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# ON MULTIPLE FOURIER COEFFICIENTS OF FUNCTIONS OF $\phi$–$\Lambda$–BOUNDED VARIATION

R. G. VYAS AND K. N. DARJI

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*Abstract.* Here, we have estimated the order of magnitude of multiple Fourier coefficients of functions of $\phi(\Lambda^1,\ldots,\Lambda^N)BV([0,2\pi]^N)$.

## 1. Introduction

In 1982, M. Schramm and D. Waterman [5] estimated the order of magnitude of Fourier coefficients of functions of $\phi\Lambda BV(\overline{\mathbb{T}})$, where $\mathbb{T}=[0,2\pi)$. Recently, V. Fülöp and F. Móricz [2] studied the order of magnitude of multiple Fourier coefficients of functions of BV($\overline{\mathbb{T}}^N$) in the sense of Vitali and Hardy. Here, we have generalized these results by estimating the order of magnitude of multiple Fourier coefficients of complex valued functions of $\phi(\Lambda^1,\ldots,\Lambda^N)BV(\overline{\mathbb{T}}^N)$.

In the sequel, $\phi$ is an increasing convex function defined on the non-negative real numbers such that $\phi(0)=0$, $\frac{\phi(x)}{x}\to 0$ as $x\to 0$, $\frac{\phi(x)}{x}\to\infty$ as $x\to\infty$ and $\mathbb{L}$ is the class of non-decreasing sequence $\Lambda=\{\lambda_n\}_{n=1}^{\infty}$ of positive numbers such that $\sum_n \frac{1}{\lambda_n}$ diverges.

The function $\phi$ is said to have property $\Delta_2$ if there is a constant $d\geqslant 2$ such that $\phi(2x)\leqslant d\phi(x)$ for all $x\geqslant 0$.

Consider a function $f$ on $\mathbb{R}^k$. For $k=1$ and $I=[a,b]$, define $\triangle f_a^b=f(I)=f(b)-f(a)$. For $k=2$, $I=[a,b]$ and $J=[c,d]$, define

$$\triangle f_{(a,c)}^{(b,d)}=f(I\times J)=f(I,d)-f(I,c)=f(b,d)-f(a,d)-f(b,c)+f(a,c).$$

DEFINITION 1. Let $\bigwedge=(\Lambda^1,\Lambda^2)$, where $\Lambda^k=\{\lambda_n^k\}_{n=1}^{\infty}\in\mathbb{L}$ for $k=1,\ 2$. A complex valued measurable function $f$ defined on a rectangle $R^2:=[a,b]\times[c,d]$ is said to be of $\phi$-$(\Lambda^1,\Lambda^2)$-bounded variation (that is, $f\in\phi(\Lambda^1,\Lambda^2)BV(R^2)$) if

$$V_{\bigwedge_\phi}(f,R^2)=\sup_{I^1,I^2}\Big(\sum_i\sum_j\frac{\phi(|f(I_i^1\times I_j^2)|)}{\lambda_i^1\lambda_j^2}\Big)<\infty,$$

where $I^1$ and $I^2$ are finite collections of nonoverlapping subintervals $\{I_i^1\}$ and $\{I_j^2\}$ in $[a,b]$ and $[c,d]$ respectively.

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Observe that a function $f \in \phi(\Lambda^1,\Lambda^2)BV(R^2)$ need not be bounded.

Consider [3, Example 1.19, p. 23] $f : [0,2\pi]^2 \to \mathbb{R}$ defined by

$$f(x_1,x_2) = \begin{cases} \frac{1}{x_1}+\frac{1}{x_2} & \text{if } x_1 \neq 0 \text{ and } x_2 \neq 0, \\ \frac{1}{x_1} & \text{if } x_1 \neq 0 \text{ and } x_2 = 0, \\ \frac{1}{x_2} & \text{if } x_1 = 0 \text{ and } x_2 \neq 0, \\ 0 & \text{if } x_1 = 0 \text{ and } x_2 = 0. \end{cases}$$

Then $f \in \phi(\Lambda^1,\Lambda^2)BV([0,2\pi]^2)$ but $f$ is not bounded on $[0,2\pi]^2$.

If $f \in \phi(\Lambda^1,\Lambda^2)BV(R^2)$ is such that the marginal functions $f(a,.) \in \phi\Lambda^2 BV([c,d])$ and $f(.,c) \in \phi\Lambda^1 BV([a,b])$ (refer [6, p. 2] for the definition of $\phi\Lambda^1 BV([a,b])$) then $f$ is said to be of $\phi$-$(\Lambda^1,\Lambda^2)^*$-bounded variation (that is, $f \in \phi(\Lambda^1,\Lambda^2)^* BV(R^2)$).

Observe that for $\phi(x) = x$ (and for $\phi(x) = x^p, p = 1$) the conditions $\frac{\phi(x)}{x} \to 0$ as $x \to 0$ and $\frac{\phi(x)}{x} \to \infty$ as $x \to \infty$ are not valid.

Note that, for $\phi(x) = x$ and $\Lambda^1 = \Lambda^2 = \{1\}$ classes $\phi(\Lambda^1,\Lambda^2)BV(R^2)$ and $\phi(\Lambda^1,\Lambda^2)^*BV(R^2)$ reduce to classes $BV_V(R^2)$, of functions of bounded variation in the sense of Vitali (refer [4, p. 279] for the definition of $BV_V(R^2)$), and $BV_H(R^2)$, of functions of bounded variation in the sense of Hardy (refer [4, p. 280] for the definition of $BV_H(R^2)$), respectively; for $\phi(x) = x$ classes $\phi(\Lambda^1,\Lambda^2)BV(R^2)$ and $\phi(\Lambda^1,\Lambda^2)^*BV(R^2)$ reduce to classes $(\Lambda^1,\Lambda^2)BV(R^2)$ (see, [1, Definition 2]) and $(\Lambda^1,\Lambda^2)^*BV(R^2)$ respectively and for $\phi(x) = x^p$ $(p \geqslant 1)$ classes $\phi(\Lambda^1,\Lambda^2)BV(R^2)$ and $\phi(\Lambda^1,\Lambda^2)^*BV(R^2)$ reduce to classes $(\Lambda^1,\Lambda^2)BV^{(p)}(R^2)$ (see, [7, Definition 1.2]) and $(\Lambda^1,\Lambda^2)^*BV^{(p)}(R^2)$ respectively.

## 2. New results for functions of two variables

For any $\mathbf{x} = (x_1,x_2) \in \overline{\mathbb{T}}^2$ and $\mathbf{k} = (k_1,k_2) \in \mathbb{Z}^2$, denote their scalar product by $\mathbf{k}\cdot\mathbf{x} = k_1x_1 + k_2x_2$.

For any $f \in L^1(\overline{\mathbb{T}}^2)$, where $f$ is $2\pi$-periodic in each variable, its Fourier series is defined as

$$f(\mathbf{x}) \sim \sum_{\mathbf{k}\in\mathbb{Z}^2} \hat{f}(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x})},$$

where

$$\hat{f}(\mathbf{k}) = \frac{1}{(2\pi)^2}\int_{\overline{\mathbb{T}}^2} f(\mathbf{x})e^{-i(\mathbf{k}\cdot\mathbf{x})}d\mathbf{x}$$

denotes the $\mathbf{k}^{th}$ Fourier coefficient of $f$.

We prove the following results.

THEOREM 1. *If $\phi$ satisfies $\Delta_2$ condition, $f \in \phi(\Lambda^1,\Lambda^2)BV(\overline{\mathbb{T}}^2) \cap L^1(\overline{\mathbb{T}}^2)$ and $\boldsymbol{k} = (k_1,k_2) \in \mathbb{Z}^2$ is such that $k_1 \cdot k_2 \neq 0$ then*

$$\hat{f}(\boldsymbol{k}) = O\left(\phi^{-1}\left(\frac{1}{\sum_{r_1=1}^{|k_1|}\sum_{r_2=1}^{|k_2|}\frac{1}{\lambda^1_{r_1}\lambda^2_{r_2}}}\right)\right). \tag{1}$$

Theorem 1 generalize the result [5, Theorem 1(ii), p. 408] for functions of two variables.

*Proof.* Since

$$\hat{f}(k_1,k_2)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(x_1,x_2)\, e^{-ik_1x_1}\, e^{-ik_2x_2}\, dx_1\, dx_2,$$

we have

$$4|\hat{f}(k_1,k_2)|=\frac{1}{4\pi^2}\left|\int_0^{2\pi}\int_0^{2\pi}\Big(f(x_1+\frac{\pi}{k_1},x_2+\frac{\pi}{k_2}\Big)-f\Big(x_1,x_2+\frac{\pi}{k_2}\Big)\right.$$
$$\left.-f\Big(x_1+\frac{\pi}{k_1},x_2\Big)+f(x_1,x_2))\, e^{-ik_1x_1}\, e^{-ik_2x_2}\, dx_1\, dx_2\right|.$$

Because of the periodicity of $f$ in each variable, we get

$$\int_0^{2\pi}\int_0^{2\pi}|\Delta f_{r_1r_2}(x_1,x_2)|\, dx_1\, dx_2$$
$$=\int_0^{2\pi}\int_0^{2\pi}\left|f\Big(x_1+\frac{\pi}{k_1},x_2+\frac{\pi}{k_2}\Big)-f\Big(x_1,x_2+\frac{\pi}{k_2}\Big)-f\Big(x_1+\frac{\pi}{k_1},x_2\Big)+f(x_1,x_2)\right| dx_1\, dx_2,$$

where

$$\Delta f_{r_1r_2}(x_1,x_2)=f\Big(x_1+\frac{r_1\pi}{k_1},x_2+\frac{r_2\pi}{k_2}\Big)-f\Big(x_1+\frac{(r_1-1)\pi}{k_1},x_2+\frac{r_2\pi}{k_2}\Big)$$
$$-f\Big(x_1+\frac{r_1\pi}{k_1},x_2+\frac{(r_2-1)\pi}{k_2}\Big)+f\Big(x_1+\frac{(r_1-1)\pi}{k_1},x_2+\frac{(r_2-1)\pi}{k_2}\Big),$$

for any $r_1,r_2\in\mathbb{Z}$.

Therefore

$$|\hat{f}(k_1,k_2)|\leqslant\frac{1}{16\pi^2}\int_0^{2\pi}\int_0^{2\pi}|\Delta f_{r_1r_2}(x_1,x_2)|\, dx_1\, dx_2$$
$$\leqslant\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}|\Delta f_{r_1r_2}(x_1,x_2)|\, dx_1\, dx_2.$$

For $c>0$, by Jensen's inequality for integrals, we have

$$\phi(c|\hat{f}(k_1,k_2)|)\leqslant\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\phi(c|\Delta f_{r_1r_2}(x_1,x_2)|)\, dx_1\, dx_2.$$

Dividing both sides of above inequality by $\lambda^1_{r_1}\,\lambda^2_{r_2}$ and then summing over $r_1=1$ to $|k_1|$ and $r_2=1$ to $|k_2|$, we get

$$\phi(c|\hat{f}(k_1,k_2)|)\left(\sum_{r_1=1}^{|k_1|}\sum_{r_2=1}^{|k_2|}\frac{1}{\lambda^1_{r_1}\,\lambda^2_{r_2}}\right)$$
$$\leqslant\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left(\sum_{r_1=1}^{|k_1|}\sum_{r_2=1}^{|k_2|}\frac{\phi(c|\Delta f_{r_1r_2}(x_1,x_2)|)}{\lambda^1_{r_1}\,\lambda^2_{r_2}}\right)dx_1\, dx_2$$
$$\leqslant V_{\Lambda_\phi}(cf,\overline{\mathbb{T}}^2), \qquad (2)$$

where $cf \in \phi(\Lambda^1,\Lambda^2)BV(\overline{\mathbb{T}}^2)$, as $\phi$ satisfies $\Delta_2$ condition.

Since $\phi$ is convex and $\phi(0)=0$, for $c \in (0,1]$ we have $\phi(cx) \leqslant c\phi(x)$ and hence we can choose sufficiently small $c \in (0,1]$ such that $V_{\Lambda_\phi}(cf,\overline{\mathbb{T}}^2) \leqslant 1$. Thus, in view of (2), we get

$$|\hat{f}(k_1,k_2)| \leqslant \frac{1}{c}\phi^{-1}\left(\frac{1}{\sum_{r_1=1}^{|k_1|}\sum_{r_2=1}^{|k_2|}\frac{1}{\lambda^1_{r_1}\lambda^2_{r_2}}}\right).$$

This completes the proof of Theorem 1. □

COROLLARY 1. *If $\phi$ satisfies $\Delta_2$ condition, $f \in \phi(\Lambda^1,\Lambda^2)^*BV(\overline{\mathbb{T}}^2)$ and $\boldsymbol{k} = (k_1,k_2) \in \mathbb{Z}^2$ is such that $k_1 \cdot k_2 \neq 0$ then (1) holds true.*

*Proof.* For any $f \in \phi(\Lambda^1,\Lambda^2)^*BV(\overline{\mathbb{T}}^2)$,

$$\begin{aligned}|f(x_1,x_2)| &\leqslant |f(x_1,x_2)-f(0,x_2)-f(x_1,0)+f(0,0)|+|f(0,x_2)-f(0,0)| \\ &\quad +|f(x_1,0)-f(0,0)|+|f(0,0)| \\ &\leqslant (\lambda^1_1\lambda^2_1)\phi^{-1}(V_{\Lambda_\phi}(f,\overline{\mathbb{T}}^2))+(\lambda^2_1)\phi^{-1}(V_{\Lambda^2_\phi}(f(0,.),\overline{\mathbb{T}}) \\ &\quad +(\lambda^1_1)\phi^{-1}(V_{\Lambda^1_\phi}(f(.,0),\overline{\mathbb{T}}))+|f(0,0)|\end{aligned}$$

implies $f$ is bounded on $\overline{\mathbb{T}}^2$. □

Since $\phi(\Lambda^1,\Lambda^2)^*BV(\overline{\mathbb{T}}^2) \subset \phi(\Lambda^1,\Lambda^2)BV(\overline{\mathbb{T}}^2)$, the Corollary 1 follows from Theorem 1.

COROLLARY 2. *If $\phi$ satisfies $\Delta_2$ condition, $f \in \phi(\Lambda^1,\Lambda^2)^*BV(\overline{\mathbb{T}}^2)$ and $\boldsymbol{k} = (k_1,0) \in \mathbb{Z}^2$ is such that $k_1 \neq 0$ then*

$$\hat{f}(\boldsymbol{k}) = O\left(\phi^{-1}\left(\frac{1}{\sum_{r_1=1}^{|k_1|}\frac{1}{\lambda^1_{r_1}}}\right)\right).$$

*Proof.* Since

$$\hat{f}(k_1,0) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(x_1,x_2)\, e^{-ik_1x_1}\, dx_1\, dx_2,$$

we have

$$2|\hat{f}(k_1,0)| = \frac{1}{4\pi^2}\left|\int_0^{2\pi}\int_0^{2\pi}\left(f\left(x_1+\frac{\pi}{k_1},x_2\right)-f(x_1,x_2)\right) e^{-ik_1x_1}\, dx_1\, dx_2\right|.$$

Because of the periodicity of $f$ in each variable, we get

$$\int_0^{2\pi}\int_0^{2\pi}|\Delta f_{r_1}(x_1,x_2)|\, dx_1\, dx_2 = \int_0^{2\pi}\int_0^{2\pi}\left|f\left(x_1+\frac{\pi}{k_1},x_2\right)-f(x_1,x_2)\right| dx_1\, dx_2,$$

where

$$\Delta f_{r_1}(x_1,x_2) = f\Big(x_1+\frac{r_1\pi}{k_1},x_2\Big) - f\Big(x_1+\frac{(r_1-1)\pi}{k_1},x_2\Big), \quad \text{for any } r_1 \in \mathbb{Z}.$$

Therefore

$$\begin{aligned}|\hat{f}(k_1,0)| &\leqslant \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi} |\Delta f_{r_1}(x_1,x_2)|\, dx_1\, dx_2 \\ &\leqslant \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} |\Delta f_{r_1}(x_1,x_2)|\, dx_1\, dx_2.\end{aligned}$$

For $c>0$, by Jensen's inequality for integrals, we have

$$\phi(c|\hat{f}(k_1,0)|) \leqslant \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \phi(c|\Delta f_{r_1}(x_1,x_2)|)\, dx_1\, dx_2.$$

Dividing both sides of above inequality by $\lambda^1_{r_1}$ and then summing over $r_1 = 1$ to $|k_1|$, we get

$$\begin{aligned}\phi(c|\hat{f}(k_1,0)|)\left(\sum_{r_1=1}^{|k_1|}\frac{1}{\lambda^1_{r_1}}\right) &\leqslant \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left(\sum_{r_1=1}^{|k_1|}\frac{\phi(c|\Delta f_{r_1}(x_1,x_2)|)}{\lambda^1_{r_1}}\right) dx_1\, dx_2 \\ &\leqslant V_{\Lambda^1_\phi}(cf(.,x_2),\overline{\mathbb{T}}).\end{aligned} \tag{3}$$

As $\phi$ is satisfying $\Delta_2$ condition and is increasing implies

$$\phi(x+y) \leqslant \phi(2\max\{x,y\}) \leqslant d\phi(\max\{x,y\}) \leqslant d(\phi(x)+\phi(y)), \quad \text{for any } x,y \geqslant 0.$$

Therefore, for any $0 < x_2 \leqslant 2\pi$,

$$V_{\Lambda^1_\phi}(f(.,x_2),[0,2\pi]) \leqslant d[\lambda^2_1 V_{\Lambda_\phi}(f,[0,2\pi]^2) + V_{\Lambda^1_\phi}(f(.,0),[0,2\pi])].$$

Thus, in view of (3), we get

$$\phi(c|\hat{f}(k_1,0)|)\left(\sum_{r_1=1}^{|k_1|}\frac{1}{\lambda^1_{r_1}}\right) \leqslant d[\lambda^2_1 V_{\Lambda_\phi}(cf,\overline{\mathbb{T}}^2) + V_{\Lambda^1_\phi}(cf(.,0),\overline{\mathbb{T}})]. \tag{4}$$

Since $\phi$ is convex and $\phi(0)=0$, so we can choose sufficiently small $c \in (0,1]$ such that $V_{\Lambda_\phi}(cf,\overline{\mathbb{T}}^2) \leqslant \frac{1}{2d\lambda_1^2}$ and $V_{\Lambda^1_\phi}(cf(.,0),\overline{\mathbb{T}}) \leqslant \frac{1}{2d}$. Hence, from (4), we have

$$|\hat{f}(k_1,0)| \leqslant \frac{1}{c}\phi^{-1}\left(\frac{1}{\sum_{r_1=1}^{|k_1|}\frac{1}{\lambda^1_{r_1}}}\right). \quad \square$$

## 3. Extension of the results for functions of several variables

Let $I^k = [a_k, b_k] \subset \mathbb{R}$, for $k = 1$ to $N$. In the above section-1, we have already define $f(I^1)$ for a function $f$ of one variable and $f(I^1 \times I^2)$ for a function $f$ of two variables. Similarly for a $N-$variables function $f$ on $\mathbb{R}^N$, by induction, defining the expression $f(I^1 \times \ldots \times I^{N-1})$ for a function of $N-1$ variables, one gets

$$f(I^1 \times \ldots \times I^N) = f(I^1 \times \ldots \times I^{N-1}, b_N) - f(I^1 \times \ldots \times I^{N-1}, a_N).$$

Observe that, $f(I^1 \times \ldots \times I^N)$ can also be express as

$$f(I^1 \times \ldots \times I^N) = \triangle f_{\mathbf{a}}^{\mathbf{b}} = f(\mathbf{b}) - f(\mathbf{a}) = \sum_{\mathbf{c}} k(\mathbf{c}) f(\mathbf{c}),$$

where $\mathbf{a} = (a_1, a_2, \cdots, a_N)$, $\mathbf{b} = (b_1, b_2, \cdots, b_N) \in \mathbb{R}^N$, the summation is over all $\mathbf{c} = (c_1, c_2 \cdots, c_N) \in \mathbb{R}^N$ such that $c_i \in \{a_i, b_i\}$, for $i = 1, \cdots, N$, and for any such $\mathbf{c}$, $k(\mathbf{c}) = k_1 \ldots k_N$, in which, for $1 \leqslant i \leqslant N$,

$$k_i = \begin{cases} 1 & \text{if } c_i = b_i, \\ -1 & \text{if } c_i = a_i. \end{cases}$$

If $N = 1$ then

$$\triangle f_{\mathbf{a}}^{\mathbf{b}} = \triangle f_{a_1}^{b_1} = \sum_{c_1} k(\mathbf{c}) f(\mathbf{c}) = f(b_1) - f(a_1),$$

while, if $N = 2$ then

$$\triangle f_{\mathbf{a}}^{\mathbf{b}} = \triangle f_{(a_1,a_2)}^{(b_1,b_2)} = \sum_{(c_1,c_2)} k(\mathbf{c}) f(\mathbf{c}) = f(b_1, b_2) + f(a_1, a_2) - f(b_1, a_2) - f(a_1, b_2).$$

Similarly, if $N = 3$ then we get

$$\begin{aligned} \triangle f_{\mathbf{a}}^{\mathbf{b}} = \triangle f_{(a_1,a_2,a_3)}^{(b_1,b_2,b_3)} &= \sum_{(c_1,c_2,c_3)} k(\mathbf{c}) f(\mathbf{c}) \\ &= f(b_1,b_2,b_3) + f(b_1,a_2,a_3) + f(a_1,b_2,a_3) + f(a_1,a_2,b_3) \\ &\quad - f(b_1,b_2,a_3) - f(a_1,b_2,b_3) - f(b_1,a_2,b_3) - f(a_1,a_2,a_3). \end{aligned}$$

Let $\bigwedge = (\Lambda^1, \cdots, \Lambda^N)$, where $\Lambda^k = \{\lambda_n^k\}_{n=1}^{\infty} \in \mathbb{L}$ for $k = 1, \ldots, N$. A complex valued measurable function $f$ defined on $R^N := \prod_{k=1}^{N} [a_k, b_k]$ is said to be of $\phi$-$(\Lambda^1, \cdots, \Lambda^N)$-bounded variation (that is, $f \in \phi(\Lambda^1, \cdots, \Lambda^N) BV(R^N)$) if

$$V_{\bigwedge_\phi}(f, R^N) = \sup_{I^1, \ldots, I^N} \Big( \sum_{k_1} \cdots \sum_{k_N} \frac{\phi(|f(I_{k_1}^1 \times \ldots \times I_{k_N}^N)|)}{\lambda_{k_1}^1 \cdots \lambda_{k_N}^N} \Big) < \infty,$$

where $I^1, \ldots, I^N$ are finite collections of non-overlapping subintervals $\{I_{k_1}^1\}, \ldots, \{I_{k_N}^N\}$ in $[a_1, b_1], \ldots, [a_N, b_N]$ respectively.

Moreover, a function $f \in \phi(\Lambda^1,\cdots,\Lambda^N)BV(R^N)$ is said to be of $\phi$-$(\Lambda^1,\cdots,\Lambda^N)^*$ bounded variation (that is, $f \in \phi(\Lambda^1,\cdots,\Lambda^N)^*BV(R^N)$) if for each of its marginal functions

$$f(x_1,\cdots,x_{i-1},a_i,x_{i+1},\cdots,x_N) \in \phi(\Lambda^1,\cdots,\Lambda^{i-1},\Lambda^{i+1},\cdots,\Lambda^N)^*BV(R^N(a_i)),$$

$\forall\, i = 1,2,\cdots,N,$ where

$$R^N(a_i) = \{(x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_N) \in \mathbb{R}^{N-1} : x_k \in [a_k,b_k] \text{ for } k = 1,\cdots,i-1,i+1,\cdots,N\}.$$

Note that, for $\phi(x) = x$ and $\Lambda^1 = \cdots = \Lambda^N = \{1\}$ classes $\phi(\Lambda^1,\cdots,\Lambda^N)BV(R^N)$ and $\phi(\Lambda^1,\cdots,\Lambda^N)^*BV(R^N)$ reduce to classes $BV_V(R^N)$ and $BV_H(R^N)$ respectively; for $\phi(x) = x$ classes $\phi(\Lambda^1,\cdots,\Lambda^N)BV(R^N)$ and $\phi(\Lambda^1,\cdots,\Lambda^N)^*BV(R^N)$ reduce to classes $(\Lambda^1,\cdots,\Lambda^N)BV(R^N)$ and $(\Lambda^1,\cdots,\Lambda^N)^*BV(R^N)$ respectively and for $\phi(x) = x^p$ $(p \geqslant 1)$ classes $\phi(\Lambda^1,\cdots,\Lambda^N)BV(R^N)$ and $\phi(\Lambda^1,\cdots,\Lambda^N)^*BV(R^N)$ reduce to classes $(\Lambda^1,\cdots,\Lambda^N)BV^{(p)}(R^N)$ and $(\Lambda^1,\cdots,\Lambda^N)^*BV^{(p)}(R^N)$ respectively.

It is easy to prove that $f \in \phi(\Lambda^1,\cdots,\Lambda^N)^*BV(R^N)$ implies it is bounded.

For any $\mathbf{x} = (x_1,\cdots,x_N) \in \overline{\mathbb{T}}^N$ and $\mathbf{k} = (k_1,\cdots,k_N) \in \mathbb{Z}^N$, denote their scalar product by

$$\mathbf{k}\cdot\mathbf{x} = k_1x_1 + \cdots + k_Nx_N.$$

For $f \in L^1(\overline{\mathbb{T}}^N)$, where $f$ is complex valued function which is $2\pi$-periodic in each variable, its Fourier series is defined as

$$f(\mathbf{x}) \sim \sum_{\mathbf{k}\in\mathbb{Z}^N} \hat{f}(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x})},$$

where

$$\hat{f}(\mathbf{k}) = \frac{1}{(2\pi)^N}\int_{\overline{\mathbb{T}}^N} f(\mathbf{x})e^{-i(\mathbf{k}\cdot\mathbf{x})}d\mathbf{x}$$

denotes the $\mathbf{k}^{th}$ Fourier coefficient of $f$.

Now, we extend the above mentioned results for higher dimensional spaces as follow.

THEOREM 2. *If $\phi$ satisfies $\Delta_2$ condition, $f \in \phi(\Lambda^1,\cdots,\Lambda^N)BV(\overline{\mathbb{T}}^N) \cap L^1(\overline{\mathbb{T}}^N)$ and $\boldsymbol{k} = (k_1,\cdots,k_N) \in \mathbb{Z}^N$ is such that $k_1\cdots k_N \neq 0$ then*

$$\hat{f}(\boldsymbol{k}) = O\left(\phi^{-1}\left(\frac{1}{\sum_{r_1=1}^{|k_1|}\cdots\sum_{r_N=1}^{|k_N|}\frac{1}{\lambda^1_{r_1}\cdots\lambda^N_{r_N}}}\right)\right). \tag{5}$$

Theorem 2 generalize the result [2, Theorem, p. 99].

COROLLARY 3. *If $\phi$ satisfies $\Delta_2$ condition, $f \in \phi(\Lambda^1,\cdots,\Lambda^N)^*BV(\overline{\mathbb{T}}^N)$ and $\boldsymbol{k} = (k_1,\cdots,k_N) \in \mathbb{Z}^N$ is such that $k_1\cdots k_N \neq 0$ then (5) holds true.*

COROLLARY 4. *If* $\phi$ *satisfies* $\Delta_2$ *condition,* $f \in \phi(\Lambda^1,\ldots,\Lambda^N)^* BV(\overline{\mathbb{T}}^N)$ *and* $\boldsymbol{k} = (k_1,\ldots,k_N) \in \mathbb{Z}^N$ *is such that* $k_j \neq 0$ *for* $(1 \leqslant) j_1 < \cdots < j_M (\leqslant N)$ *and* $k_j = 0$ *for* $(1 \leqslant) l_1 < \cdots < l_{N-M} (\leqslant N)$*, where* $\{l_1,\cdots,l_{N-M}\}$ *is the complementary set of* $\{j_1,\cdots, j_M\}$ *with respect to* $\{1,\cdots,N\}$*, then*

$$\hat{f}(\boldsymbol{k}) = O\left(\phi^{-1}\left(\frac{1}{\sum_{r_1=1}^{|k_{j_1}|}\cdots\sum_{r_M=1}^{|k_{j_M}|}\frac{1}{\lambda_{r_1}^{j_1}\ldots\lambda_{r_M}^{j_M}}}\right)\right).$$

Corollary 4 generalize the result [2, Corollary, p. 103].

Extended results of this section can be prove in the same way as we proved the results in section 2.

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*R. G. Vyas*
*Department of Mathematics, Faculty of Science*
*The Maharaja Sayajirao University of Baroda*
*Vadodara, Gujarat, India*
*e-mail:* drrgvyas@yahoo.com

*K. N. Darji*
*Department of Science and Humanities*
*Tatva Institute of Technological Studies*
*Modasa, Sabarkantha, Gujarat, India*
*e-mail:* darjikiranmsu@gmail.com