\delta)} + \int_{\mathbf{Q}\setminus I(\mathbf{0},\delta)} \right) f(\mathbf{x}) T_{\mathbf{M_k}}(\mathbf{x}) e^{-i(\mathbf{n_k}\cdot\mathbf{x})} d\mathbf{x} \\ &= I_1 + I_2, \ \text{say}. \end{aligned} \tag{4}$$

Now

$$
\begin{aligned}
|I_2| &= \frac{1}{(2\pi)^2}\left|\int_{\mathbf{Q}\setminus I(\mathbf{0},\delta)} f(\mathbf{x})T_{\mathbf{M_k}}(\mathbf{x})e^{-i(\mathbf{n_k}\cdot\mathbf{x})}d\mathbf{x}\right| \\
&\leqslant \frac{1}{(2\pi)^2}A_2 e^{(-\delta(\mathbf{1}\cdot\mathbf{M_k})/(8e))}\int_{\mathbf{Q}\setminus I(\mathbf{0},\delta)}|f(\mathbf{x})|d\mathbf{x} \\
&\leqslant \frac{1}{(2\pi)^2}A_2 e^{(-\delta(\mathbf{1}\cdot\mathbf{M_k})/(8e))}||f||_1. \qquad (5)
\end{aligned}
$$

In view of (3), for each $j=1,2$, we have

$$
-\frac{\delta}{8e}\cdot M_{k_j}^{(j)} < -\frac{\delta}{8e}\cdot\frac{8e}{\delta}\cdot\ln|n_{k_j}^{(j)}| = -\ln|n_{k_j}^{(j)}|,
$$

and therefore

$$
e^{-\frac{\delta}{8e}(\mathbf{1}\cdot\mathbf{M_k})} = e^{-\frac{\delta}{8e}(M_{k_1}^{(1)}+M_{k_2}^{(2)})} < e^{-\ln|n_{k_1}^{(1)}|}e^{-\ln|n_{k_2}^{(2)}|} = \frac{1}{|n_{k_1}^{(1)}n_{k_2}^{(2)}|}.
$$

Using this in (5) we get

$$
I_2 = O\left(\frac{1}{|n_{k_1}^{(1)}n_{k_2}^{(2)}|}\right). \qquad (6)
$$

Now we estimate I_1. Choose $\mathbf{n_k}$ such that each $|n_{k_j}^{(j)}|$ is sufficiently large so that (3) holds, and such that $2\pi/|n_{k_j}^{(j)}| < \delta$ for $j=1,2$. Again, for simplicity, we put $n_{k_1}^{(1)} = u$ and $n_{k_2}^{(2)} = v$. Then $2\pi/|u| < \delta$, $2\pi/|v| < \delta$, and there are unique non-negative integers α and β such that

$$
\alpha\frac{2\pi}{|u|} \leqslant \delta < (\alpha+1)\frac{2\pi}{|u|};\quad \beta\frac{2\pi}{|v|} \leqslant \delta < (\beta+1)\frac{2\pi}{|v|}.
$$

Therefore

$$
0 \leqslant \delta - \alpha\frac{2\pi}{|u|} < \frac{2\pi}{|u|};\quad 0 \leqslant \delta - \beta\frac{2\pi}{|v|} < \frac{2\pi}{|v|}. \qquad (7)
$$

Since $0 < \alpha\frac{2\pi}{|u|}, \beta\frac{2\pi}{|v|} \leqslant \delta$, say, $J = \left[-\alpha\frac{2\pi}{|u|}, \alpha\frac{2\pi}{|u|}\right]\times\left[-\beta\frac{2\pi}{|v|}, \beta\frac{2\pi}{|v|}\right] \subset I(\mathbf{0},\delta)$. Therefore we can write I_1 as

$$
\begin{aligned}
I_1 &= \frac{1}{(2\pi)^2}\int_{I(\mathbf{0},\delta)}(fT_{\mathbf{M_k}})(\mathbf{x})e^{-i(\mathbf{n_k}\cdot\mathbf{x})}d\mathbf{x} \\
&= \frac{1}{(2\pi)^2}\left(\int_J + \int_{I(\mathbf{0},\delta)\setminus J}\right)(fT_{\mathbf{M_k}})(\mathbf{x})e^{-i(\mathbf{n_k}\cdot\mathbf{x})}d\mathbf{x} \\
&= I_{11} + I_{12},\ \text{say}. \qquad (8)
\end{aligned}
$$

Next we estimate I_{11}. Note that e^{-iux} and e^{-ivy} are periodic functions of periods $2\pi/|u|$ and $2\pi/|v|$ respectively. Thus by putting

$$a_r = r\frac{2\pi}{|u|}\ \ (r=-\alpha,-\alpha+1,\dots,\alpha);\quad b_s = s\frac{2\pi}{|v|}\ \ (s=-\beta,-\beta+1,\dots,\beta)$$

we get

$$\int_{a_{r-1}}^{a_r} e^{-iux}dx = 0 \quad (r=-\alpha+1,-\alpha+2,\dots,\alpha) \tag{9}$$

and

$$\int_{b_{s-1}}^{b_s} e^{-ivy}dy = 0 \quad (s=-\beta+1,-\beta+2,\dots,\beta). \tag{10}$$

Define three functions f_1, f_2, f_3 on J by setting

$$f_1(x,y) = (fT_{\mathbf{M_k}})(x,b_{s-1})\ \ (a_{-\alpha}\leqslant x< a_\alpha;\ b_{s-1}\leqslant y<b_s)$$

for $s=-\beta+1,-\beta+2,\dots,\beta$;

$$f_2(x,y) = (fT_{\mathbf{M_k}})(a_{r-1},y)\ \ (a_{r-1}\leqslant x< a_r;\ b_{-\beta}\leqslant y<b_\beta)$$

for $r=-\alpha+1,-\alpha+2,\dots,\alpha$; and

$$f_3(x,y) = (fT_{\mathbf{M_k}})(a_{r-1},b_{s-1})\ \ (a_{r-1}\leqslant x< a_r;\ b_{s-1}\leqslant y<b_s)$$

for $r=-\alpha+1,-\alpha+2,\dots,\alpha$; $s=-\beta+1,-\beta+2,\dots,\beta$. Since $f\in\phi\mathrm{BV_H}(I(\mathbf{0},\delta))$, $J\subset I(\mathbf{0},\delta)$ and $T_{\mathbf{M_k}}$ is a trigonometric polynomial, each $f_i\in\phi\mathrm{BV_H}(J)$ and hence $(fT_{\mathbf{M_k}}-f_1-f_2+f_3)\in\phi\mathrm{BV_H}(J)\subset \mathrm{L}^1(J)$. Further in view of Fubini's theorem and the relations (9) and (10) we have

$$\begin{aligned}\int_J f_1(\mathbf{x})e^{-i(\mathbf{n_k}\cdot\mathbf{x})}d\mathbf{x} &= \int_{a_{-\alpha}}^{a_\alpha}\int_{b_{-\beta}}^{b_\beta} f_1(x,y)e^{-iux}e^{-ivy}dxdy\\ &= \int_{a_{-\alpha}}^{a_\alpha}\left[\sum_{s=-\beta+1}^{\beta}(fT_{\mathbf{M_k}})(x,b_{s-1})\int_{b_{s-1}}^{b_s}e^{-ivy}dy\right]e^{-iux}dx=0,\\ \int_J f_2(\mathbf{x})e^{-i(\mathbf{n_k}\cdot\mathbf{x})}d\mathbf{x} &= \int_{a_{-\alpha}}^{a_\alpha}\int_{b_{-\beta}}^{b_\beta} f_2(x,y)e^{-iux}e^{-ivy}dxdy\\ &= \int_{b_{-\beta}}^{b_\beta}\left[\sum_{r=-\alpha+1}^{\alpha}(fT_{\mathbf{M_k}})(a_{r-1},y)\int_{a_{r-1}}^{a_r}e^{-iux}dx\right]e^{-ivy}dy=0\end{aligned}$$

and

$$\begin{aligned}\int_J f_3(\mathbf{x})e^{-i(\mathbf{n_k}\cdot\mathbf{x})}d\mathbf{x} &= \int_{a_{-\alpha}}^{a_\alpha}\int_{b_{-\beta}}^{b_\beta} f_3(x,y)e^{-iux}e^{-ivy}dxdy\\ &= \sum_{r=-\alpha+1}^{\alpha}\sum_{s=-\beta+1}^{\beta}(fT_{\mathbf{M_k}})(a_{r-1},b_{s-1})\left[\int_{a_{r-1}}^{a_r}e^{-iux}dx\right]\left[\int_{b_{s-1}}^{b_s}e^{-ivy}dy\right]\\ &= 0.\end{aligned}$$

Using these equations in the expression for I_{11} we get

$$\begin{aligned}(2\pi)^2|I_{11}| &= \left|\int_J (fT_{\mathbf{M_k}})(\mathbf{x})e^{-i(\mathbf{n_k}\cdot\mathbf{x})}d\mathbf{x}\right| \\ &= \left|\int_{a_{-\alpha}}^{a_\alpha}\int_{b_{-\beta}}^{b_\beta}(fT_{\mathbf{M_k}}-f_1-f_2+f_3)(x,y)e^{-iux}e^{-ivy}dxdy\right| \\ &\leqslant \int_{a_{-\alpha}}^{a_\alpha}\int_{b_{-\beta}}^{b_\beta}|(fT_{\mathbf{M_k}}-f_1-f_2+f_3)(x,y)|dxdy. \qquad (11)\end{aligned}$$

Now, by Jensen's inequality, for $c>0$

$$\begin{aligned}&\phi\left(\left(2a_\alpha\cdot 2b_\beta\right)^{-1}\cdot c\cdot\int_{a_{-\alpha}}^{a_\alpha}\int_{b_{-\beta}}^{b_\beta}|(fT_{\mathbf{M_k}}-f_1-f_2+f_3)(x,y)|dxdy\right) \\ &\leqslant \left(2a_\alpha\cdot 2b_\beta\right)^{-1}\int_{a_{-\alpha}}^{a_\alpha}\int_{b_{-\beta}}^{b_\beta}\phi\left(c|(fT_{\mathbf{M_k}}-f_1-f_2+f_3)(x,y)|\right)dxdy \\ &= \left(4a_\alpha b_\beta\right)^{-1}\sum_{r=-\alpha+1}^{\alpha}\sum_{s=-\beta+1}^{\beta}\int_{a_{r-1}}^{a_r}\int_{b_{s-1}}^{b_s}\phi\left(c|(fT_{\mathbf{M_k}}-f_1-f_2+f_3)(x,y)|\right)dxdy \\ &= \left(4a_\alpha b_\beta\right)^{-1}\sum_{r=-\alpha+1}^{\alpha}\sum_{s=-\beta+1}^{\beta}\int_{a_{r-1}}^{a_r}\int_{b_{s-1}}^{b_s}\phi(c|(fT_{\mathbf{M_k}})(x,y)-(fT_{\mathbf{M_k}})(x,b_{s-1}) \\ &\qquad -(fT_{\mathbf{M_k}})(a_{r-1},y)+(fT_{\mathbf{M_k}})(a_{r-1},b_{s-1})|)dxdy \\ &\leqslant \left(4a_\alpha b_\beta\right)^{-1}\sum_{r=-\alpha+1}^{\alpha}\sum_{s=-\beta+1}^{\beta}V_\phi(cfT_{\mathbf{M_k}};[a_{r-1},a_r]\times[b_{s-1},b_s])(a_r-a_{r-1})(b_s-b_{s-1}) \\ &\leqslant \left(4a_\alpha b_\beta\right)^{-1}\frac{(2\pi)^2}{|uv|}V_\phi(cfT_{\mathbf{M_k}};J) \\ &\leqslant \left(4\left(\delta-\frac{2\pi}{|u|}\right)\left(\delta-\frac{2\pi}{|v|}\right)\right)^{-1}\frac{(2\pi)^2}{|uv|}V_\phi(cfT_{\mathbf{M_k}};I(\mathbf{0},\delta)),\end{aligned}$$

in view of Lemma 3. Since ϕ is convex and $\phi(0)=0$, we have $\phi(ax)\leqslant a\phi(x)$ for $0<a<1$ and for all $x\geqslant 0$. Therefore, choosing c in $(0,1)$ so small that

$$\left(4\left(\delta-\frac{2\pi}{|u|}\right)\left(\delta-\frac{2\pi}{|v|}\right)\right)^{-1}\cdot(2\pi)^2\cdot V_\phi(cfT_{\mathbf{M_k}};I(\mathbf{0},\delta))\leqslant 1,$$

we get

$$\begin{aligned}\int_{a_{-\alpha}}^{a_\alpha}\int_{b_{-\beta}}^{b_\beta}|(fT_{\mathbf{M_k}}-f_1-f_2+f_3)(x,y)|dxdy &\leqslant \left(\frac{4a_\alpha b_\beta}{c}\right)\phi^{-1}\left(\frac{1}{|uv|}\right) \\ &\leqslant \left(\frac{4\delta^2}{c}\right)\phi^{-1}\left(\frac{1}{|uv|}\right). \qquad (12)\end{aligned}$$

In view of (11) and (12) we get

$$I_{11} = O\left(\phi^{-1}\left(\frac{1}{|uv|}\right)\right). \tag{13}$$

Finally, we have

$$I_{12} = I_{121} + I_{122} + I_{123} + I_{124} + I_{125} + I_{126} + I_{127} + I_{128}, \tag{14}$$

where $I_{121}, \ldots, I_{128}$ are integrals of the function $(1/(2\pi)^2)(fT_{\mathbf{M_k}})(\mathbf{x})e^{-i(\mathbf{n_k}\cdot\mathbf{x})}$ over the rectangles $[-\delta, a_{-\alpha}] \times [-\delta, b_{-\beta}]$, $[-\delta, a_{-\alpha}] \times [b_\beta, \delta]$, $[a_\alpha, \delta] \times [-\delta, b_{-\beta}]$, $[a_\alpha, \delta] \times [b_\beta, \delta]$, $[a_{-\alpha}, a_\alpha] \times [-\delta, b_{-\beta}]$, $[a_{-\alpha}, a_\alpha] \times [b_\beta, \delta]$, $[-\delta, a_{-\alpha}] \times [b_{-\beta}, b_\beta]$ and $[a_\alpha, \delta] \times [b_{-\beta}, b_\beta]$ respectively.

Since $f \in \phi\mathrm{BV_H}(I(\mathbf{0}, \delta))$, it is bounded there and as $T_{\mathbf{M_k}}$ is a trigonometric polynomial, there is a constant $M \geqslant 0$ such that $|(fT_{\mathbf{M_k}})(\mathbf{x})| \leqslant M$ for all $\mathbf{x} \in I(\mathbf{0}, \delta)$. Therefore we have

$$|I_{121}| \leqslant \frac{M}{(2\pi)^2}\int_{-\delta}^{a-\alpha}\int_{-\delta}^{b_{-\beta}} dxdy = \frac{M}{(2\pi)^2}(a_{-\alpha}+\delta)(b_{-\beta}+\delta) \leqslant \frac{M}{(2\pi)^2}\cdot\frac{2\pi}{|u|}\cdot\frac{2\pi}{|v|},$$

showing that $I_{121} = O\left(\frac{1}{|uv|}\right)$.

Similarly, we have $I_{122}, I_{123}, I_{124} = O\left(\frac{1}{|uv|}\right)$.

Now we estimate I_{125}. We may assume without loss of generality that $-\delta < b_{-\beta}$, that is, $b_\beta < \delta$, because otherwise $-\delta = b_{-\beta}$ and then $I_{125} = 0$. Define a function h on $[a_{-\alpha}, a_\alpha] \times [-\delta, b_{-\beta}] = J'$, say, by setting

$$h(x,y) = (fT_{\mathbf{M_k}})(a_{r-1}, y) \ \ (a_{r-1} \leqslant x < a_r;\ -\delta \leqslant y < b_{-\beta})$$

for $r = -\alpha+1, -\alpha+2, \ldots, \alpha$. Since $f \in \phi\mathrm{BV_H}(I(\mathbf{0}, \delta))$, $J' \subset I(\mathbf{0}, \delta)$ and $T_{\mathbf{M_k}}$ is a trigonometric polynomial, $h \in \phi\mathrm{BV_H}(J')$ and hence $(fT_{\mathbf{M_k}} - h) \in \phi\mathrm{BV_H}(J') \subset \mathrm{L}^1(J')$. Further in view of Fubini's theorem and (9) we have

$$\begin{aligned}
&\int_{a_{-\alpha}}^{a_\alpha}\int_{-\delta}^{b_{-\beta}} h(x,y)e^{-iux}e^{-ivy}dxdy \\
&\qquad = \sum_{r=-\alpha+1}^{\alpha}\int_{a_{r-1}}^{a_r}\int_{-\delta}^{b_{-\beta}} h(x,y)e^{-iux}e^{-ivy}dxdy \\
&\qquad = \sum_{r=-\alpha+1}^{\alpha}\int_{-\delta}^{b_{-\beta}}\left[(fT_{\mathbf{M_k}})(a_{r-1},y)\left\{\int_{a_{r-1}}^{a_r} e^{-iux}dx\right\}e^{-ivy}\right]dy = 0.
\end{aligned}$$

Thus

$$\begin{aligned}
(2\pi)^2|I_{125}| &= \left|\int_{a_{-\alpha}}^{a_\alpha}\int_{-\delta}^{b_{-\beta}}(fT_{\mathbf{M_k}})(x,y)e^{-iux}e^{-ivy}dxdy\right| \\
&= \left|\int_{a_{-\alpha}}^{a_\alpha}\int_{-\delta}^{b_{-\beta}}(fT_{\mathbf{M_k}} - h)(x,y)e^{-iux}e^{-ivy}dxdy\right| \\
&\leqslant \int_{a_{-\alpha}}^{a_\alpha}\int_{-\delta}^{b_{-\beta}}\left|(fT_{\mathbf{M_k}} - h)(x,y)\right|dxdy. \qquad (15)
\end{aligned}$$

Now, again by Jensen's inequality, for $c>0$

$$
\begin{aligned}
&\phi\left(\frac{c}{2a_\alpha(b_{-\beta}+\delta)}\int_{a_{-\alpha}}^{a_\alpha}\int_{-\delta}^{b_{-\beta}}\left|(fT_{\mathbf{M_k}}-h)(x,y)\right|dxdy\right)\\
&\leqslant \frac{1}{2a_\alpha(\delta-b_\beta)}\int_{a_{-\alpha}}^{a_\alpha}\int_{-\delta}^{b_{-\beta}}\phi\left(c|(fT_{\mathbf{M_k}}-h)(x,y)|\right)dxdy\\
&=\frac{1}{2a_\alpha(\delta-b_\beta)}\sum_{r=-\alpha+1}^{\alpha}\int_{a_{r-1}}^{a_r}\int_{-\delta}^{b_{-\beta}}\phi\left(c|(fT_{\mathbf{M_k}}-h)(x,y)|\right)dxdy\\
&=\frac{1}{2a_\alpha(\delta-b_\beta)}\int_{-\delta}^{b_{-\beta}}\left[\sum_{r=-\alpha+1}^{\alpha}\int_{a_{r-1}}^{a_r}\phi(c|(fT_{\mathbf{M_k}})(x,y)-(fT_{\mathbf{M_k}})(a_{r-1},y)|)dx\right]dy\\
&\leqslant \frac{1}{2a_\alpha(\delta-b_\beta)}\int_{-\delta}^{b_{-\beta}}\left[\sum_{r=-\alpha+1}^{\alpha}V_\phi\left((cfT_{\mathbf{M_k}})(\cdot,y);[a_{r-1},a_r]\right)(a_r-a_{r-1})\right]dy\\
&\leqslant \frac{1}{2a_\alpha(\delta-b_\beta)}\frac{2\pi}{|u|}\int_{-\delta}^{b_{-\beta}}V_\phi\left((cfT_{\mathbf{M_k}})(\cdot,y);[a_{-\alpha},a_\alpha]\right)dy\\
&\leqslant \frac{1}{2a_\alpha(\delta-b_\beta)}\frac{2\pi}{|u|}\int_{-\delta}^{b_{-\beta}}d[V_\phi\left(cfT_{\mathbf{M_k}};[a_{-\alpha},a_\alpha]\times[-\delta,b_{-\beta}]\right)\\
&\qquad\qquad\qquad\qquad +V_\phi\left((cfT_{\mathbf{M_k}})(\cdot,-\delta);[a_{-\alpha},a_\alpha]\right)]dy\\
&=\frac{d[V_\phi\left(cfT_{\mathbf{M_k}};[a_{-\alpha},a_\alpha]\times[-\delta,b_{-\beta}]\right)+V_\phi\left((cfT_{\mathbf{M_k}})(\cdot,-\delta);[a_{-\alpha},a_\alpha]\right)]}{2a_\alpha(\delta-b_\beta)}\frac{2\pi}{|u|}(b_{-\beta}+\delta)\\
&\leqslant \frac{\pi d[V_\phi\left(cfT_{\mathbf{M_k}};[a_{-\alpha},a_\alpha]\times[-\delta,b_{-\beta}]\right)+V_\phi\left((cfT_{\mathbf{M_k}})(\cdot,-\delta);[a_{-\alpha},a_\alpha]\right)]}{(\delta-2\pi/|u|)}\frac{1}{|u|},
\end{aligned}
\tag{16}
$$

in view of Lemma 3 (for a function of one variable) and Lemma 2. Since ϕ is convex and $\phi(0)=0$, now we can choose $c\in(0,1)$ such that

$$\frac{\pi dV_\phi\left(cfT_{\mathbf{M_k}};[a_{-\alpha},a_\alpha]\times[-\delta,b_{-\beta}]\right)}{(\delta-2\pi/|u|)}\leqslant\frac{1}{2}$$

and

$$\frac{\pi dV_\phi\left((cfT_{\mathbf{M_k}})(\cdot,-\delta);[a_{-\alpha},a_\alpha]\right)}{(\delta-2\pi/|u|)}\leqslant\frac{1}{2}.$$

Using these inequalities in (16) we get

$$\phi\left(\frac{c}{2a_\alpha(b_{-\beta}+\delta)}\int_{a_{-\alpha}}^{a_\alpha}\int_{-\delta}^{b_{-\beta}}\left|(fT_{\mathbf{M_k}}-h)(x,y)\right|dxdy\right)\leqslant\frac{1}{|u|},$$

which implies that

$$
\begin{aligned}
\int_{a_{-\alpha}}^{a_\alpha}\int_{-\delta}^{b_{-\beta}}\left|(fT_{\mathbf{M_k}}-h)(x,y)\right|dxdy &\leqslant \frac{2a_\alpha(b_{-\beta}+\delta)}{c}\phi^{-1}\left(\frac{1}{|u|}\right)\\
&\leqslant \frac{2\delta}{c}\cdot\frac{2\pi}{|v|}\cdot\phi^{-1}\left(\frac{1}{|u|}\right).
\end{aligned}
\tag{17}
$$

Since ϕ is convex and $\phi(0)=0$, it follows that

$$\alpha\phi^{-1}(y) \leqslant \phi^{-1}(\alpha y) \text{ for } 0 \leqslant \alpha \leqslant 1 \text{ and } y \geqslant 0. \tag{18}$$

As $0 < 1/|v| < 1$, from (17) and (18) we get

$$\int_{a_{-\alpha}}^{a_\alpha}\int_{-\delta}^{b_{-\beta}} \left|(fT_{\mathbf{M_k}}-h)(x,y)\right| dxdy \leqslant \frac{4\pi\delta}{c}\cdot\phi^{-1}\left(\frac{1}{|uv|}\right).$$

Using this inequality in (15) we get

$$I_{125} = O\left(\phi^{-1}\left(\frac{1}{|uv|}\right)\right).$$

Similar arguments shows that

$$I_{126}, I_{127}, I_{128} = O\left(\phi^{-1}\left(\frac{1}{|uv|}\right)\right).$$

Since ϕ is convex, $\phi(0)=0$, and $\phi(1)=1$, it follows that $\phi(x) \leqslant x$ and hence $x \leqslant \phi^{-1}(x)$ for $0 \leqslant x \leqslant 1$. In particular, we have

$$\frac{1}{|uv|} \leqslant \phi^{-1}\left(\frac{1}{|uv|}\right). \tag{19}$$

Using estimates of $I_{121},\ldots,I_{128}$ in (14), in view of (19), we obtain

$$I_{12} = O\left(\phi^{-1}\left(\frac{1}{|uv|}\right)\right). \tag{20}$$

The proof of the theorem is now completed in view of (4), (6), (8), (13), (19), and (20).

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