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ASYMPTOTIC EXPANSIONS AND COMPARISON OF BIVARIATE PARAMETER MEANS

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Abstract. The subject of this paper is the analysis of bivariate parameter means: general power mean, generalized logarithmic mean, Gini mean and Stolarsky mean. Asymptotical analysis of these means are made and series of corresponding coefficients are calculated. Using these information, a necessary conditions for the comparison of these means are derived. This approach enables better understanding of relations between these means.

1. Introduction

We are interested in analysis of bivariate means, and theirs behaviour when the data are translated by some large quantity x. In other words, we will analyse the asymptotic behaviour of the function $F(x+s,x+t)$, where F is bivariate mean and x tends to ∞. By a bivariate mean we understand a function $M\colon \mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ which satisfy

$$\min(s,t)\leqslant M(s,t)\leqslant \max(s,t).$$

It follows that $M(s,s)=s$ for all $s>0$. Means considered here will be homogeneous and symmetric.

This paper is a continuation of the previous paper [11] where similar problems were studied for the arithmetic, quadratic, harmonic, geometric, logarithmic, identric and some other particular means. Here, we will cover more general parameter means: power mean, generalized logarithmic, Stolarsky mean and Gini mean. We expect to obtain asymptotic expansions of the form

$$F(x+s,x+t)=x+\frac{s+t}{2}+\sum_{n=2}^{\infty}c_n(t,s)x^{-n+1}.$$

Here, c_n are polynomials of two variables of degree n, which depend, of course, also on the parameters of the involved mean.

Such expansions have many important properties. First, they are introduction for the similar analysis for general n-variable means. Further, they reveal many important properties of the means under consideration, for example, the comparison of various means. In [17], [19]–[22] similar problems were studied.

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Because of the homogeneity property of bivariate means, we have

$$F(x+s,x+t) = xF\left(1+\frac{s}{x}, 1+\frac{t}{x}\right)$$

so, the asymptotic expansion is essentially equivalent to the power series expansion of the function $F(1+s,1+t)$ for small values of s and t. It will be shown that it is sufficient to consider symmetric case $F(1-t,1+t)$. In the paper [15] authors consider expansion of the function $F(1,1+t)$, but this is nonsymmetric case and results are not so clear as in our approach.

The notation will be much simpler with s and t being replaced by variables α and β, $t = \alpha+\beta$ and $s = \alpha-\beta$. Then

$$\alpha = \frac{t+s}{2}, \qquad \beta = \frac{t-s}{2}.$$

We shall use also

$$\gamma = st = \alpha^2 - \beta^2, \qquad \delta = \frac{s^2+t^2}{2} = \alpha^2+\beta^2.$$

In all examples, the asymptotic expansions will be stated in terms of α and β.

Finally, let us denote

$$S_n = t^n - s^n, \qquad T_n = \tfrac{1}{2}(s^n+t^n).$$

These sequences can be calculated by recursive relations

$$S_n = 2\alpha S_{n-1} - \gamma S_{n-2}, \qquad n \geqslant 2,$$

where $S_0 = 0$ and $S_1 = 2\beta$, and

$$T_n = 2\alpha T_{n-1} - \gamma T_{n-2}, \qquad n \geqslant 2,$$

where $T_0 = 1$ and $T_1 = \alpha$.

The following lemmas about functional transformations of asymptotic series will be used later, see e.g. [5] and [14]:

LEMMA 1.1. *Let functions $f(x)$ and $g(x)$ have following asymptotic expansions ($a_0 \neq 0, b_0 \neq 0$) as $x \to \infty$:*

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}, \qquad g(x) \sim \sum_{n=0}^{\infty} b_n x^{-n}.$$

Then asymptotic expansion of their quotient $f(x)/g(x)$ reads as

$$\frac{f(x)}{g(x)} \sim \sum_{n=0}^{\infty} c_n x^{-n},$$

where coefficients c_n are defined by

$$c_n = \frac{1}{b_0}\left(a_n - \sum_{k=1}^{n} b_k c_{n-k}\right).$$

LEMMA 1.2. *Let $g(x)$ be a function with asymptotic expansion, $a_0 = 1$ and*

$$g(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}.$$

Then for all real p we have

$$[g(x)]^p \sim \sum_{n=0}^{\infty} P_n(p) x^{-n},$$

where

$$\begin{aligned} P_0(p) &= 1, \\ P_n(p) &= \frac{1}{n} \sum_{k=1}^{n} [k(1+p) - n] a_k P_{n-k}(p). \end{aligned} \tag{1.1}$$

2. Generalized logarithmic mean

Let r be a real number. The generalized logarithmic mean is defined for all $s,t > 0$ by

$$L_r(s,t) = \begin{cases} \left(\dfrac{t^{r+1} - s^{r+1}}{(r+1)(t-s)} \right)^{1/r}, & r \neq -1, 0, \\ \dfrac{t-s}{\log t - \log s}, & r = -1, \\ \dfrac{1}{e} \left(\dfrac{t^t}{s^s} \right)^{1/(t-s)}, & r = 0. \end{cases}$$

THEOREM 2.1. *Generalized logarithmic mean can be expanded into asymptotic series*

$$L_r(x+s, x+t) = x \sum_{n=0}^{\infty} c_n x^{-n}$$

where sequence (c_n) is defined by $c_0 = 1$ and

$$c_n = \frac{1}{2n} \sum_{k=1}^{n} \left(\frac{k}{r} - \frac{n}{r+1} \right) \binom{r+1}{k+1} \frac{S_{k+1}}{\beta} c_{n-k}. \tag{2.1}$$

Proof. We can write

$$\begin{aligned} L_r(x+s, x+t) &= \left[\frac{(x+t)^{r+1} - (x+s)^{r+1}}{(r+1)(t-s)} \right]^{1/r} \\ &= x \left[\sum_{k=0}^{\infty} \binom{r+1}{k+1} \frac{S_{k+1}}{2(r+1)\beta} x^{-k} \right]^{1/r}. \end{aligned}$$

Now we can apply Lemma 1.2 for the calculation of the power of an asymptotic series. The result is the procedure given in (2.1). □

It might seem that the proof of previous theorem is incomplete since we did not comment the cases $r=-1$ and $r=0$. But there is no need to treat those cases separately given that the value of the mean for $r=-1$ is a limit

$$L_{-1}(s,t)=\lim_{r\to -1} L_r(s,t)$$

and also

$$L_0(s,t)=\lim_{r\to 0} L_r(s,t).$$

Coefficients c_n from 2.1 behave well for all values of r and therefore by the following lemma the coefficients in asymptotic expansion of the $L_{-1}(x+s,x+t)$ and $L_0(x+s,x+t)$ can be calculated by taking limits $r\to -1$ and $r\to 0$ in (2.1).

LEMMA 2.2. *Let u be a real parameter, $R>0$, and $f_u\colon \mathbb{C}\to\mathbb{C}$ be analytic functions on $\{|z|>R\}$, such that*

1. $f_u(z)=\sum_{n=0}^{\infty} a_n(u)z^{-n}$, *for* $|z|>R$,
2. $\lim_{u\to u_0} a_n(u)=a_n$, *for each* $n\in\mathbb{N}$,
3. $\lim_{u\to u_0} f_u(z)=f(z)$, *for* $|z|>R$,
4. $f(z)=\sum_{n=0}^{\infty} b_n z^{-n}$, *for* $|z|>R$.

Then $a_n=b_n$.

Proof. Let Γ be a closed Jordan arc in annulus $\{|z|>R\}$ which surround the origin. Being analytic function, for the coefficients of the Laurent expansion of f_u we have

$$a_n(u)=\frac{1}{2\pi i}\int_\Gamma \frac{f_u(\xi)}{\xi^{n+1}}d\xi \to \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi^{n+1}}d\xi=b_n.$$

This proves the lemma. □

Asymptotic expansions for all means mentioned before are in fact expansion into Laurent series of the corresponding analytic continuation, se we can apply the above lemma.

The first few coefficients in asymptotic expansion of generalized logarithmic mean are

$$\begin{aligned}
c_0&=1,\\
c_1&=\alpha,\\
c_2&=\tfrac{1}{6}(r-1)\beta^2,\\
c_3&=-\tfrac{1}{6}(r-1)\alpha\beta^2,\\
c_4&=-\tfrac{1}{360}(r-1)\beta^2[(2r^2+5r-13)\beta^2-60\alpha^2],\\
c_5&=\tfrac{1}{120}(r-1)\alpha\beta^2[(2r^2+5r-13)\beta^2-20\alpha^2].
\end{aligned}$$

Taking $r=-1$ and $r=0$ we obtain the asymptotic expansion of logarithmic and identric mean:

$$L(x+s,x+t)=x+\alpha-\frac{\beta^2}{3x}+\frac{\alpha\beta^2}{3x^2}-\frac{\beta^2(15\alpha^2+4\beta^2)}{45x^3}+\frac{\alpha\beta^2(5\alpha^2+4\beta^2)}{15x^4},$$
$$I(x+s,x+t)=x+\alpha-\frac{\beta^2}{6x}+\frac{\alpha\beta^2}{6x^2}-\frac{\beta^2(60\alpha^2+13\beta^2)}{360x^3}+\frac{\alpha\beta^2(20\alpha^2+13\beta^2)}{120x^4}.$$

This agrees with expansions obtained in [11].

3. Power mean

The r-th power mean is defined for all $s,t>0$ by

$$M_r(s,t)=\begin{cases}\left(\dfrac{t^r+s^r}{2}\right)^{1/r}, & r\neq 0,\\ \sqrt{st}, & r=0.\end{cases}$$

The important particular cases of this mean are arithmetic mean $A=M_1$, quadratic mean $Q=M_2$ and harmonic mean $H=M_{-1}$. Geometric mean $G=M_0$ is obtained as limit of means M_r for $r\to 0$.

THEOREM 3.1. *The power mean has the asymptotic expansion of the form*

$$M_r(x+s,x+t)=x\sum_{k=0}^{\infty}c_kx^{-k}$$

where $c_0=1$ *and*

$$c_n=\frac{1}{n}\sum_{k=1}^{n}\left[k\left(1+\frac{1}{r}\right)-n\right]\binom{r}{k}T_kc_{n-k}.$$

Proof. We have

$$M_r(x+s,x+t)=\left(\frac{(x+s)^r+(x+t)^r}{2}\right)^{1/r}$$
$$=x\left[\sum_{n=0}^{\infty}\binom{r}{n}\frac{T_n}{x^n}\right]^{1/r}.$$

Hence, the asymptotic expansion can be derived using algorithm of Lemma 1.2, as in the previous theorem. □

The first few coefficients are

$$c_0=1,$$
$$c_1=\alpha,$$

$$
\begin{aligned}
c_2 &= \tfrac{1}{2}(r-1)\beta^2, \\
c_3 &= -\tfrac{1}{2}(r-1)\alpha\beta^2, \\
c_4 &= \tfrac{1}{24}(r-1)\beta^2(12\alpha^2+(3+r-2r^2)\beta^2), \\
c_5 &= -\tfrac{1}{8}(r-1)\alpha\beta^2(4\alpha^2+(3+r-2r^2)\beta^2).
\end{aligned}
$$

4. Stolarsky mean

The Stolarsky means, also called the extended means, is a class of two-parameter means introduced by Stolarsky in [23]. Their properties were studied by Leach and Sholander in [16] and [17] and further by Páles and others.

The extended mean of order p,r is defined for all $s,t>0$ by

$$
E_{p,r}(s,t) = \left[\frac{r(t^p-s^p)}{p(t^r-s^r)}\right]^{1/(p-r)}, \qquad p \neq r,\ p,r \neq 0. \tag{4.1}
$$

It is symmetric both on t and s as well on p and r. Therefore, we may suppose that $s \leqslant t$ and $r \leqslant p$. The excluded cases are obtained by limit procedure:

$$
\begin{aligned}
E_{r,r}(s,t) &= \frac{1}{e^{1/r}}\left(\frac{t^{t^r}}{s^{s^r}}\right)^{1/(t^r-s^r)}, \qquad r = p \neq 0, \\
E_{0,r}(s,t) &= \left[\frac{t^r-s^r}{r(\log t - \log s)}\right]^{1/r}, \qquad r \neq 0, \\
E_{0,0}(s,t) &= \sqrt{st}.
\end{aligned}
$$

Let us denote

$$
a_n(q) = \binom{q}{n+1}\frac{t^{n+1}-s^{n+1}}{q(t-s)}. \tag{4.2}
$$

Then the asymptotic expansion of Stolarsky mean can be obtained by the following algorithm.

THEOREM 4.1. *Let $r \neq p$, $r,p \neq 0$. The Stolarsky mean has the asymptotic expansion of the form*

$$
E_{p,r}(x+s,x+t) = x\sum_{n=0}^{\infty} c_n x^{-n}
$$

where (c_n) is obtained by following algorithm, $c_0 = 1$ and

$$
\begin{aligned}
b_0 &= 1, \\
b_n &= a_n(p) - \sum_{k=1}^{n} a_k(r) b_{n-k}, \qquad n \geqslant 1 \qquad (4.3)\\
c_n &= \frac{1}{n}\sum_{k=1}^{n}\left[k\left(1+\frac{1}{p-r}\right)-n\right] b_k c_{n-k}, \qquad n \geqslant 1. \qquad (4.4)
\end{aligned}
$$

Proof. The Stolarsky mean can be written as

$$
\begin{aligned}
E_{p,r}(x+s,x+t) &= x\left[\frac{r((1+\frac{t}{x})^p-(1+\frac{s}{x})^p)}{p((1+\frac{t}{x})^r-(1+\frac{s}{x})^r)}\right]^{1/(p-r)} \\
&= x\left[\frac{\sum\limits_{n=1}^{\infty} r\binom{p}{n}(t^n-s^n)x^{-n}}{\sum\limits_{n=1}^{\infty} p\binom{r}{n}(t^n-s^n)x^{-n}}\right]^{1/(p-r)} \\
&= x\left[\frac{\sum\limits_{n=0}^{\infty}\binom{p}{n+1}\dfrac{t^{n+1}-s^{n+1}}{p(t-s)}x^{-n}}{\sum\limits_{n=0}^{\infty}\binom{r}{n+1}\dfrac{t^{n+1}-s^{n+1}}{r(t-s)}x^{-n}}\right]^{1/(p-r)} \\
&= x\left[\frac{\sum\limits_{n=0}^{\infty} a_n(p)x^{-n}}{\sum\limits_{n=0}^{\infty} a_n(r)x^{-n}}\right]^{1/(p-r)}
\end{aligned}
$$

This expression can be transformed into asymptotic sequence by two steps. In the first one, coefficients (b_n) of the above ratio are calculated using the algorithm in Lemma 1.1, the procedure is written in (4.3). In the second step, the asymptotic series of the power of obtained asymptotical expansion is calculated using Lemma 1.2. The procedure is given in (4.4). □

Although at first sight we can be afraid that this two step procedure will give complicated coefficients, this is not the case. The coefficients are very nice. Here is the list of the first few terms.

$$
\begin{aligned}
c_0 &= 1, \\
c_1 &= \alpha, \\
c_2 &= \tfrac{1}{6}(-3+p+r)\beta^2, \\
c_3 &= -\tfrac{1}{6}(-3+p+r)\alpha\beta^2, \\
c_4 &= \tfrac{1}{360}\beta^2\Big[60(p+r-3)\alpha^2+(-2(p+r)(p^2+r^2) \\
&\qquad +5(p+r)^2+10(p+r)-45)\beta^2\Big], \\
c_5 &= -\tfrac{1}{120}\alpha\beta^2\Big[20(p+r-3)\alpha^2+(-2(p+r)(p^2+r^2) \\
&\qquad +5(p+r)^2+10(p+r)-45)\beta^2\Big].
\end{aligned}
$$

The explicit formula of Stolarsky mean in the special cases when $r=p$, $r=0$ or $p=0$ is too complicated to be expanded into asymptotical series, but as in the case of generalized logarithmic mean, there is no need for any further calculations. In the coefficients obtained here there is no ambiguity for these values of parameters and we can use the list above for any value of r and p.

For example, if $r = 0$ and $p = 1$ Stolarsky mean reduces to logarithmic mean $L(s,t)$ and for $r = p = 0$ we obtain expansion of geometric mean

$$G(x+s,x+t) = \sqrt{(x+s)(x+t)}$$
$$= x + \alpha - \frac{\beta^2}{2x} + \frac{\alpha\beta^2}{2}x^2 - \frac{\beta^2(4\alpha^2+\beta^2)}{8x^3} + \frac{\alpha\beta^2(4\alpha^2+3\beta^2)}{8x^4}.$$

5. Gini means

The Gini means are defined for all $s,t > 0$ by

$$G_{p,r}(s,t) = \begin{cases} \left(\dfrac{t^p+s^p}{t^r+s^r}\right)^{\frac{1}{p-r}}, & p \neq r, \\ \exp\left(\dfrac{s^p\log s + t^p \log t}{s^p+t^p}\right), & p = r \neq 0, \\ \sqrt{st}, & p = r = 0. \end{cases} \tag{5.1}$$

for parameters p and r. These means were first introduced by C. Gini in 1938. For some references on Gini means see [2].

Some of the special cases of the Gini means are power mean $G_{0,r} = M_r$ and Lehmer mean $G_{r+1,r}$.

Let us derive an algorithm for asymptotic expansion of the Gini mean. Let

$$a_k(q) = \binom{q}{k}\frac{t^k+s^k}{2}.$$

THEOREM 5.1. *Let $r \neq p$ and $r,p \neq 0$. The Gini mean has asymptotic expansion*

$$G_{p,r}(x+t,x+s) = x\sum_{n=0}^{\infty} c_n x^{-n},$$

where coefficients c_n are obtained by the following algorithm:

$$c_0 = 1;$$
$$c_n = \frac{1}{n}\sum_{k=1}^{n}\left[k\left(1+\frac{1}{p-r}\right)-n\right] b_k c_{n-k}, \tag{5.2}$$

and

$$b_n = a_n(p) - \sum_{k=1}^{n} a_k(r) b_{n-k}.$$

Proof.

$$
\begin{aligned}
G_{p,r}(x+t,x+s) &= x\left(\frac{(1+\frac{t}{x})^p+(1+\frac{s}{x})^p}{(1+\frac{t}{x})^r+(1+\frac{s}{x})^r}\right)^{\frac{1}{p-r}}\\
&= x\left(\frac{\sum_{n=0}^{\infty}\binom{p}{n}(t^n+s^n)x^{-n}}{\sum_{n=0}^{\infty}\binom{r}{n}(t^n+s^n)x^{-n}}\right)^{\frac{1}{p-r}}\\
&= x\left(\frac{\sum_{n=0}^{\infty}a_n(p)x^{-n}}{\sum_{n=0}^{\infty}a_n(r)x^{-n}}\right)^{\frac{1}{p-r}}
\end{aligned}
$$

The rest of the proof is same as in Theorem 4.1. □

Using this theorem we get

$$
\begin{aligned}
&c_0 = 1,\\
&c_1 = \alpha,\\
&c_2 = \tfrac{1}{2}(p+r-1)\beta^2,\\
&c_3 = -\tfrac{1}{2}(p+r-1)\alpha\beta^2,\\
&c_4 = \tfrac{1}{24}\beta^2[12(p+r-1)\alpha^2\\
&\qquad +(-3-2p^3+p^2(3-2r)+2r+3r^2-2r^3+p(2+6r-2r^2))\beta^2],\\
&c_5 = \tfrac{1}{8}\alpha\beta^2[-4(-1+p+r)\alpha^2\\
&\qquad +(3+2p^3-2r-3r^2+2r^3+p^2(-3+2r)+2p(-1-3r+r^2))\beta^2],\\
&\ \vdots
\end{aligned}
$$

COROLLARY 5.2. (Lehmer mean) *The asymptotic expansion of Lehmer mean*

$$G_{r+1,r}(s,t) = \frac{t^{r+1}+s^{r+1}}{t^r+s^r}$$

reads as folows:

$$G_{r+1,r}(x+s,x+t) = x\sum_{n=0}^{\infty}a_n x^{-n}$$

where

$$
\begin{aligned}
c_0 &= 1,\\
c_n &= \frac{1}{n}\sum_{k=1}^{n}(2k-n)b_k c_{n-k},\\
b_n &= a_n(r+1)-\sum_{k=1}^{n}a_k(r)b_{n-k}.
\end{aligned}
$$

The first few coefficients are

$$\begin{aligned}
c_0 &= 1,\\
c_1 &= \alpha,\\
c_2 &= r\beta^2,\\
c_3 &= -r\alpha\beta^2,\\
c_4 &= \tfrac{1}{3}r\beta^2[3\alpha^2-(r^2-1)\beta^2],\\
c_5 &= r\alpha\beta^2[-\alpha^2+(r^2-1)\beta^2],\\
c_6 &= \frac{1}{15}r\beta^2[15\alpha^4-30(r^2-1)\alpha^2\beta^2+(2r^4-5r^2+3)\beta^4],\\
&\;\vdots
\end{aligned}$$

6. Comparison of means

In this section, we will establish asymptotic inequalities, including order of inequality, between observed means analysing coefficients in their asymptotic expansion. There are relevant information about asymptotic series and order symbol $\mathscr{O}$ in [13]. As a consequence, we will get a method for obtaining necessary conditions for inequalities between means.

In the sequel, we shall use the following definition

DEFINITION 6.1. Let F_1 and F_2 be any two means, and

$$F_1(x+s,x+t)-F_2(x+s,x+t)=c_k(t,s)x^{-k+1}+\mathscr{O}(x^{-k}). \tag{6.1}$$

If $c_k(s,t)>0$ for all s and t, then we say that mean F_1 is *asymptotically* greater than mean F_2, and write

$$F_1 \succ F_2.$$

Of course, this is equivalent to

$$F_2 \prec F_1.$$

Suppose

$$F_1(x-\beta,x+\beta)-F_2(x-\beta,x+\beta)=c_k(0,\beta)x^{-k+1}+\mathscr{O}(x^{-k})$$

where $c_k(0,\beta)>$ for all $\beta>0$. Then

$$\begin{aligned}
F_1(x+\alpha-\beta,x+\alpha+\beta)-F_2(x+\alpha-\beta,x+\alpha+\beta) &= c_k(0,\beta)(x+\alpha)^{-k+1}+\mathscr{O}(x^{-k})\\
&= c_k(0,\beta)x^{-k+1}+\mathscr{O}(x^{-k}).
\end{aligned}$$

Hence, it is sufficient to observe the case $\alpha=0$.

Asymptotical inequalities are necessary condition for the usual inequalities between means: if $F_1 \geqslant F_2$, then $F_1 \succ F_2$. Namely, for x large enough, the sign of the difference $F_1(x+s,x+t)-F_2(x+s,x+t)$ is the same as the sign of the first term in its asymptotic expansion. See [8] for detailed analysis of relation between asymptotic and true inequalities between means.

In the case $\alpha=0$ we have derived:

$$L_r(x-\beta,x+\beta)=x+\frac{(r-1)\beta^2}{6x}-\frac{(r-1)(2r^2+5r-13)\beta^4}{360x^3}+\cdots \tag{6.2}$$

$$M_r(x-\beta,x+\beta)=x+\frac{(r-1)\beta^2}{2x}-\frac{(r-1)(r+1)(2r-3)\beta^4}{24x^3}+\cdots \tag{6.3}$$

$$E_{p,r}(x-\beta,x+\beta)=x+\frac{(p+r-3)\beta^2}{6x}+\frac{((p+r-3)(7+11p-2p^2-r-2r^2)-12(p-1)(p-2))\beta^4}{360x^3}+\cdots \tag{6.4}$$

$$G_{p,r}(x-\beta,x+\beta)=x+\frac{(p+r-1)\beta^2}{2x}+\frac{((p+r-1)(3+5p-2p^2+r-2r^2)-4p(p-1))\beta^4}{2x^3}+\cdots \tag{6.5}$$

As explained in [8], other necessary conditions will be derived observing Laurent series of difference $F_1(s,\frac{1}{s})-F_2(s,\frac{1}{s})$ near $s=0$. It is easy to derive the following expansions:

$$G_{p,r}\left(s,\tfrac{1}{s}\right)=\begin{cases} s^{-1}\left(1+\frac{1}{p-r}s^{2p}-\frac{1}{p-r}s^{2r}+\cdots\right), & 0<p<r,\\ s^{\frac{p+r}{p-r}}\left(1+\frac{1}{p-r}s^{-2p}-\frac{1}{p-r}s^{2r}+\cdots\right), & p<0<r,\\ s\left(1-\frac{1}{p-r}s^{-2r}+\frac{1}{p-r}s^{-2p}+\cdots\right), & p<r<0. \end{cases} \tag{6.6}$$

$$E_{p,r}\left(s,\tfrac{1}{s}\right)=\begin{cases} \left(\frac{r}{p}\right)^{\frac{1}{p-r}}s^{-1}\left(1-\frac{1}{p-r}s^{2p}+\frac{1}{p-r}s^{2r}+\cdots\right), & 0<p<r,\\ \left(\frac{r}{-p}\right)^{\frac{1}{p-r}}s^{\frac{p+r}{p-r}}\left(1-\frac{1}{p-r}s^{-2p}+\frac{1}{p-r}s^{2r}+\cdots\right), & p<0<r,\\ \left(\frac{r}{p}\right)^{\frac{1}{p-r}}s\left(1+\frac{1}{p-r}s^{-2r}-\frac{1}{p-r}s^{-2p}+\cdots\right), & p<r<0. \end{cases} \tag{6.7}$$

$$L_q\left(s,\tfrac{1}{s}\right)=\begin{cases} \left(\frac{1}{q+1}\right)^{\frac{1}{q}}s^{-1}\left(1-\frac{1}{q}s^{2(q+1)}+\cdots\right)\left(1+\frac{1}{q}s^2+\cdots\right), & 0<q+1,\\ \left(-\frac{1}{q+1}\right)^{\frac{1}{q}}s^{\frac{q+2}{q}}\left(1-\frac{1}{q}s^{-2(q+1)}+\cdots\right)\left(1+\frac{1}{q}s^2+\cdots\right), & q+1<0. \end{cases} \tag{6.8}$$

First, we compare two Gini means.

6.1. Gini means

We have

$$G_{p,r}(x-\beta,x+\beta)-G_{u,v}(x-\beta,x+\beta)=\frac{(p+r-u-v)\beta^2}{2}x^{-1}+\mathscr{O}(x^{-3})$$

Thus, we have asymptotic inequality of order $\mathscr{O}(x^{-1})$

$$G_{p,r}\succ G_{u,v},\qquad \text{for } p+r>u+v. \tag{6.9}$$

Let $p+r=u+v$. Then

$$G_{p,r}(x-\beta,x+\beta)-G_{p+r-v,v}(x-\beta,x+\beta)=\frac{(p+r)(p-v)(r-v)\beta^4}{6x^3}+\mathscr{O}(x^{-5}),$$

and we conclude

$$G_{p,r}\succ G_{p+r-v,v},\qquad \text{for } (p+r)(p-v)(r-v)>0.$$

and the approximation is of order $\mathscr{O}(x^{-3})$.

We could go further by choosing parameters such that coefficient by x^{-3} vanishes, but in this case that leads to identities

$$\begin{aligned}G_{-r,r}-G_{-v,v}&=0,\\ G_{v,r}-G_{r,v}&=0,\\ G_{p,v}-G_{p,v}&=0.\end{aligned}$$

Suppose $G_{p,r}\leqslant G_{u,v}$. Without loss of generality we may assume that $p<r$ and $u<v$. If $p,r,u,v>0$ then

$$s^{-1}\left(1+\frac{1}{p-r}s^{2p}+\cdots\right)\leqslant s^{-1}\left(1+\frac{1}{u-v}s^{2u}+\cdots\right)$$

and since s is close to 0 it follows $p\leqslant u$. Similarly, if $p,r,u,v<0$ we obtain $r\leqslant v$. Otherwise, it is necessary that leading power in Taylor expansion of $G_{p,r}$ be greater than leading power in Taylor expansion of $G_{u,v}$. Furthermore, in each of three cases that power equals $-\frac{|x|-|y|}{x-y}$ where x,y stands for either p,r or u,v. Hence, $\frac{|p|-|r|}{p-r}\leqslant\frac{|u|-|v|}{u-v}$.

We have just proved the necessity of conditions given in the following theorem from [22].

THEOREM A. *Let p,r,u,v be arbitrary real numbers with $p\neq r$ and $u\neq v$. Then inequality*

$$G_{p,r}(s,t)\leqslant G_{u,v}(s,t)$$

holds for all positive s and t if an only if

$$p+r\leqslant u+v \tag{6.10}$$

and

$$m(p,r) \leqslant m(u,v), \tag{6.11}$$

where

$$m(x,y) = \begin{cases} \min(x,y) & \text{if } 0 \leqslant \min(p,r,u,v), \\ \frac{|x|-|y|}{x-y} & \text{if } \min(p,r,u,v) < 0 < \max(p,r,u,v), \\ \max(x,y) & \text{if } \max(p,r,u,v) \leqslant 0. \end{cases}$$

6.2. Gini mean and power mean.

Although power mean is one of Gini means, we will compare those two means separately.

$$G_{p,r}(x-\beta,x+\beta) - M_q(x-\beta,x+\beta) = \frac{(-q+p+r)\beta^2}{2x} + \mathscr{O}(x^{-3}).$$

Thus

$$G_{p,r} \succ M_q, \quad \text{for } p+r > q. \tag{6.12}$$

If $q = p+r$, we have

$$G_{p,r}(x-\beta,x+\beta) - M_{p+r}(x-\beta,x+\beta) = \frac{pr(p+r)\beta^4}{6x^3} + \mathscr{O}(x^{-5}).$$

And if $pr(p+r) = 0$ then $G_{p,r} - M_{p+r} = 0$.

Similarly as in the case of two Gini means, we can compare Laurent series (6.6) with Laurent series of power mean M_q. We give the necessary conditions in the following theorem.

THEOREM 6.2. *Let p,q,r be real parameters, with $p \neq r$. If the inequality*

$$M_q(s,t) \leqslant G_{p,r}(s,t)$$

holds for all $s,t > 0$, then the following conditions must be satisfied:

1. $q \leqslant p+r$,
2. $0 < p < r$ *or* $q < 0$, $p < 0 < r$.

Notice that these conditions are also the sufficient ones which we see using Theorem A with $M_q = G_{0,q}$.

6.3. Gini mean and logarithmic mean.

$$G_{p,r}(x-\beta,x+\beta) - L_q(x-\beta,x+\beta) = \frac{(3p+3r-q-2)\beta^2}{6x} + \mathscr{O}(x^{-3})$$

We have

$$G_{p,r} \succ L_q, \qquad \text{for } p+r > \tfrac{1}{3}(q+2). \tag{6.13}$$

If $p+r = \frac{1}{3}(q+2)$, then

$$G_{(q+2-3r)/3,r}(x-\beta, x+\beta) - L_q(x-\beta, x+\beta) = \\ = \frac{(q+2)\left(-1-q+2q^2+15(2+q)r-45r^2\right)\beta^4}{810x^3} + \mathscr{O}(x^{-5}).$$

Therefore, in order to $G_{p,r} \succ L_q$ be true, we must have the following

$$(q+2)\left(-1-q+2q^2+15(2+q)r-45r^2\right) \geqslant 0.$$

Let

$$(q+2)\left(-1-q+2q^2+15(2+q)r-45r^2\right) = 0.$$

In that case either $q=-2$ which gives

$$G_{-r,r} - L_{-2} = 0,$$

or

$$r = \frac{1}{30}\left(10+5q \pm \sqrt{5}\sqrt{16+16q+13q^2}\right)$$

wherefrom it follows

$$G_{(q+2-3r)/3,r}(x-\beta, x+\beta) - L_q(x-\beta, x+\beta) = \\ = -\frac{4(q-1)(q+2)(2q+1)(17q^2+59q+59)\beta^6}{1913625x^5} + \mathscr{O}(x^{-7}).$$

Thus,

$$G_{(q+2-3r)/3,r} \prec L_q, \qquad \text{for } (q-1)(q+2)(2q+1) > 0.$$

If also

$$(q-1)(q+2)(2q+1) = 0,$$

then

$$G_{(q+2-3r)/3,r} - L_q = 0.$$

Now we present the main necessary conditions for comparison of Gini and generalized logarithmic mean.

THEOREM 6.3. *Let p,r,q be real parameters, $p \neq r$, $q \neq -1,0$. If*

$$G_{p,r}(s,t) \geqslant L_q(s,t) \tag{6.14}$$

holds for all $s,t>0$ then following conditions must be satisfied:

1. $p+r \geqslant \frac{1}{3}(q+2)$,
2. $0<p<r$ *or* $p<0<r$, $q+1<0$, $\frac{p+r}{p-r} \leqslant \frac{q+2}{q}$.

If $p+r=\frac{1}{3}(q+2)$, *then*

Proof. First condition follows from asymptotic inequality (6.13). Second necessary condition follows from expansions (6.6) and (6.8). Namely, if $0<p<r$ and $q+1>0$ then leading term in series expansion of $G_{p,r}\left(s,\frac{1}{s}\right)$ is s^{-1} and leading term of $L_q\left(s,\frac{1}{s}\right)$ is $\left(\frac{1}{q+1}\right)^{1/q}$ so it must be $1\geqslant\left(\frac{1}{q+1}\right)^{1/q}$ which is always true for $q+1>0$. If $0<p<r$ and $q+1<0$ then Gini mean is greater than generalized logarithmic mean only if leading power of the first one is smaller than leading power of second one, that is $-1\leqslant(q+2)/q$ which is also always true for $q+1<0$. In the same way we conclude that in case of $p<0<r$ it must be $q+1<0$ and $(p+r)/(p-r)\leqslant(q+2)/q$. Similarly, we see that the $p<r<0$ is not possible. □

6.4. Power mean and logarithmic mean.

$$M_r(x-\beta,x+\beta)-L_q(x-\beta,x+\beta)=\frac{(3r-q-2)\beta^2}{6x}+\mathscr{O}(x^{-3}).$$

We have

$$M_r\succ L_q,\qquad \text{for } 3r>q+2.$$

Let $3r=q+2$. Then

$$M_r(x-\beta,x+\beta)-L_{3r-2}(x-\beta,x+\beta)=\frac{r(r-1)(2r-1)\beta^4}{30x^3}+\mathscr{O}(x^{-5})$$

and we have

$$M_r\succ L_{3r-2},\qquad \text{for } r(r-1)(2r-1)>0.$$

Let

$$r(r-1)(2r-1)=0.$$

It can easily be seen that

$$M_0(x-\beta,x+\beta)-L_{-2}(x-\beta,x+\beta)=\sqrt{x^2-\beta^2}-\sqrt{x^2-\beta^2}=0,$$

$$M_1(x-\beta,x+\beta)-L_1(x-\beta,x+\beta)=x-x=0$$

and

$$M_{1/2}(x-\beta,x+\beta)-L_{-1/2}(x-\beta,x+\beta)=\frac{1}{2}\left(x+\sqrt{x^2-\beta^2}\right)-\frac{\beta^2}{2x-2\sqrt{x^2-\beta^2}}=0.$$

Stolarsky in [24] proved the following theorem:

THEOREM B. *If* $-1<a<\frac{1}{2}$ *or* $2<a$ *then*

$$L_{a-1}\leqslant M_{(a+1)/3}.$$

If $a<-1$ *or* $\frac{1}{2}<a<2$, *then reversed inequality holds.*

This coincides with our observation. Indeed, for $q=a-1$ and $r=\frac{a+1}{3}$ we have $3r=q+2$, and the condition $-1<a<\frac{1}{2}$ or $2<a$ is equivalent to $r(r-1)(2r-1)>0$.

6.5. Stolarsky mean.

$$E_{p,r}(x-\beta,x+\beta)-E_{u,v}(x-\beta,x+\beta)=\frac{(p+r-u-v)\beta^2}{6x}+\mathscr{O}(x^{-3})$$

The following asymptotic inequalities of order $\mathscr{O}(x^{-1})$ and $\mathscr{O}(x^{-3})$ are valid

$$E_{p,r}\succ E_{u,v}\quad \text{for } p+r>u+v, \tag{6.15}$$

and

$$E_{p,r}\succ E_{u,v}\quad \text{for } (p+r)(p-v)(r-v)>0,$$

since

$$E_{p,r}(x-\beta,x+\beta)-E_{p+r-v,v}(x-\beta,x+\beta)=\frac{(p+r)(p-v)(r-v)\beta^4}{90x^3}+\mathscr{O}(x^{-5}).$$

In [21] Páles proved the following theorem.

THEOREM C. *Let p,r,u,v be arbitrary with $p\neq r$ and $u\neq w$. Then*

$$E_{p,r}(s,t)\leqslant E_{u,v}(s,t)$$

is satisfied for all $s,t>0$ if and only if

$$r+s\leqslant u+v \tag{6.16}$$

and

$$e(r,s)\leqslant e(u,v), \tag{6.17}$$

where

$$e(x,y)=\begin{cases}\frac{x-y}{\log(x/y)}, & xy<0, x\neq y,\\ 0, & xy=0,\end{cases}$$

if $0\leqslant \min(p,r,u,v)$ or $\max(p,r,u,v)\leqslant 0$, and

$$e(x,y)=\frac{|x|-|y|}{x-y}\ \textit{for}\ x\neq y$$

if $\min(p,r,u,v)<0<\max(p,r,u,v)$.

Note that (6.16) is a consequence of asymptotic inequality (6.15) and (6.17) follows from (6.7) as in the case of comparison of two Gini means.

6.6. Gini mean and Stolarsky mean.

$$G_{p,r}(x-\beta,x+\beta)-E_{u,v}(x-\beta,x+\beta)=\frac{(3p+3r-u-v)\beta^2}{6x}+\mathscr{O}(x^{-3})$$

We have

$$G_{p,r}\succ E_{u,v},\qquad \text{for } 3p+3r>u+v.$$

Let $3p+3r=u+v$. Then we have

$$G_{p,r}(x-\beta,x+\beta)-E_{3p+3r-v,v}(x-\beta,x+\beta)= \\ =\frac{(p+r)(2p^2+9pr+2r^2-3pv-3rv+v^2)\beta^4}{30x^3}+\mathscr{O}(x^{-5}).$$

The following asymptotic inequality of order $\mathscr{O}(x^{-3})$ holds

$$G_{p,r}\succ E_{3p+3r-v,v},\qquad \text{for } (p+r)(2p^2+9pr+2r^2-3pv-3rv+v^2)>0.$$

Let

$$(p+r)(2p^2+9pr+2r^2-3pv-3rv+v^2).$$

If $p+r=0$, then

$$G_{-r,r}(x-\beta,x+\beta)-E_{-v,v}(x-\beta,x+\beta)=0.$$

If

$$2p^2+9pr+2r^2-3pv-3rv+v^2=0,$$

or equivalent

$$v=\frac{1}{2}\left(3(p+r)\pm\sqrt{p^2-18pr+r^2}\right),$$

then

$$G_{p,r}(x-\beta,x+\beta)-E_{3p+3r-v,v}(x-\beta,x+\beta)= \\ =\frac{4pr(p+r)\left(3p^2+5pr+3r^2\right)\beta^6}{315x^5}+\mathscr{O}(x^{-7}).$$

Now we want to obtain even better approximation by putting

$$pr(p+r)\left(3p^2+5pr+3r^2\right)=0.$$

We have the following possibilities

$$p=0,v=r;\ p=0,v=2r;\ r=0,v=p;\ r=0,v=2p;\ p+r=0$$

which all give $G_{p,r}(x-\beta,x+\beta)-E_{3p+3r-v,v}(x-\beta,x+\beta)=0$.

In [19] was shown that $E_{p,r}(s,t) < G_{p,r}(s,t)$ for all $s,t > 0$, $t \neq s$, if and only if $p+r > 0$. In that paper were also given sufficient conditions for inequality

$$E_{u,v} \leqslant G_{p,r}$$

which are

$$\min(u,v) \leqslant \min(p,r), \ \max(u,v) \leqslant \max(p,r) \text{ and } p+r \geqslant 0.$$

In [7] authors gave the necessary conditions for comparison of Gini and Stolarsky mean with different parameters as follows.

THEOREM D. *Suppose that the inequality*

$$G_{p,r}(s,t) \leqslant E_{u,v}(s,t).$$

holds for any positive s,t. *Then*

1. $3(p+r) \leqslant u+v$
2. $\min(p,r) \leqslant 0$ *and if* $\min(p,r) = 0 < \max(p,r)$ *then* $\max(p,r) < \log 2 \cdot l(u,v)$,
3. $\mu(p,r) \leqslant \mu(u,v)$,

where

$$l(u,v) = \begin{cases} \frac{u-v}{\log(u/v)}, & 0 < uv, u \neq v, \\ u, & 0 < uv, u = v, \\ 0, & \textit{otherwise}, \end{cases}$$

and

$$\mu(u,v) = \begin{cases} \frac{|u|-|v|}{u-v}, & u \neq v, \\ \operatorname{sgn}(u), & u = v. \end{cases}$$

These conditions can be obtained analogously as in the subsections 6.1 and 6.5.

6.7. Power mean and Stolarsky mean.

The necessary condition for comparison of these two means is as follows:

THEOREM 6.4. *Let* p,r,q *be real parameters. If*

$$M_q(s,t) \leqslant E_{p,r}(s,t) \tag{6.18}$$

holds for all $s,t > 0$ *then it is necessarily* $3q \leqslant p+r$ *and one of the following conditions must be satisfied:*

1. *if* $0 < q,\ 0 < p < r$ *or* $q < 0,\ p < r < 0$ *then* $q \leqslant \log 2 \frac{r-p}{\log \frac{r}{p}}$,
2. $q > 0,\ 0 < p < r$ *or* $q < 0,\ p < 0 < r$.

The proof follows from the known asymptotic expansions and Laurent series of considered means. By observing Theorem C we notice that these conditions are also sufficient for (6.18) to hold, since $M_q = E_{q,2q}$.

6.8. Stolarsky mean and logarithmic mean.

$$E_{p,r}(x-\beta,x+\beta)-L_q(x-\beta,x+\beta)=\frac{(p+r-q-2)\beta^2}{6x}+\mathscr{O}(x^{-3}).$$

We have

$$E_{p,r}\succ L_q,\qquad \text{for } p+r>q+2.$$

Let $p+r=q+2$.

$$E_{p,r}(x-\beta,x+\beta)-L_{p+r-2}(x-\beta,x+\beta)=\frac{(p-1)(r-1)(p+r)\beta^4}{90x^3}+\mathscr{O}(x^{-5}).$$

Then

$$E_{p,r}\succ L_q,\qquad \text{for } (p-1)(r-1)(p+r)>0.$$

THEOREM 6.5. *Let p,r,q be real parameters, $p\neq r$, $p,r\neq 0$, $q\neq -1,0$. If the inequality*

$$E_{p,r}(s,t)\geqslant L_q(s,t) \tag{6.19}$$

holds for all $s,t>0$ then

$$p+r>q+2 \tag{6.20}$$

and one of the following conditions must be satisfied

1. $0<p<r,\ q+1>0,\ \left(\frac{r}{p}\right)^{\frac{1}{p-r}}\geqslant\left(\frac{1}{q+1}\right)^{\frac{1}{q}}$,
2. $0<p<r,\ q+1<0$,
3. $p<0<r,\ q+1<0,\ \frac{p+r}{p-r}\leqslant\frac{q+2}{q}$.

Proof of this theorem is the same as the proof of Theorem 6.3.

REMARK 6.6. Asymptotic expansions are also convenient for finding an intersection of different classes of parameter means. For example, equating coefficients in asymptotic expansion of Gini and Stolarsky mean we obtained parameters for which those means can be identical. That problem was also studied in [15] for Stolarsky and Lehmer mean and in [1] for Stolarsky and Gini means.

REFERENCES

[1] H. ALZER, S. RUSCHEWEYH, *On the intersection of two-parameter mean value families*, Proc. Amer. Math. Soc., **129** 9 (2001), 2655–2662.

[2] P. S. BULLEN, *Handbook of Means and Their Inequalities*, Kluwer Academic Publishers, 2003.

[3] P. S. BULLEN, D. S. MITRINOVIĆ, P. M. VASIĆ, *Means and theirs inequalities*, D Reidel, Dordrecht, 1988.

[4] CHAO-PING CHEN, *Asymptotic representations for Stolarsky, Gini and the generalized Muirhead means*, JIPAM **11** 4.

[5] CHAO-PING CHEN, NEVEN ELEZOVIĆ AND LENKA VUKŠIĆ, *Asymptotic formulae associated with the Wallis power function and digamma function*, Journal of Classical Analysis **2**, 2 (2013), 151–166.
[6] P. CZINDER, ZS. PÁLES, *Local monotonicity properties of two-variable Gini means and the comparison theorem revisited*, J. Math. Anal. Appl. no. 301 (2005) 427–438.
[7] P. CZINDER, ZS. PÁLES, *Some comparison inequalities for Gini and Stolarsky means*, Math. Inequal. Appl. vol. 9 no. 4 (2006) 607–616.
[8] N. ELEZOVIĆ, *Asymptotic inequalities and comparison of classical means*, J. Math. Inequal. (to appear).
[9] N. ELEZOVIĆ, C. GIORDANO AND J. PEČARIĆ, *The best bounds in Gautschi's inequalities*, Math. Inequal. Appl., **3** (2000), 239–252.
[10] N. ELEZOVIĆ AND J. PEČARIĆ, *Differential and integral f-means and applications to digamma function*, Math. Inequal. Appl. **3** (2000), 189–196.
[11] N. ELEZOVIĆ AND L. VUKŠIĆ, *Asymptotic expansions of bivariate classical means and related inequalities*, J. Math. Inequal. **8**, 4 (2014), 707–724.
[12] N. ELEZOVIĆ AND L. VUKŠIĆ, *Asymptotic expansions of integral means and applications to the ratio of gamma functions*, Appl. Math. Comput. (2014), `http://dx.doi.org/10.1016/j.amc.2014.02.026`
[13] A. ERDÉLYI, *Asymptotic expansions*, Dover Publications, 1956.
[14] H. W. GOULD, *Coefficient identities for powers of Taylor and Dirichlet series*, The American Mathematical Monthly, **81**, 1 (1974) 3–14.
[15] H. W. GOULD, M. E. MAYS, *Series expansions of means*, J. Math. Anal. Appl. **101** (1984), 611–621.
[16] E. B. LEACH, M. C. SHOLANDER, *Extended mean values*, The American Mathematical Monthly, **85**, 2 (1978) 84–90.
[17] E. B. LEACH, M. C. SHOLANDER, *Extended mean values II*, J. Math. Anal. Appl. **92** (1983) 207–223.
[18] D. S. MITRINOVIĆ, J. E. PEČARIĆ, A. M. FINK, *Classical and New Inequalities in Analysis*, D Reidel, Dordrecht, 1993.
[19] E. NEUMAN, ZS. PÁLES, *On comparison of Stolarsky and Gini means*, J. Math. Anal. Appl. **278** (2003) 274–284.
[20] E. NEUMAN, J. SANDOR, *Inequalities involving Stolarsky and Gini means*, Mathematica Pannonica 14/1 (2003) 29–44.
[21] ZS. PÁLES, *Inequalities for differences of powers*, J. Math. Anal. Appl. **131** (1988) 271–281.
[22] ZS. PÁLES, *Inequalities for sums of powers*, J. Math. Anal. Appl. **131** (1988) 265–270.
[23] K. B. STOLARSKY, *Generalizations of the Logarithmic Mean*, Mathematics Magazine **48**, 2 (1975), 87–92.
[24] K. B. STOLARSKY, *The power and generalized logarithmic means*, Amer. Math. Monthly **87**, 7 (1980), 545–548.

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