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SOME DYNAMIC INEQUALITIES OF HARDY TYPE ON TIME SCALES

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Abstract. In this paper we prove some new dynamic inequalities of Hardy type on time scales. The main results will be proved using algebraic inequalities, Hölder inequality and Keller's chain rule on time scales.

1. Introduction

The classical Hardy inequality states that if $f \geqslant 0$ and integrable over any finite interval $(0,x)$ and f^p is integrable and convergent over $(0,\infty)$ and $p>1$, then

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx. \tag{1}$$

The constant $(p/(p-1))^p$ is the best possible. The discrete version [5] of (1) is

$$\sum_{n=1}^\infty \left(\frac{1}{n}\sum_{k=1}^n a_k\right)^p \leqslant \left(\frac{p}{p-1}\right)^p \sum_{n=1}^\infty a_n, \ (a_n>0, \quad p>1).$$

We refer the reader to the books [11, 12, 16] and the papers [1, 4, 9, 10, 13, 14, 15, 18, 19] for various generalizations and extensions of these results. Hardy's inequality (1) was generalized by Hardy himself in [6] and he showed that for any integrable function $f(x)>0$ on $(0,\infty)$, $p>1$, then

$$\int_0^\infty \frac{1}{x^m}\left(\int_0^x f(t)dt\right)^p dx \leqslant \left(\frac{p}{m-1}\right)^p \int_0^\infty \frac{1}{x^{m-p}} f^p(x)dx, \ \ m>1 \tag{2}$$

and

$$\int_0^\infty \frac{1}{x^m}\left(\int_x^\infty f(t)dt\right)^p dx \leqslant \left(\frac{p}{1-m}\right)^p \int_0^\infty \frac{1}{x^{m-p}} f^p(x)dx, \ m<1. \tag{3}$$

Recently, a number of dynamic inequalities of Hardy type on time scales was established in [17, 20, 21, 22]. Hardy type inequalities on time scales not only give a unification of continuous and discrete inequalities of Hardy type but also can be extended

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to different types of time scales. The general idea is to prove a result for a dynamic inequality where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which may be an arbitrary closed subset of the real numbers $\mathbb{R}$. In [20] the author proved the time scale version of (1) and proved that if $p > 1$ and g is a nonnegative and such that the delta integral $\int_a^\infty (g(t))^p \Delta t$ exits as a finite number, then

$$\int_a^\infty \left(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} g(t)\Delta t\right)^p \Delta x \leqslant \left(\frac{p}{p-1}\right)^p \int_a^\infty g^p(x)dx; \tag{4}$$

here $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. If in addition $\mu(t)/t \to 0$ as $t \to \infty$, then the constant is the best possible (here $\mu(t) := \sigma(t) - t$).

In [17] the authors established a new inequality with weighted functions and they proved that if $u \in C_{rd}([a,b],\mathbb{R})$ (the set of rd–continuous functions) is a nonnegative function such that the delta integral $\int_t^b \frac{u(s)}{(s-a)(\sigma(s)-a)}\Delta s$ exists as a finite number and the function v is defined by

$$v(t) = (t-a)\int_t^b \frac{u(s)}{(s-a)(\sigma(s)-a)}\Delta s, \ t \in [a,b],$$

and $\Phi : (c,d) \to \mathbb{R}$, is continuous and convex, where $c,d \in \mathbb{R}$, then the inequality

$$\int_a^b u(t)\Phi\left(\frac{1}{(\sigma(t)-a)}\int_a^{\sigma(t)} g(s)\Delta s\right)\frac{\Delta t}{t-a} \leqslant \int_a^b v(t)\Phi(g(t))\frac{\Delta t}{t-a}, \tag{5}$$

holds for all delta integrable functions $g \in C_{rd}([a,b],\mathbb{R})$ such that $g(t) \in (c,d)$. The inequality (5) can be considered as the time scale version of the (Hardy-Knopp type) inequality

$$\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x f(t)dt\right)\frac{dx}{x} \leqslant \int_0^\infty \Phi(f(x))\frac{dx}{x}, \tag{6}$$

that was proved by Kaijser et al [10], where Φ is a convex function on $(0,\infty)$.

Our aim in this paper is to prove some new inequalities of Hardy type on time scales using the chain rule, Hölder's inequality and some algebraic inequalities. These inequalities contain the inequalities (2) and (3) and some new discrete inequalities.

2. Main results

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by:

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

where $\sup\emptyset = \inf\mathbb{T}$. A point $t \in \mathbb{T}$, is said to be left–dense if $\rho(t) = t$ and $t > \inf\mathbb{T}$, is right–dense if $\sigma(t) = t$, is left–scattered if $\rho(t) < t$ and right–scattered if $\sigma(t) > t$. A function $g:\mathbb{T}\to\mathbb{R}$ is said to be right–dense continuous (rd–continuous) provided g is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd–continuous functions is denoted by $C_{rd}(\mathbb{T})$. We assume that $\sup\mathbb{T}=\infty$, and define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$. For more details of time scale analysis, we refer the reader to the two books by Bohner and Peterson [2], [3].

The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t):=\sigma(t)-t$, and for any function $f:\mathbb{T}\to\mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. We will assume that $\sup\mathbb{T}=\infty$, and define the time scale interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}}:=[a,b]\cap\mathbb{T}$. Fix $t\in\mathbb{T}$ and let $x:\mathbb{T}\to\mathbb{R}$. Define $x^{\Delta}(t)$ to be the number (if it exists) with the property that given any $\varepsilon>0$ there is a neighborhood U of t with

$$|[x(\sigma(t))-x(s)]-x^{\Delta}(t)[\sigma(t)-s]|\leqslant\varepsilon|\sigma(t)-s|,\quad \text{for all } s\in U.$$

In this case, we say $x^{\Delta}(t)$ is the (delta) derivative of x at t and that x is (delta) differentiable at t.

We will frequently use the results in the following theorem which is due to Hilger [8]. Assume that $g:\mathbb{T}\to\mathbb{R}$ and let $t\in\mathbb{T}$.
(i) If g is Δ-differentiable at t, then g is continuous at t.
(ii) If g is continuous at t and t is right-scattered, then g is Δ-differentiable at t with $g^{\Delta}(t)=\frac{g(\sigma(t))-g(t)}{\mu(t)}$.
(iii) If g is Δ-differentiable and t is right-dense, then

$$g^{\Delta}(t)=\lim_{s\to t}\frac{g(t)-g(s)}{t-s}.$$

(iv) If g is Δ-differentiable at t, then $g(\sigma(t))=g(t)+\mu(t)g^{\Delta}(t)$.

Note that if $\mathbb{T}=\mathbb{R}$ then

$$\sigma(t)=t,\quad \mu(t)=0,\quad f^{\Delta}(t)=f'(t),\quad \int_a^b f(t)\Delta t=\int_a^b f(t)dt$$

if $\mathbb{T}=\mathbb{Z}$, then

$$\sigma(t)=t+1,\quad \mu(t)=1,\quad f^{\Delta}(t)=\Delta f(t),\quad \int_a^b f(t)\Delta t=\sum_{t=a}^{b-1}f(t),$$

if $\mathbb{T}=h\mathbb{Z}$, $h>0$, then $\sigma(t)=t+h$, $\mu(t)=h$, and

$$y^{\Delta}(t)=\Delta_h y(t):=\frac{y(t+h)-y(t)}{h},\quad \int_a^b f(t)\Delta t=\sum_{k=0}^{\frac{b-a-h}{h}}f(a+kh)h,$$

and if $\mathbb{T}=\{t: t=q^k,\ k\in\mathbb{N}_0,\ q>1\}$, then $\sigma(t)=qt$, $\mu(t)=(q-1)t$,

$$x^{\Delta}(t)=\Delta_q x(t)=\frac{(x(qt)-x(t))}{(q-1)t},\quad \int_{t_0}^{\infty}f(t)\Delta t=\sum_{k=n_0}^{\infty}f(q^k)\mu(q^k),$$

where $t_0 = q^{n_0}$, and if $\mathbb{T} = \mathbb{N}_0^2 := \{n^2 : n \in \mathbb{N}_0\}$, then $\sigma(t) = (\sqrt{t}+1)^2$,

$$\mu(t) = 1+2\sqrt{t}, \quad \Delta_N y(t) = \frac{y((\sqrt{t}+1)^2) - y(t)}{1+2\sqrt{t}}.$$

In this paper we will refer to the (delta) integral which we can define as follows: If $G^{\Delta}(t) = g(t)$, then the Cauchy (delta) integral of g is defined by $\int_a^t g(s)\Delta s := G(t) - G(a)$. It can be shown (see [2]) that if $g \in C_{rd}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^t g(s)\Delta s$ exists, $t_0 \in \mathbb{T}$, and satisfies $G^{\Delta}(t) = g(t)$, $t \in \mathbb{T}$. An infinite integral is defined as

$$\int_a^{\infty} f(t)\Delta t = \lim_{b\to\infty} \int_a^b f(t)\Delta t.$$

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma} \neq 0$, here $g^{\sigma} = g \circ \sigma$) of two differentiable function f and g

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta} = fg^{\Delta} + f^{\Delta}g^{\sigma}, \text{ and } \left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}}. \tag{7}$$

We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided $1+\mu(t)p(t) \neq 0$, $t \in \mathbb{T}$. The chain rule formula that we will use in this paper is

$$(x^{\gamma}(t))^{\Delta} = \gamma \int_0^1 [hx^{\sigma} + (1-h)x]^{\gamma-1}\, dh x^{\Delta}(t), \tag{8}$$

which is a simple consequence of Keller's chain rule [2, Theorem 1.90]. Using the fact that $g(\sigma(t)) = g(t) + \mu(t)g^{\Delta}(t)$, then we obtain

$$(x^{\gamma}(t))^{\Delta} = \gamma \int_0^1 \left[x + h\mu(t)x^{\Delta}(t)\right]^{\gamma-1} dh x^{\Delta}(t). \tag{9}$$

The integration by parts formula is given by

$$\int_a^b u(t)v^{\Delta}(t)\Delta t = [u(t)v(t)]_a^b - \int_a^b u^{\Delta}(t)v^{\sigma}(t)\Delta t. \tag{10}$$

To prove the main results, we will use the following Hölder Inequality [2, Theorem 6.13]. Let a, $b \in \mathbb{T}$. For $u, v \in C_{rd}(\mathbb{T}, \mathbb{R})$, we have

$$\int_a^b |u(t)v(t)|\Delta t \leqslant \left[\int_a^b |u(t)|^q \Delta t\right]^{\frac{1}{q}} \left[\int_a^b |v(t)|^p \Delta t\right]^{\frac{1}{p}}, \tag{11}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Throughout the paper, we will assume that the functions are nonnegative rd-continuous positive functions, $\Delta-$differentiable, locally delta integrable and the left hand sides of the inequalities exit if the right hand side exist. We will assume also that

$$\frac{s}{\sigma(s)} \geqslant \frac{1}{K}, \quad \text{for } s \in [a,\infty)_{\mathbb{T}}, \tag{12}$$

for some constant $K > 0$. Now, we are ready to state and prove the main results in this paper and we begin with the case when $p/q \geqslant 2$.

THEOREM 2.1. *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$ and $p > q > 0$ such that $p/q \geqslant 2$ and $\gamma > 1$ and define*

$$\Lambda(t) := \frac{1}{t}\int_a^t \frac{f(s)g(s)}{s}\Delta s, \quad \textit{for any} \quad t \in [a,\infty)_{\mathbb{T}}. \tag{13}$$

If $f(t)g(t) \geqslant t\Lambda(t)$ for $t \in [a,\infty)_{\mathbb{T}}$ and

$$1 + \frac{p\left(2^{p/q-2}\right)K^{\gamma-1}}{q(\gamma-1)} \geqslant \frac{1}{m} > 0, \tag{14}$$

(here K is as in (12)) for some constant $m > 0$, then

$$\begin{aligned}\int_a^\infty \frac{1}{t^\gamma}&\left[\Lambda^{\frac{p}{q}}(t) - \frac{2^{\frac{p}{q}-2}m}{\gamma-1}\mu^{\frac{p}{q}-1}(t)\left(\Lambda^\Delta(t)\right)^{\frac{p}{q}}\right]\Delta t \\ &\leqslant \frac{2^{\frac{p}{q}-2}pmK^\gamma}{q(\gamma-1)}\left[\int_a^\infty \frac{1}{t^\gamma}\left(\frac{f(t)g(t)}{\sigma(t)}\right)^{\frac{p}{q}}\Delta t\right]^{\frac{q}{p}}\left[\int_a^\infty \frac{\Lambda^{\frac{p}{q}}(t)}{t^\gamma}\Delta t\right]^{\frac{p-q}{p}}. \end{aligned} \tag{15}$$

Proof. Using the integration by parts formula (10) on the term $\int_a^\infty \frac{1}{t^\gamma}(\Lambda^\sigma(t))^{p/q}\Delta t$ with

$$u^\Delta(t) = \frac{1}{t^\gamma}, \quad \text{and} \quad v^\sigma(t) = (\Lambda^\sigma(t))^{p/q},$$

we have that

$$\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q}}{t^\gamma}\Delta t = uv|_a^\infty + \int_a^\infty (-u(t))\left(\Lambda^{p/q}(t)\right)^\Delta \Delta t, \tag{16}$$

where

$$u(t) = \int_t^\infty \left(\frac{-1}{s^\gamma}\right)\Delta s. \tag{17}$$

Using the chain rule (8), we see that

$$\begin{aligned}\left(\frac{-1}{s^{\gamma-1}}\right)^\Delta &= (\gamma-1)\int_0^1 \frac{1}{[h\sigma(s)+(1-h)s]^\gamma}dh \\ &\geqslant (\gamma-1)\int_0^1 \frac{1}{[h\sigma(s)+(1-h)\sigma(s)]^\gamma}dh \\ &= \int_0^1 \left(\frac{\gamma-1}{\sigma^\gamma(s)}\right)dh = \frac{\gamma-1}{\sigma^\gamma(s)}. \end{aligned} \tag{18}$$

From (12) and (18), we have that

$$\left(\frac{-1}{s^{\gamma-1}}\right)^{\Delta} \geqslant \frac{\gamma-1}{K^{\gamma}s^{\gamma}}.$$

Then

$$\int_t^{\infty}\frac{-1}{s^{\gamma}}\Delta s \geqslant \frac{-K^{\gamma}}{\gamma-1}\int_t^{\infty}\left(\frac{-1}{s^{\gamma-1}}\right)^{\Delta}\Delta s = \frac{K^{\gamma}}{\gamma-1}\left(\frac{1}{s^{\gamma-1}}\right)\Big|_t^{\infty} = \frac{-K^{\gamma}}{\gamma-1}\left(\frac{1}{t^{\gamma-1}}\right). \tag{19}$$

Hence

$$-u(t) = -\int_t^{\infty}\left(\frac{-1}{s^{\gamma}}\right)\Delta s \leqslant \frac{K^{\gamma}}{\gamma-1}\left(\frac{1}{t^{\gamma-1}}\right). \tag{20}$$

Again by applying (9), we have

$$\left(\Lambda^{p/q}(t)\right)^{\Delta} = \frac{p}{q}\int_0^1\left[\Lambda+\mu h\Lambda^{\Delta}\right]^{\frac{p}{q}-1}dh\Lambda^{\Delta}(t). \tag{21}$$

From (16), (20) and (21), we have (note that $u(\infty)=0$ and $\Lambda(a)=0$) that

$$\int_a^{\infty}\frac{1}{t^{\gamma}}\left(\Lambda^{\sigma}(t)\right)^{p/q}\Delta t \leqslant \frac{pK^{\gamma}}{q(\gamma-1)}\int_a^{\infty}\frac{1}{t^{\gamma-1}}\int_0^1\left[\Lambda+\mu h\Lambda^{\Delta}\right]^{\frac{p}{q}-1}dh\Lambda^{\Delta}(t)\Delta t. \tag{22}$$

From the definition of $\Lambda(t)$ we see that

$$\Lambda^{\Delta}(t) = \left(\frac{1}{t}\int_a^t\frac{f(s)g(s)}{s}\Delta s\right)^{\Delta} = \frac{f(t)g(t)}{t\sigma(t)} - \frac{\int_a^t\frac{f(s)g(s)}{s}\Delta s}{t\sigma(t)} = \frac{f(t)g(t)-t\Lambda(t)}{t\sigma(t)} \geqslant 0. \tag{23}$$

Applying the inequality

$$a^{\lambda}+b^{\lambda} \leqslant (a+b)^{\lambda} \leqslant 2^{\lambda-1}(a^{\lambda}+b^{\lambda}), \quad \text{if} \quad a,\ b\geqslant 0,\ \lambda\geqslant 1, \tag{24}$$

on the term $\left[\Lambda+h\mu\Lambda^{\Delta}\right]^{(p/q)-1}$, we see that

$$(p/q)\int_0^1\left[\Lambda+h\mu\Lambda^{\Delta}\right]^{\frac{p}{q}-1}dh \leqslant \left(\frac{p}{q}\right)2^{(p/q)-2}\Lambda^{\frac{p}{q}-1}(t)+2^{(p/q)-2}(\mu\Lambda^{\Delta})^{\frac{p}{q}-1},\ p/q\geqslant 2. \tag{25}$$

Substituting (23) and (25) into (22), we get

$$\int_a^{\infty}\frac{\left(\Lambda^{\sigma}(t)\right)^{p/q}}{t^{\gamma}}\Delta t \leqslant \frac{p\left(2^{p/q-2}\right)K^{\gamma}}{q(\gamma-1)}\int_a^{\infty}\frac{\left(\Lambda(t)\right)^{p/q-1}}{t^{\gamma-1}}\left[\frac{f(t)g(t)-t\Lambda(t)}{t\sigma(t)}\right]\Delta t$$
$$+\frac{2^{(p/q)-2}K^{\gamma}}{\gamma-1}\int_a^{\infty}\frac{(\mu(t))^{p/q-1}\left(\Lambda^{\Delta}(t)\right)^{p/q}}{t^{\gamma-1}}\Delta t,$$

so

$$\int_a^\infty \left[\frac{(\Lambda^\sigma(t))^{p/q}}{t^\gamma} + \frac{2^{p/q-2}pK^\gamma}{q(\gamma-1)}\frac{1}{t^\gamma}\frac{t}{\sigma(t)}(\Lambda(t))^{p/q}\right]\Delta t$$
$$\leqslant \frac{p\left(2^{p/q-2}\right)K^\gamma}{q(\gamma-1)}\int_a^\infty \frac{(\Lambda(t))^{p/q-1}}{t^\gamma}\left[\frac{f(t)g(t)}{\sigma(t)}\right]\Delta t$$
$$+\frac{2^{(p/q)-2}K^\gamma}{\gamma-1}\int_a^\infty \frac{(\mu(t))^{p/q-1}\left(\Lambda^\Delta(t)\right)^{p/q}}{t^{\gamma-1}}\Delta t.$$

This implies (note $(\Lambda^\sigma(t))^{p/q} > (\Lambda(t))^{p/q}$ since $\Lambda^\Delta(t) > 0$) that

$$\int_a^\infty \frac{1}{t^\gamma}(\Lambda(t))^{p/q}\left[1+\frac{p2^{p/q-2}K^\gamma}{q(\gamma-1)}\frac{t}{\sigma(t)}\right]\Delta t$$
$$\leqslant \frac{2^{p/q-2}pK^\gamma}{q(\gamma-1)}\int_a^\infty \left[\frac{(t^\gamma)^{(p-q)/p}}{t^\gamma}\frac{f(t)g(t)}{\sigma(t)}\right]\left[(t^\gamma)^{-(p-q)/p}\Lambda^{(p-q)/q}(t)\right]\Delta t$$
$$+\frac{2^{p/q-2}K^\gamma}{\gamma-1}\int_a^\infty \frac{\mu^{\frac{p}{q}-1}(t)}{t^{\gamma-1}}(\Lambda^\Delta(t))^{p/q}\Delta t. \tag{26}$$

Applying the Hölder inequality (11) on the term

$$\int_a^\infty \left[\frac{(t^\gamma)^{(p-q)/p}}{t^\gamma}\frac{f(t)g(t)}{\sigma(t)}\right]\left[(t^\gamma)^{-(p-q)/p}\Lambda^{(p-q)/q}(t)\right]\Delta t,$$

with indices p/q and $p/(p-q)$, we see that

$$\int_a^\infty \left[\frac{(t^\gamma)^{(p-q)/p}}{t^\gamma}\frac{f(t)g(t)}{\sigma(t)}\right]\left[(t^\gamma)^{-(p-q)/p}\Lambda^{(p-q)/q}(t)\right]\Delta t$$
$$\leqslant \left[\int_a^\infty \left[\frac{(t^\gamma)^{(p-q)/p}}{t^\gamma}\frac{f(t)g(t)}{\sigma(t)}\right]^{p/q}\Delta t\right]^{\frac{q}{p}}\left[\int_a^\infty \frac{\Lambda^{p/q}(t)}{t^\gamma}\Delta t\right]^{\frac{p-q}{p}}. \tag{27}$$

Substituting (27) into (26), we have

$$\int_a^\infty \frac{1}{t^\gamma}(\Lambda(t))^{\frac{p}{q}}\left[1+\frac{p\left(2^{p/q-2}\right)K^{\gamma-1}}{q(\gamma-1)}\right]\Delta t$$
$$\leqslant \frac{2^{\frac{p}{q}-2}(p/q)K^\gamma}{\gamma-1}\left[\int_a^\infty \left[\frac{(t^\gamma)^{(p-q)/p}}{t^\gamma}\frac{f(t)g(t)}{\sigma(t)}\right]^{\frac{p}{q}}\Delta t\right]^{\frac{q}{p}}\left[\int_a^\infty \frac{\Lambda^{\frac{p}{q}}(t)}{t^\gamma}\Delta t\right]^{\frac{p-q}{p}}$$
$$+\frac{2^{\frac{p}{q}-2}K^\gamma}{\gamma-1}\int_a^\infty \frac{\mu^{p/q-1}(t)}{t^{\gamma-1}}(\Lambda^\Delta(t))^{p/q}\Delta t.$$

Hence from assumption (14) we have

$$\frac{1}{m}\int_a^\infty \frac{1}{t^\gamma}(\Lambda(t))^{p/q}\Delta t$$
$$\leqslant \frac{2^{\frac{p}{q}-2}pK^\gamma}{q(\gamma-1)}\left[\int_a^\infty \left[\frac{(t^\gamma)^{(p-q)/p}}{t^\gamma}\frac{f(t)g(t)}{\sigma(t)}\right]^{p/q}\Delta t\right]^{\frac{q}{p}}\left[\int_a^\infty \frac{\Lambda^{p/q}(t)}{t^\gamma}\Delta t\right]^{\frac{p-q}{p}}$$
$$+\frac{2^{\frac{p}{q}-2}K^\gamma}{\gamma-1}\int_a^\infty \frac{\mu^{\frac{p}{q}-1}(t)}{t^{\gamma-1}}(\Lambda^\Delta(t))^{\frac{p}{q}}\Delta t.$$

This implies that

$$\int_a^\infty \frac{1}{t^\gamma}(\Lambda(t))^{p/q}\Delta t-\frac{2^{p/q-2}m}{\gamma-1}\int_a^\infty \frac{\mu^{p/q-1}(t)}{t^{\gamma-1}}\left(\Lambda^\Delta(t)\right)^{p/q}\Delta t$$
$$\leqslant \frac{2^{p/q-2}pmK^\gamma}{q(\gamma-1)}\left[\int_a^\infty \frac{1}{t^\gamma}\left(\frac{f(t)g(t)}{\sigma(t)}\right)^{p/q}\Delta t\right]^{\frac{q}{p}}\left[\int_a^\infty \frac{\Lambda^{p/q}(t)}{t^\gamma}\Delta t\right]^{\frac{p-q}{p}},$$

and on simplification, we get the desired inequality (15). The proof is complete. □

THEOREM 2.2. *Let $\mathbb{T}$ be a time scale with $a,\ b\in\mathbb{T}$ and $p>q>0$ such that $p/q\geqslant 2$ and $\gamma>1$. Let $\Lambda(t)$ be defined as in (13). If $f(t)g(t)\geqslant t\Lambda(t)$ for $t\in[a,\infty)_{\mathbb{T}}$ and*

$$1+\frac{pK^{\gamma-1}}{q(\gamma-1)}\left(\frac{\Lambda(t)}{\Lambda^\sigma(t)}\right)^{\frac{p}{q}}\geqslant\frac{1}{m}>0, \tag{28}$$

for some constant $m>0$, then

$$\int_a^\infty \frac{1}{t^\gamma}\left(\frac{1}{\sigma(t)}\int_a^{\sigma(t)}\frac{f(s)g(s)}{s}\Delta s\right)^{\frac{p}{q}}\Delta t\leqslant\left(\frac{pmK^\gamma}{q(\gamma-1)}\right)^{\frac{p}{q}}\int_a^\infty \frac{1}{t^\gamma}\left(\frac{f(t)g(t)}{\sigma(t)}\right)^{\frac{p}{q}}\Delta t. \tag{29}$$

Proof. We proceed as in the proof of Theorem 2.1 to get that

$$\int_a^\infty \frac{1}{t^\gamma}(\Lambda^\sigma(t))^{p/q}\Delta t\leqslant\frac{K^\gamma}{\gamma-1}\int_a^\infty \frac{1}{t^{\gamma-1}}(\Lambda^{p/q}(t))^\Delta\Delta t. \tag{30}$$

Applying the chain rule ([2, Theorem 1.87])

$$F^\Delta(g(t))=F'(g(c))g^\Delta(t),\text{ where } c\in[t,\sigma(t)],$$

on the term $\left(\Lambda^{p/q}(t)\right)^\Delta$, we see that

$$\left(\Lambda^{p/q}(t)\right)^\Delta=\frac{p}{q}\Lambda^{\frac{p}{q}-1}(c)\Lambda^\Delta(t),\ \text{ for } c\in[t,\sigma(t)]. \tag{31}$$

Using (23), we see that $\Lambda^{\sigma}(t) \geqslant \Lambda(c)$, since $\sigma(t) \geqslant c$. Substituting this into (30) and using (23), we have that

$$\begin{aligned}
&\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q}}{t^\gamma}\Delta t \\
&\leqslant \frac{pK^\gamma}{q(\gamma-1)}\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q-1}}{t^{\gamma-1}}\left[\frac{f(t)g(t)}{t\sigma(t)}-\frac{\Lambda(t)}{\sigma(t)}\right]\Delta t \\
&= \frac{pK^\gamma}{q(\gamma-1)}\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q-1}}{t^\gamma}\frac{f(t)g(t)}{\sigma(t)}\Delta t - \frac{pK^\gamma}{q(\gamma-1)}\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q-1}\Lambda(t)}{t^\gamma}\frac{t}{\sigma(t)}\Delta t \\
&\leqslant \frac{pK^\gamma}{q(\gamma-1)}\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q-1}}{t^\gamma}\frac{f(t)g(t)}{\sigma(t)}\Delta t - \frac{pK^{\gamma-1}}{q(\gamma-1)}\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q}}{t^\gamma}\left(\frac{\Lambda(t)}{\Lambda^\sigma(t)}\right)^{p/q}\Delta t.
\end{aligned}$$

Hence

$$\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q}}{t^\gamma}\left[1+\frac{pK^{\gamma-1}}{q(\gamma-1)}\left(\frac{\Lambda(t)}{\Lambda^\sigma(t)}\right)^{p/q}\right]\Delta t \leqslant \frac{p}{q(\gamma-1)}\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q-1}}{t^\gamma}\frac{f(t)g(t)}{\sigma(t)}\Delta t. \tag{32}$$

Using assumption (28), we get that

$$\begin{aligned}
\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q}}{t^\gamma}\Delta t &\leqslant \frac{pmK^\gamma}{q(\gamma-1)}\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q-1}}{t^\gamma}\frac{f(t)g(t)}{\sigma(t)}\Delta t \\
&= \frac{pmK^\gamma}{q(\gamma-1)}\int_a^\infty \left[\frac{(t^\gamma)^{(p-q)/p}}{t^\gamma}\frac{f(t)g(t)}{\sigma(t)}\right]\left[(t^\gamma)^{-(p-q)/p}\Lambda^{(p-q)/q}(t)\right]\Delta t.
\end{aligned}$$

Applying the Hölder inequality (11) on the right hand side with indices p/q and $p/(p-q)$, we get that

$$\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q}}{t^\gamma} \leqslant \left(\frac{pmK^\gamma}{q(\gamma-1)}\right)^{p/q}\int_a^\infty \frac{1}{t^\gamma}\left(\frac{f(t)g(t)}{\sigma(t)}\right)^{p/q}\Delta t,$$

which is the desired inequality (29). The proof is complete. □

In the following, we will use the chain rule formula

$$(\Lambda^{p/q}(t))^\Delta = \frac{p}{q}\int_0^1 [h\Lambda^\sigma+(1-h)\Lambda]^{\frac{p}{q}-1}\,dh\Lambda^\Delta(t), \tag{33}$$

instead of the formula

$$(\Lambda^{p/q}(t))^\Delta = \frac{p}{q}\int_0^1 \left[\Lambda+\mu h\Lambda^\Delta\right]^{\frac{p}{q}-1}\,dh\Lambda^\Delta(t),$$

and inequality (2.18).

THEOREM 2.3. *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$ and $p > q > 0$ such that $p/q \geqslant 2$ and $\gamma > 1$. Let $\Lambda(t)$ be defined as in (13). If $f(t)g(t) \geqslant t\Lambda(t)$ for $t \in [a,\infty)_{\mathbb{T}}$ and*

$$1 + \frac{2^{p/q-1}K^{\gamma-1}}{\gamma-1}\frac{\Lambda(t)}{\Lambda^{\sigma}(t)} \geqslant \frac{1}{m} > 0, \tag{34}$$

for some constant $m > 0$, then

$$\int_a^{\infty} \frac{1}{t^{\gamma}}(\Lambda^{\sigma}(t))^{p/q}\Delta t \leqslant \left(\frac{2^{\frac{p}{q}-1}mK^{\gamma}}{\gamma-1}\right)^{\frac{p}{q}} \int_a^{\infty} \frac{1}{t^{\gamma}}\left(\frac{f(t)g(t)}{\sigma(t)}\right)^{p/q} \Delta t. \tag{35}$$

Proof. Proceeding as in the proof of Theorem 2.1 to get that

$$\int_a^{\infty} \frac{1}{t^{\gamma}}(\Lambda^{\sigma}(t))^{p/q}\Delta t \leqslant \frac{K^{\gamma}}{\gamma-1}\int_a^{\infty} \frac{(\Lambda^{p/q}(t))^{\Delta}}{t^{\gamma-1}}\Delta t. \tag{36}$$

From (33) and (24), we see that

$$\begin{aligned}(\Lambda^{p/q}(t))^{\Delta} &\leqslant 2^{\frac{p}{q}-2}\frac{p}{q}\int_0^1 \left[(h\Lambda^{\sigma})^{\frac{p}{q}-1} + (1-h)^{\frac{p}{q}-1}\Lambda^{\frac{p}{q}-1}\right] dh\Lambda^{\Delta}(t) \\ &= 2^{\frac{p}{q}-2}\left[(\Lambda^{\sigma})^{\frac{p}{q}-1} + \Lambda^{\frac{p}{q}-1}\right]\Lambda^{\Delta}(t).\end{aligned} \tag{37}$$

From the definition of $\Lambda(t)$ and since $\Lambda^{\Delta}(t) \geqslant 0$, we have that

$$(\Lambda^{p/q}(t))^{\Delta} \leqslant 2^{\frac{p}{q}-1}(\Lambda^{\sigma}(t))^{\frac{p}{q}-1}\Lambda^{\Delta}(t). \tag{38}$$

This, (23) and (36) implies that

$$\int_a^{\infty} \frac{1}{t^{\gamma}}(\Lambda^{\sigma}(t))^{p/q}\Delta t \leqslant \frac{2^{\frac{p}{q}-1}K^{\gamma}}{(\gamma-1)}\int_a^{\infty} \frac{1}{t^{\gamma-1}}(\Lambda^{\sigma}(t))^{\frac{p}{q}-1}\left[\frac{f(t)g(t)-t\Lambda(t)}{t\sigma(t)}\right]\Delta t. \tag{39}$$

Thus

$$\begin{aligned}&\int_a^{\infty} \frac{1}{t^{\gamma}}(\Lambda^{\sigma}(t))^{p/q}\left[1 + \frac{\left(2^{(p/q)-1}\right)K^{\gamma-1}}{(\gamma-1)}\frac{\Lambda(t)}{\Lambda^{\sigma}(t)}\right]\Delta t \\ &\leqslant \frac{2^{\frac{p}{q}-1}K^{\gamma}}{\gamma-1}\int_a^{\infty}\left[\frac{(t^{\gamma})^{(p-q)/p}}{t^{\gamma}}\frac{f(t)g(t)}{\sigma(t)}\right]\left[(t^{\gamma})^{-(p-q)/p}(\Lambda^{\sigma}(t))^{(p-q)/q}\right]\Delta t.\end{aligned} \tag{40}$$

Applying the Hölder inequality (11) on the right hand side with indices p/q and $p/(p-q)$, we see that

$$\begin{aligned}&\int_a^{\infty}\left[\frac{(t^{\gamma})^{(p-q)/p}}{t^{\gamma}}\frac{f(t)g(t)}{\sigma(t)}\right]\left[(t^{\gamma})^{-(p-q)/p}(\Lambda^{\sigma}(t))^{(p-q/q)}\right]\Delta t \\ &\leqslant \left[\int_a^{\infty}\left[\frac{(t^{\gamma})^{(p-q)/p}}{t^{\gamma}}\frac{f(t)g(t)}{\sigma(t)}\right]^{p/q}\Delta t\right]^{\frac{q}{p}}\left[\int_a^{\infty}\frac{(\Lambda^{\sigma}(t))^{p/q}(t)}{t^{\gamma}}\Delta t\right]^{\frac{p-q}{p}}.\end{aligned} \tag{41}$$

Substituting (41) into (40) and using (34), we have

$$\int_a^\infty \frac{1}{t^\gamma}(\Lambda^\sigma(t))^{p/q}\Delta t$$
$$\leqslant \frac{2^{\frac{p}{q}-1}mK^\gamma}{\gamma-1}\left[\int_a^\infty \left[\frac{(t^\gamma)^{(p-q)/p}}{t^\gamma}\frac{f(t)g(t)}{\sigma(t)}\right]^{p/q}\Delta t\right]^{\frac{q}{p}}\left[\int_a^\infty \frac{(\Lambda^\sigma)^{p/q})(t)}{t^\gamma}\Delta t\right]^{\frac{p-q}{p}}.$$

This implies that

$$\left[\int_a^\infty \frac{1}{t^\gamma(t)}(\Lambda^\sigma(t))^{p/q}\Delta t\right]^{1-\frac{p-q}{p}} \leqslant \frac{2^{\frac{p}{q}-1}mK^\gamma}{\gamma-1}\left[\int_a^\infty \left[\frac{(t^\gamma)^{(p-q)/p}}{t^\gamma(t)}\frac{f(t)g(t)}{\sigma(t)}\right]^{p/q}\Delta t\right]^{\frac{q}{p}}.$$

Hence

$$\int_a^\infty \frac{1}{t^\gamma}(\Lambda^\sigma(t))^{p/q}\Delta t \leqslant \left(\frac{2^{\frac{p}{q}-1}mK^\gamma}{\gamma-1}\right)^{\frac{p}{q}}\int_a^\infty \frac{1}{t^\gamma}\left(\frac{f(t)g(t)}{\sigma(t)}\right)^{p/q}\Delta t,$$

which is the desired inequality (15). The proof is complete. □

When $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{N}$ we establish from Theorems 2.1–2.3 some new differential and discrete inequalities. We begin with Theorem 2.1 when $\mathbb{T}=\mathbb{R}$. In this case (note that $\mu(t)=0$ and $\sigma(t)=t$) Theorem 2.1 reduces to the following corollary after replacing p/q by $\lambda \geqslant 2$.

COROLLARY 2.1. *Let $a\in\mathbb{R}^+$, $\lambda\geqslant 2$ and $\gamma>1$ be real numbers and f and g are nonnegative real valued functions on $[a,\infty)_{\mathbb{R}}$. If $f(t)g(t)\geqslant t\Lambda(t)$, where*

$$\Lambda(t)=\frac{1}{t}\int_a^t \frac{f(s)g(s)}{s}ds,$$

and $m\geqslant(\gamma-1)/(\gamma-1+\lambda 2^{\lambda-2})$, then

$$\int_a^\infty \frac{1}{t^\gamma}\left[\frac{1}{t}\int_a^t \frac{f(s)g(s)}{s}ds\right]^\lambda dt \leqslant \left(\frac{2^{\lambda-2}\lambda m}{\gamma-1}\right)^\lambda \int_a^\infty \frac{1}{t^\gamma}\left(\frac{f(t)g(t)}{t}\right)^\lambda dt,\ \lambda\geqslant 2. \quad (42)$$

REMARK. From Corollary 2.1, when $f(t)=t$, the inequality (42) reduces to

$$\int_a^\infty \frac{1}{t^\gamma}\left(\frac{1}{t}\int_a^t g(s)ds\right)^\lambda dt \leqslant \left(\frac{2^{\lambda-2}\lambda m}{\gamma-1}\right)^\lambda \int_a^\infty \frac{1}{t^\gamma}g^\lambda(t)dt,\ \lambda\geqslant 2. \quad (43)$$

From Theorem 2.2 when $\mathbb{T}=\mathbb{R}$, we have the following corollary after replacing p/q by $\lambda\geqslant 2$.

COROLLARY 2.2. *Let $a \in \mathbb{R}^+$, $\lambda \geqslant 2$ and $\gamma > 1$ be real numbers and f and g are nonnegative real valued functions on $[a,\infty)_{\mathbb{R}}$. If $f(t)g(t) \geqslant t\Lambda(t)$, where*

$$\Lambda(t) = \frac{1}{t}\int_a^t \frac{f(s)g(s)}{s}ds,$$

and $m \geqslant (\gamma-1)/(\gamma-1+\lambda)$, then

$$\int_a^\infty \frac{1}{t^\gamma}\left(\frac{1}{t}\int_a^t \frac{f(s)g(s)}{s}ds\right)^\lambda dt \leqslant \left(\frac{\lambda m}{\gamma-1}\right)^\lambda \int_a^\infty \frac{1}{t^\gamma}\left(\frac{f(t)g(t)}{t}\right)^\lambda dt. \tag{44}$$

From Corollary 2.2 if we put $f(t) = t$ and $m = (\gamma-1)/(\gamma-1+\lambda)$, then (44) reduces to a Hardy type inequality (1) of the form

$$\int_a^\infty \frac{1}{t^\gamma}\left(\frac{1}{t}\int_a^t g(s)ds\right)^\lambda dt \leqslant \left(\frac{\lambda}{\lambda+\gamma-1}\right)^\lambda \int_a^\infty \frac{1}{t^\gamma}(g(t))^\lambda dt. \tag{45}$$

From Theorem 2.3 when $\mathbb{T} = \mathbb{R}$, we have the following corollary after replacing p/q by $\lambda \geqslant 2$.

COROLLARY 2.3. *Let $a \in \mathbb{R}^+$, $\lambda \geqslant 2$ and $\gamma > 1$ be real numbers and f and g are nonnegative real valued functions on $[a,\infty)_{\mathbb{R}}$. If $f(t)g(t) \geqslant t\Lambda(t)$, where*

$$\Lambda(t) = \frac{1}{t}\int_a^t \frac{f(s)g(s)}{s}ds,$$

and $m \geqslant (\gamma-1)/(\gamma-1+2^{\lambda-1})$, then

$$\int_a^\infty \frac{1}{t^\gamma}\left[\frac{1}{t}\int_a^t \frac{f(s)g(s)}{s}ds\right]^\lambda dt \leqslant \left(\frac{2^{\lambda-1}m}{\gamma-1}\right)^\lambda \int_a^\infty \frac{1}{t^\gamma}\left(\frac{f(t)g(t)}{t}\right)^\lambda dt, \ \lambda \geqslant 2.$$

When $\mathbb{T} = \mathbb{N}$, we have from Theorem 2.1 the following discrete inequality.

COROLLARY 2.4. *Let $a \in \mathbb{N}$ and $\lambda \geqslant 2$ and $\gamma > 1$. Let $f(n)$ and $g(n)$ be nonnegative sequences and define*

$$\Lambda(n) := \frac{1}{n}\sum_{s=a}^{n-1} \frac{f(s)g(s)}{s}. \tag{46}$$

If $f(n)g(n) \geqslant n\Lambda(n)$ and

$$1 + \frac{2^{\lambda-2}\lambda}{\gamma-1}K^{\gamma-1} \geqslant \frac{1}{m} > 0,$$

for some constants $m > 0$, $K \geqslant 1$, then

$$\sum_{n=a}^\infty \frac{1}{n^\gamma}\left[\Lambda^\lambda(n) - \frac{2^{\lambda-2}m}{\gamma-1}(\Delta\Lambda(n))^\lambda\right] \leqslant \frac{\lambda 2^{\lambda-2}mK^\gamma}{\gamma-1}\left[\sum_{n=a}^\infty \frac{1}{n^\gamma}\left(\frac{f(n)g(n)}{(n+1)}\right)^\lambda\right]^{\frac{1}{\lambda}}\left[\sum_{n=a}^\infty \frac{\Lambda^\lambda(n)}{n^\gamma}\right]^{1-\frac{1}{\lambda}}.$$

When $\mathbb{T} = \mathbb{N}$, we have the following discrete inequality as a special case of Theorem 2.2.

COROLLARY 2.5. *Let $a \in \mathbb{N}$ and $\lambda \geqslant 2$ and $\gamma > 1$. Let $f(n)$ and $g(n)$ be nonnegative sequences and $\Lambda(n)$ be defined as in (46). If $f(n)g(n) \geqslant n\Lambda(n)$ and*

$$\left[1+\frac{\lambda K^{\gamma-1}}{\gamma-1}\left(\frac{\Lambda(n)}{\Lambda(n+1)}\right)^{\lambda}\right] \geqslant \frac{1}{m} > 0,$$

for some constants m and $K \geqslant 1$, then

$$\sum_{n=a}^{\infty}\frac{1}{n^{\gamma}}\left(\frac{\sum_{s=a}^{n}\frac{f(s)g(s)}{s}}{(n+1)}\right)^{\lambda} \leqslant \left(\frac{\lambda m K^{\gamma}}{\gamma-1}\right)^{\lambda}\sum_{n=a}^{\infty}\frac{1}{n^{\gamma}}\left(\frac{f(n)g(n)}{(n+1)}\right)^{\lambda}.$$

When $\mathbb{T} = \mathbb{N}$, we have the following result as a special case of Theorem 2.3.

COROLLARY 2.6. *Let $a \in \mathbb{N}$, $\lambda \geqslant 2$ and $\gamma > 1$ and $f(n)$ and $g(n)$ be nonnegative sequences. If $f(n)g(n) \geqslant n\Lambda(n)$ and*

$$1+\frac{2^{\lambda-1}K^{\gamma-1}}{\gamma-1}\frac{\Lambda(n)}{\Lambda(n+1)} \geqslant \frac{1}{m} > 0,$$

for some constants m and $K \geqslant 1$, then

$$\sum_{n=a}^{\infty}\frac{1}{n^{\gamma}}\left(\frac{1}{(n+1)}\sum_{s=a}^{n}\frac{f(s)g(s)}{s}\right)^{\lambda} \leqslant \left(\frac{2^{\lambda-1} m K^{\gamma}}{\gamma-1}\right)^{\lambda}\sum_{n=a}^{\infty}\frac{1}{n^{\gamma}}\left(\frac{f(n)g(n)}{(n+1)}\right)^{\lambda}.$$

In the following, we consider the case when $p/q \leqslant 2$ and prove new inequalities of Hardy type on time scales. To prove these results, we need the inequality

$$2^{r-1}\left(a^{r}+b^{r}\right) \leqslant (a+b)^{r} \leqslant \left(a^{r}+b^{r}\right), \quad \text{where} \quad a, b \geqslant 0 \quad \text{and} \quad 0 \leqslant r \leqslant 1. \tag{47}$$

Applying this inequality when $r = p/q - 1 < 1$, instead of the inequality (24) that has been used in the proof of Theorem 2.1, we see that

$$(p/q)\int_{0}^{1}\left[\Lambda+h\mu\Lambda^{\Delta}\right]^{(p/q)-1}dh \leqslant (p/q)\Lambda^{p/q-1}+(\mu\Lambda^{\Delta})^{p/q-1}, \quad p/q \leqslant 2.$$

Proceeding as in the proof of Theorem 2.1, we can prove the following result.

THEOREM 2.4. *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$ and $p,\ q > 0$ such that $p/q \leqslant 2$ and $\gamma > 1$. Let $\Lambda(t)$ be defined as in (13). If $f(t)g(t) \geqslant t\Lambda(t)$ for $t \in [a,\infty)_{\mathbb{T}}$ and*

$$1+\frac{pK^{\gamma-1}}{q(\gamma-1)} \geqslant \frac{1}{m} > 0, \tag{48}$$

for some constant $m > 0$, then

$$\int_{a}^{\infty}\frac{\Lambda^{p/q}(t)}{t^{\gamma}}\Delta t - \int_{a}^{\infty}\frac{mK^{\gamma}\mu^{p/q-1}(t)}{t^{\gamma-1}}(\Lambda^{\Delta}(t))^{p/q}\Delta t$$
$$\leqslant \frac{pmK^{\gamma}}{q(\gamma-1)}\left[\int_{a}^{\infty}\frac{1}{t^{\gamma}}\left(\frac{f(t)g(t)}{\sigma(t)}\right)^{p/q}\Delta t\right]^{\frac{q}{p}}\left[\int_{a}^{\infty}\frac{\Lambda^{p/q}(t)}{t^{\gamma}}\Delta t\right]^{\frac{p-q}{p}}.$$

As in the proof of Theorem 2.2 one can also prove the following theorem.

THEOREM 2.5. *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$ and $p,\ q > 0$ such that $p/q \leqslant 2$ and $\gamma > 1$. Let $\Lambda(t)$ be defined as in (13). If $f(t)g(t) \geqslant t\Lambda(t)$ for $t \in [a,\infty)_{\mathbb{T}}$ and*

$$1 + \frac{2K^{\gamma-1}}{(\gamma-1)} \frac{\Lambda(t)}{\Lambda^{\sigma}(t)} \geqslant \frac{1}{m} > 0,$$

for some constant $m > 0$, then

$$\int_a^\infty \frac{1}{t^\gamma} (\Lambda^\sigma(t))^{p/q} \Delta t \leqslant \left(\frac{2mK^\gamma}{\gamma-1}\right)^{\frac{p}{q}} \int_a^\infty \frac{1}{t^\gamma} \left(\frac{f(t)g(t)}{\sigma(t)}\right)^{p/q} \Delta t.$$

In the following, we prove a new class of inequalities on time scales when $\gamma < 1$ by using the new operator

$$\Omega(t) := \frac{1}{t} \int_t^\infty \frac{f(s)g(s)}{s} \Delta s \quad \textit{for any} \quad t \in [a,\infty)_{\mathbb{T}}, \tag{49}$$

instead of the function $\Lambda(t)$ that has been used in the proofs of the above theorems.

THEOREM 2.6. *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$ and $p > q > 0$ such that $p/q \geqslant 2$ and $\gamma < 1$. Let $\Omega(t)$ be defined as in (49). Then*

$$\begin{aligned} &\int_a^\infty \frac{(\Omega^\sigma(t))^{p/q}}{t^\gamma} \left[1 - \frac{(p/q)K}{(1-\gamma)} \left(\frac{\Omega(t)}{(\Omega^\sigma(t))}\right)^{p/q} \frac{t}{\sigma(t)}\right] \Delta t \\ &\leqslant \frac{(p/q)}{1-\gamma} \left[\int_a^\infty \frac{1}{t^\gamma} \left(\frac{f(t)g(t)}{\sigma(t)}\right)^{p/q} \Delta t\right]^{\frac{q}{p}} \left[\int_a^\infty \frac{\Omega^{p/q}(t)}{t^\gamma} \Delta t\right]^{\frac{p-q}{p}}. \end{aligned} \tag{50}$$

Proof. As in the proof of Theorem 2.2, we see that

$$\int_a^\infty \frac{1}{t^\gamma} (\Omega^\sigma)^{p/q}(t) \Delta t \leqslant \frac{(p/q)}{1-\gamma} \int_a^\infty v^\sigma(t)(-\Omega^\Delta(t)) \int_0^1 \left[\Omega + \mu(t) h \Omega^\Delta\right]^{\frac{p}{q}-1} dh \Delta t, \tag{51}$$

where $v(t) = \int_a^t (1/s^\gamma) \Delta s$. Using the chain rule (8) and using the fact that $\sigma(s) \geqslant s$ and (12), we have

$$\begin{aligned} \left(s^{1-\gamma}\right)^\Delta &= (1-\gamma) \int_0^1 [h\sigma(s) + (1-h)s]^{-\gamma} dh = (1-\gamma) \int_0^1 \frac{1}{[h\sigma(s) + (1-h)s]^\gamma} dh \\ &\geqslant (1-\gamma) \int_0^1 \frac{1}{[h\sigma(s) + (1-h)\sigma(s)]^\gamma} dh = \frac{1-\gamma}{\sigma^\gamma(s)} \\ &= \frac{(1-\gamma)s^\gamma}{\sigma^\gamma(s) s^\gamma} \geqslant \frac{(1-\gamma)}{s^\gamma} \frac{1}{K^\gamma}. \end{aligned}$$

This implies that

$$\begin{aligned} v^\sigma(t) &= \int_a^{\sigma(t)} \frac{1}{s^\gamma} \Delta s \leqslant \frac{K^\gamma}{1-\gamma} \int_a^{\sigma(t)} \left(\frac{1}{s^{\gamma-1}}\right)^\Delta \Delta t \\ &= \frac{K^\gamma}{1-\gamma} \frac{1}{(\sigma(t))^{\gamma-1}} - \frac{K^\gamma}{1-\gamma} \frac{1}{a^{\gamma-1}} \leqslant \frac{K^\gamma}{1-\gamma} (\sigma(t))^{1-\gamma}. \end{aligned} \tag{52}$$

From (12) and (52), we have

$$v^{\sigma}(t) \leqslant \frac{K^{\gamma}}{1-\gamma}(Kt)^{1-\gamma} = \frac{K}{(1-\gamma)}\frac{1}{t^{\gamma-1}}. \tag{53}$$

Combining (51), (53) and using the facts that $\Omega(\infty)=0$ and $v(a)=0$, we get that

$$\int_a^{\infty} \frac{1}{t^{\gamma}}(\Omega^{\sigma})^{p/q}(t)\Delta t \leqslant \frac{(p/q)K}{1-\gamma}\int_a^{\infty}\frac{(-\Omega^{\Delta}(t))}{t^{\gamma-1}}\int_0^1 \left[\Omega+\mu(t)h\Omega^{\Delta}\right]^{\frac{p}{q}-1} dh\Delta t.$$

Now

$$-\Omega^{\Delta}(t) = -\left[\frac{1}{t}\int_t^{\infty}\frac{f(s)g(s)}{s}\Delta s\right]^{\Delta} = \left[\frac{f(t)g(t)+\int_t^{\infty}\frac{f(s)g(s)}{s}\Delta s}{t\sigma(t)}\right] \geqslant 0. \tag{54}$$

Since $\Omega^{\Delta}(t)\leqslant 0,$ we see that

$$\int_0^1 \left[\Omega+\mu(t)h\Omega^{\Delta}\right]^{\frac{p}{q}-1} dh \leqslant \Omega^{\frac{p}{q}-1}(t). \tag{55}$$

Substituting (54) and (55) into (51), we have

$$\begin{aligned}
&\int_a^{\infty}\frac{1}{t^{\gamma}}(\Omega^{\sigma}(t))^{p/q}\Delta t \\
&\leqslant \frac{(p/q)K}{1-\gamma}\int_a^{\infty}\frac{1}{t^{\gamma-1}}(\Omega(t))^{\frac{p}{q}-1}\left[\frac{f(t)g(t)+t\Omega(t)}{t\sigma(t)}\right]\Delta t \\
&= \frac{(p/q)K}{1-\gamma}\int_a^{\infty}\frac{1}{t^{\gamma}}(\Omega(t))^{\frac{p}{q}-1}\frac{f(t)g(t)}{\sigma(t)}\Delta t + \frac{(p/q)K}{1-\gamma}\int_a^{\infty}\frac{1}{t^{\gamma}}(\Omega(t))^{\frac{p}{q}}\frac{t}{\sigma(t)}\Delta t.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\int_a^{\infty}\frac{1}{t^{\gamma}}(\Omega^{\sigma}(t))^{p/q}\Delta t - \frac{(p/q)}{(1-\gamma)}\int_a^{\infty}\frac{1}{t^{\gamma}}(\Omega(t))^{\frac{p}{q}}\frac{t}{\sigma(t)}\Delta t \\
&\leqslant \frac{(p/q)}{1-\gamma}\int_a^{\infty}\frac{1}{t^{\gamma}}(\Omega(t))^{\frac{p}{q}-1}\frac{f(t)g(t)}{\sigma(t)}\Delta t \\
&= \frac{(p/q)}{1-\gamma}\int_a^{\infty}\left[\frac{(t^{\gamma})^{(p-q)/p}}{t^{\gamma}}\frac{f(t)g(t)}{\sigma(t)}\right]\left[\frac{(\Omega(t))^{(p-q)/q}}{(t^{\gamma})^{(p-q)/p}}\right]\Delta t.
\end{aligned} \tag{56}$$

Applying the Hölder inequality (11) on the right hand side with indices p/q and $p/(p-q)$, we see that

$$\begin{aligned}
&\int_a^{\infty}\left[\frac{(t^{\gamma})^{(p-q)/p}}{t^{\gamma}}\frac{f(t)g(t)}{\sigma(t)}\right]\left[\frac{(\Omega(t))^{(p-q)/q}}{(t^{\gamma})^{(p-q)/p}}\right]\Delta t \\
&\leqslant \left[\int_a^{\infty}\left[\frac{(t^{\gamma})^{(p-q)/p}}{t^{\gamma}}\frac{f(t)g(t)}{\sigma(t)}\right]^{p/q}\Delta t\right]^{\frac{q}{p}}\left[\int_a^{\infty}\frac{\Omega^{p/q}(t)}{t^{\gamma}}\Delta t\right]^{\frac{p-q}{p}},
\end{aligned} \tag{57}$$

Substituting (57) into (56), we get that

$$\int_a^\infty \frac{1}{t^\gamma}\left[(\Omega^\sigma(t))^{p/q} - \frac{(p/q)K}{(1-\gamma)}\frac{1}{t^{\gamma-1}}\Omega^{p/q}(t)\frac{t}{\sigma(t)}\right]\Delta tt$$
$$\leqslant \frac{(p/q)K}{1-\gamma}\left[\int_a^\infty \frac{1}{t^\gamma}\left(\frac{f(t)g(t)}{\sigma(t)}\right)^{p/q}\Delta t\right]^{\frac{q}{p}}\left[\int_a^\infty \frac{\Omega^{p/q}(t)}{t^\gamma}\Delta t\right]^{\frac{p-q}{p}},$$

which is the desired inequality (50). The proof is complete. □

Again, applying the inequality (47) on the term $[h\Omega^\sigma + (1-h)\Omega)]^{\frac{p}{q}-1}$, when $p/q \leqslant 2$ we see that

$$\frac{p}{q}\int_0^1 [h\Omega^\sigma + (1-h)\Omega)]^{\frac{p}{q}-1}\,dh \leqslant \frac{p}{q}\int_0^1 \left[h^{\frac{p}{q}-1}(\Omega^\sigma)^{\frac{p}{q}-1} + (1-h)^{\frac{p}{q}-1}\Omega^{\frac{p}{q}-1}\right]dh$$
$$= \left[(\Omega^\sigma)^{\frac{p}{q}-1} + \Omega^{\frac{p}{q}-1}\right] \leqslant 2\Omega^{\frac{p}{q}-1}(t). \tag{58}$$

Proceeding as in the proof of Theorem 2.6, we can prove the following theorem.

THEOREM 2.7. *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$ and $p > q > 0$ such that $p/q \leqslant 2$ and $\gamma < 1$. Let $\Omega(t)$ be defined as in (49). Then*

$$\int_a^\infty \frac{(\Omega^\sigma(t))^{p/q}}{t^\gamma}\left[1 - \frac{2K}{(1-\gamma)}\left(\frac{\Omega(t)}{(\Omega^\sigma(t))}\right)^{p/q}\frac{t}{\sigma(t)}\right]\Delta t$$
$$\leqslant \frac{2K}{1-\gamma}\left[\int_a^\infty \frac{1}{t^\gamma}\left(\frac{f(t)g(t)}{\sigma(t)}\right)^{p/q}\Delta t\right]^{\frac{q}{p}}\left[\int_a^\infty \frac{\Omega^{p/q}(t)}{t^\gamma}\Delta t\right]^{\frac{p-q}{p}}.$$

One can use Theorems 2.6 and 2.7 to derive some differential and discrete inequalities $\mathbb{T} = \mathbb{R}$ and when $\mathbb{T} = \mathbb{N}$. The details are left to the interested reader.

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