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UPPER BOUNDS FOR THE COVERING NUMBER OF CENTRALLY SYMMETRIC CONVEX BODIES IN $\mathbb{R}^n$

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Abstract. The covering number $c(K)$ of a convex body K is the least number of smaller homothetic copies of K needed to cover K. We provide new upper bounds for $c(K)$ when K is centrally symmetric by introducing and studying the generalized α-blocking number $\beta_2^{\alpha}(K)$ of K. It is shown that when a centrally symmetric convex body K is sufficiently close to a centrally symmetric convex body K', then $c(K)$ is bounded by $\beta_2^{\alpha}(K')$ from above, where α is a properly chosen number. Related results in Minkowski geometry are also presented.

1. Introduction

For an integer $m > 1$ we denote by $[m]$ the set $\{1,\cdots,m\}$. The *interior*, *boundary*, and *closure* of a set $A \subset \mathbb{R}^n$ is denoted by $\mathrm{int}A$, $\mathrm{bd}A$, and $\mathrm{cl}A$, respectively. Let A and B be two sets in $\mathbb{R}^n$, and λ be a real number. Set

$$A+B := \{x+y : \quad x \in A, y \in B\} \quad \text{and} \quad \lambda A := \{\lambda x : \quad x \in A\}.$$

For each positive number λ and each point $x \in \mathbb{R}^n$, the set

$$\lambda A + x := \lambda A + \{x\}$$

is called a *homothetic copy* of A. If $\lambda \in (0,1)$, then $\lambda A + x$ is called a *smaller homothetic copy* of A. The set $A + x$ is called a *translate* of A. Let $A_1,\cdots,A_m$ be some sets in $\mathbb{R}^n$. If

$$A \subseteq \bigcup_{i=1}^{m} A_i,$$

then we say that A is *covered* by these m sets.

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A compact convex set in $\mathbb{R}^n$ having interior points is called a *convex body*. The set of convex bodies in $\mathbb{R}^n$ is denoted by $\mathscr{K}^n$, and the set of centrally symmetric convex bodies in $\mathscr{K}^n$ is denoted by $\mathscr{C}^n$. Let x be a boundary point of a convex body $K \in \mathscr{K}^n$, u be a vector in $\mathbb{R}^n$ distinct from the *origin* o (called a *direction*), and p be a point exterior to K. We say that x is *illuminated by the direction* u if there exists a positive number λ such that $x+\lambda u \in \mathrm{int}K$; x is *illuminated by the point* p if the ray starting from p and passing through x intersects $\mathrm{int}K$ while the segment between p and x does not. Let $U = \{u_i : \quad i \in [m]\}$ be a set of directions, and $P = \{p_i : \quad i \in [k]\}$ be a set of points exterior to K. We say that $\mathrm{bd}K$ is *illuminated* by U (by P, resp.) if each point in $\mathrm{bd}K$ is illuminated by one of the directions in U (one of the points in P, resp.).

For a convex body K in $\mathscr{K}^n$, the *covering number* $c(K)$ of K is the least number of translates of $\mathrm{int}K$ needed to cover K. It turns out (cf. Chapter V of [8]) that $c(K)$

- equals the least number of smaller homothetic copies of K required to cover K;
- equals the least cardinality of a set of directions that can illuminate $\mathrm{bd}K$;
- equals the least cardinality of a set of points exterior to K that can illuminate $\mathrm{bd}K$.

Concerning the upper bound for $c(K)$, we have the following well-known conjecture due to Hadwiger [15] and also to Gohberg and Markus [14]:

CONJECTURE 1. (Hadwiger's covering conjecture) *If $K \in \mathscr{K}^n$, then*

$$n+1 \leqslant c(K) \leqslant 2^n,$$

and the equality $c(K) = 2^n$ holds if and only if K is a parallelotope.

This conjecture is completely confirmed only when $n = 2$, and is essentially open even when $n = 3$. It is also verified for several classes of convex bodies, such as convex bodies having smooth boundary, belt bodies, and convex bodies of constant width (the latter when $n \geqslant 16$). See the monographs [8], [10], and [3], and the surveys [21] and [2] for more information about this conjecture and for further references. See also [28] and [4] for more recent results concerning this conjecture.

In this paper we shall consider the upper bound of $c(K)$ when $K \in \mathscr{C}^n$. An important progress in this direction has been made by Lassak [19]. He proved that $c(K) \leqslant 8$ holds for each $K \in \mathscr{C}^3$. A general upper bound for $c(K)$ when $K \in \mathscr{C}^n$ was obtained by Zong [27]. Zong proved that, for each $K \in \mathscr{C}^n$, $c(K) \leqslant \beta(K)$, where $\beta(K)$, called the *blocking number* of K, is the least integer m such that there exist m non-overlapping translates $K_1, \cdots, K_m$ of K satisfying the following properties:

1. K_i touches K at $\mathrm{bd}K$, i.e., $\emptyset \neq K_i \cap K \subset \mathrm{bd}K$, for each $i \in [m]$;
2. $K_1, \cdots, K_m$ block any other translate of K from touching K, i.e., if K' is a translate of K touching K at $\mathrm{bd}K$, then there exists an $i \in [m]$ such that $K' \cap \mathrm{int}K_i \neq \emptyset$.

For each $K \in \mathscr{K}^n$, $\beta(K) = \beta(K-K)$ and $\beta(K)$ is bounded by the kissing number of K from above (cf. [27] and [26]). In general, $\beta(K)$ is not easy to compute. Several known examples are:

- if K is an n-dimensional parallelotope, then $\beta(K) = 2^n$, cf. [12, Theorem 4];
- if K is the unit ball of $\mathbb{R}^3$, then $\beta(K) = 6$, cf. [12, Theorem 5];
- if K is the unit ball of $\mathbb{R}^4$, then $\beta(K) = 9$, cf. [12, Theorem 7];
- if K is a regular octahedron then, $\beta(K) = 6$, cf. [25, Theorem 1.1].

In [27], the following conjecture was posed.

CONJECTURE 2. *For each convex body K in $\mathbb{R}^n$ we have*

$$2n \leqslant \beta(K) \leqslant 2^n,$$

and the equality $\beta(K) = 2^n$ holds if and only if K is a parallelotope.

Clearly, if Conjecture 2 is true, then the Hadwiger's covering conjecture is also true for centrally symmetric convex bodies.

We shall provide a new upper bound for $c(K)$ when $K \in \mathscr{C}^n$ by introducing and studying the generalized α-blocking number $\beta_2^\alpha(K)$ of K (cf. Definition 9 in Section 3). A convex body $K \in \mathscr{C}^n$ will be viewed as a ball of a *Minkowski space* (a real finite dimensional Banach space) so that we can apply several tools and known and new results from the geometry of Minkowski spaces, some of which are listed in the next section. We show that when $K \in \mathscr{C}^n$ is sufficiently close to $K' \in \mathscr{C}^n$, then $c(K)$ is bounded by $\beta_2^\alpha(K')$, where α is a properly chosen number related to K (cf. Theorem 13 in Section 3). This fact can be used to overcome the upper semi-continuity of $c(K)$. When

$$\alpha = \frac{2}{2 - RC(X)} - 1,$$

where X is the Minkowski space corresponding to K (cf. Section 2 below) and $RC(X)$ is a number determined by X which is closely related to illuminating and covering K (cf. Definition 7 in Section 2), we obtain a new constant $\beta'(K) := \beta_2^\alpha(K)$ (cf. Definition 15 in Section 3) satisfying

$$1 + n \leqslant c(K) \leqslant \beta'(K) \leqslant \beta_2^1(K) \leqslant \beta(K).$$

Clearly, the upper bound $\beta'(K)$ of $c(K)$ is tighter than $\beta(K)$. Estimations of $RC(X)$ are presented in Section 4.

2. Preliminaries

Let K be a convex body in $\mathscr{C}^n$ centered at the origin o. Then K induces a norm $\|\cdot\|$ via its *Minkowski functional*

$$\|x\| := \inf\{\lambda > 0 : \quad x \in \lambda K\}$$

and is the *unit ball* B_X of the Minkowski space $X = (\mathbb{R}^n, \|\cdot\|)$. Each Minkowski space appearing in this paper has this form. The *unit sphere* S_X of X, i.e., the set of unit vectors of X, is precisely the boundary bd$K$ of K. A unit vector in X is also called a *direction*. For two linearly independent unit vectors u and v, we set

$$\operatorname{arc}(u,v) := \left\{ \frac{\alpha u+\beta v}{\|\alpha u+\beta v\|} : \quad \alpha,\beta \geqslant 0, \alpha+\beta>0 \right\}.$$

For two distinct points x and y in X, we denote by $[x,y]$ the *segment* between x and y, i.e.,

$$[x,y] = \{\lambda x + (1-\lambda)y : \quad \lambda \in [0,1]\}.$$

Since $c(K) = c(T(K))$ holds for each non-singular affine transformation T, we shall use the *Banach-Mazur distance*

$$d(K_1,K_2) := \min\{\gamma \geqslant 1 : \ K_1 \subset T(K_2) \subset \gamma K_1 + x, \ x \in \mathbb{R}^n, \ T \in \mathscr{A}^n\}$$

to measure the difference between two convex bodies K_1 and K_2 in $\mathscr{K}^n$, where $\mathscr{A}^n$ is the set of non-degenerate affine transforms from $\mathbb{R}^n$ to $\mathbb{R}^n$.

If X and Y are two Minkowski spaces, then $d(B_X,B_Y)$ equals to the *Banach-Mazur distance* $d(X,Y)$ between X and Y, which is defined by

$$d(X,Y) := \min\{\|T\| \cdot \left\|T^{-1}\right\| : \quad T : X \mapsto Y \text{ is a linear isomorphism}\}.$$

2.1. Birkhoff orthogonality, shadow boundaries

Let x and y be two vectors in X. If

$$\|x+\alpha y\| \geqslant \|x\|$$

holds for each real number α, then x is said to be *Birkhoff orthogonal to* y ($x \perp_B y$), cf. [5], [18], and [1]. Clearly, if $x,y \neq o$, then $x \perp_B y$ if and only if the line $\{x+\alpha y : \alpha \in \mathbb{R}\}$ supports $\|x\| B_X$ (meets $\|x\| B_X$ but not its interior). This concept is closely related to the shadow boundary (see the definition below) of B_X.

DEFINITION 1. (cf. p. 161 in [20]) Let K be a convex body in $\mathscr{K}^n$, $u \in \mathbb{R}^n$ be a point distinct from the origin o. The *shadow boundary* $\mathrm{Sbd}_K(u)$ of K in the direction of u is the set of points $z \in \mathrm{bd}K$ such that the line $\{z+\lambda u : \quad \lambda \in \mathbb{R}\}$ supports K.

PROPOSITION 2. (cf. Statement 1 and Lemma 1 in [17]) *Let* $K \in \mathscr{K}^n$, $u \in \mathbb{R}^n \setminus \{o\}$. *Then* bd$K$ *is the union of three disjoint subsets of it, namely,* $\mathrm{Sbd}_K(u)$,

$$K_u^+ := \{z \in \mathrm{bd}K : \quad \exists \lambda > 0 \ s.t. \ z - \lambda u \in \mathrm{int}K\},$$

and

$$K_u^- := \{z \in \mathrm{bd}K : \quad \exists \lambda > 0 \ s.t. \ z + \lambda u \in \mathrm{int}K\}.$$

The sets K_u^+ *and* K_u^- *are path-connected,* $(\mathrm{cl}K_u^+) \setminus K_u^+$ *and* $(\mathrm{cl}K_u^-) \setminus K_u^-$ *are two closed connected* $(n-2)$*-dimensional subsets of* $\mathrm{Sbd}_K(u)$ *separating* bd$K$.

Clearly, K_u^+ and K_u^- is the set of boundary points of K that can be illuminated by the direction $-u$ and u, respectively; each point from $\mathrm{Sbd}_K(u)$ cannot be illuminated either by u or by $-u$.

For a direction u in X, we denote by $\mathrm{IC}_X(u)$ the set of points from S_X that can be illuminated by the direction u. Clearly,

$$-u \in \mathrm{IC}_X(u) \quad \text{and} \quad u \in \mathrm{IC}_X(-u).$$

By Definition 1 and Proposition 2 we obtain the following proposition.

PROPOSITION 3. *Let u be a direction in X. Then S_X is the union of three disjoint sets $\mathrm{Sbd}_{B_X}(u)$, $\mathrm{IC}_X(u)$, and $\mathrm{IC}_X(-u)$. Moreover,*

$$\mathrm{Sbd}_{B_X}(u) = \{z \in S_X : \quad z \perp_B u\}.$$

The following corollaries follow directly from Proposition 3, the first one of which is closely related to a problem posed by Lassak in [19]: whether the boundary of a convex body $K \in \mathscr{C}^n$, which is not a parallelotope, can be illuminated by $2^{n-1} - 1$ pairs of opposite directions?

COROLLARY 4. *The set $\{\pm u_i : \quad i \in [m]\}$ of m pairs of directions illuminate S_X if and only if*

$$\bigcap_{i=1}^{m} \mathrm{Sbd}_{B_X}(u_i) = \emptyset.$$

COROLLARY 5. *S_X can be illuminated by a set $\{u_i : \quad i \in [m]\}$ of m directions if and only if*

$$S_X \subseteq \bigcup_{i=1}^{m} \mathrm{IC}_X(u_i).$$

For a set $A \subset S_X$ and a point $x \in S_X$, we put

$$r'(x,A) = \sup\{\gamma : \quad (x+\gamma B_X) \cap S_X \subseteq A\}.$$

PROPOSITION 6. *For each direction u in X, we have*

1. $r'(u, \mathrm{IC}_X(-u)) = \inf\{\|u - z\| : \quad z \in \mathrm{Sbd}_{B_X}(u)\}$*;*
2. $r'(u, \mathrm{IC}_X(-u)) \geqslant 1$*, and equality holds if and only if there exists a unit vector z such that*
$$[z-u, z] \subset S_X; \tag{1}$$
3. $r'(u, \mathrm{IC}_X(-u)) \leqslant 2$*, and equality holds if and only if the unit sphere of each two-dimensional subspace L of X containing u is a parallelogram having u as a vertex.*

Proof. 1. Put
$$\alpha := \inf\{\|u-z\| : \quad z \in \mathrm{Sbd}_{B_X}(u)\}.$$
It is clear that, for each $z \in \mathrm{Sbd}_{B_X}(u)$,
$$r'(u, \mathrm{IC}_X(-u)) \leqslant \|u-z\|.$$
Thus
$$r'(u, \mathrm{IC}_X(-u)) \leqslant \alpha.$$

Let γ be an arbitrary number in $(0,\alpha)$, and z be an arbitrary point in $(u+\gamma B_X) \cap S_X$. Then, by the monotonocity lemma (cf. [22, Proposition 31]),
$$\mathrm{arc}(u,z) \cap \mathrm{Sbd}_{B_X}(u) = \emptyset.$$
If $z \notin \mathrm{IC}_X(-u)$, then none of the sets $\mathrm{arc}(u,z) \cap \mathrm{IC}_X(-u)$ and $\mathrm{arc}(u,z) \cap \mathrm{IC}_X(u)$ is empty. Since $\mathrm{arc}(u,z)$ is a curve connecting u and z, this is impossible. Therefore $z \in \mathrm{IC}_X(-u)$. It follows that
$$r'(u, \mathrm{IC}_X(-u)) \geqslant \alpha.$$

2. From the first part of the proof, Proposition 3, and the definition of Birkhoff orthogonality it follows that
$$r'(u, \mathrm{IC}_X(-u)) \geqslant 1.$$

If $r'(u, \mathrm{IC}_X(-u)) = 1$, then there exists a point z in $\mathrm{Sbd}_{B_X}(u)$ such that $\|z-u\| = 1$. Since the line $\{z+\lambda u : \quad \lambda \in \mathbb{R}\}$ is a supporting line of B_X,
$$[z-u, z] \subset B_X \cap \{z+\lambda u : \quad \lambda \in \mathbb{R}\} \subset S_X.$$
Conversely, suppose that there exists a unit vector z satisfying (1). It is clear that $z \perp_B u$. Therefore
$$1 \leqslant r'(u, \mathrm{IC}_X(-u)) \leqslant \|u-z\| = 1.$$

3. We only need to characterize the case when $r'(u, \mathrm{IC}_X(-u)) = 2$.

First suppose that $r'(u, \mathrm{IC}_X(-u)) = 2$. Let L be an arbitrary two-dimensional subspace of X containing u, and w be an arbitrary point in $\mathrm{Sbd}_{B_X}(u) \cap L$. Then
$$2 = \inf\{\|u-z\| : \quad z \in \mathrm{Sbd}_{B_X}(u)\} \leqslant \min\{\|w-u\|, \|w+u\|\} \leqslant 2.$$
Thus
$$\|w+u\| = \|w-u\| = 2.$$
It is not difficult to see that the unit sphere S_L of L is a parallelogram having u as a vertex.

Conversely, suppose that the unit sphere of each two-dimensional subspace of X containing u is a parallelogram having u as a vertex. Let z be an arbitrary point in $\mathrm{Sbd}_{B_X}(u)$ and L be the subspace spanned by u and z. Then, since S_L is a parallelogram having u as a vertex, z is also a vertex of S_L. It follows that $\|z-u\| = 2$. Thus $r'(u, \mathrm{IC}_X(-u)) = 2$. $\square$

DEFINITION 7. For a Minkowski space X, set

$$RC(X) := \inf\{r'(u, \mathrm{IC}_X(-u)) : \quad u \in S_X\}.$$

THEOREM 8. *If there exist m unit vectors $u_1, u_2, \cdots, u_m$ such that*

$$S_X \subset \bigcup_{i=1}^{m} \mathrm{int}(u_i + RC(X) \cdot B_X),$$

then $c(B_X) \leqslant m$.

Proof. By the definition of $RC(X)$, we have

$$\begin{aligned} S_X &\subseteq \bigcup_{i=1}^{m} \mathrm{int}(u_i + RC(X) \cdot B_X) \\ &\subseteq \bigcup_{i=1}^{m} \mathrm{int}(u_i + r'(u_i, \mathrm{IC}_X(-u_i)) \cdot B_X) \\ &\subseteq \bigcup_{i=1}^{m} \mathrm{IC}_X(-u_i). \end{aligned}$$

Then Corollary 5 implies that $c(B_X) \leqslant m$. □

In Section 3 we shall estimate $c(B_X)$ with the help of $RC(X)$. We will provide more properties of this constant in Section 4.

3. Generalized blocking number

We begin with introducing the generalized α-blocking number.

DEFINITION 9. Let K be a convex body in $\mathbb{R}^n$ satisfying $o \in \mathrm{int}K$, and

$$x_1, x_2, \cdots, x_m$$

be m points in X. We say that $\mathscr{B} := \{x_i + \alpha K : \quad i \in [m]\}$ is a *generalized α-blocking configuration* of K for a positive number α if

1. each element of $\mathscr{B}$ touches K at its boundary,
2. $\mathscr{B}$ blocks any other translate of αK from touching K.

The least cardinality of a generalized α-blocking configuration of K is called the *generalized α-blocking number* of K and denoted by $\beta_2^{\alpha}(K)$.

The name "generalized α-blocking number" comes from two observations. On the one hand, in contrast to $\beta(K)$ and $\beta_2(K)$ defined in [26], the homothetic copies of K involved in Definition 9 need not to be translates of K. On the other hand, compared with the generalized blocking number defined in [9], these homothetic copies need not be non-overlapping.

LEMMA 10. *Let α be a positive number, and $x_1, x_2, \cdots, x_m$ be m points in X. Then*

$$\mathscr{B} := \{x_i + \alpha B_X : \quad i \in [m]\}$$

is a generalized α-blocking configuration of B_X if and only if

1. $\|x_i\| = 1 + \alpha$ *holds for each $i \in [m]$, and*
2. $(1+\alpha)S_X \subset \bigcup_{i=1}^{m} \operatorname{int}(2\alpha B_X + x_i)$.

Proof. Clearly, the set $x + \alpha B_X$ touches B_X if and only if $\|x\| = 1 + \alpha$. Thus each set in $\mathscr{B}$ touches B_X if and only if $\|x_i\| = 1 + \alpha$ holds for each $i \in [m]$. It follows that $\mathscr{B}$ blocks any other translate of αB_X from touching B_X if and only if, for each point $x \in (1+\alpha)S_X$,

$$x \in \bigcup_{i=1}^{m} \operatorname{int}(2\alpha B_X + x_i).$$

The proof is complete. □

The following is a direct corollary.

COROLLARY 11. *Let α be a positive number, and $x_1, x_2, \cdots, x_m$ be m points in X. Then $\mathscr{B} := \{x_i + \alpha B_X : \quad i \in [m]\}$ is a generalized α-blocking configuration of B_X if and only if*

$$S_X \subset \bigcup_{i=1}^{m} \operatorname{int}\left(\frac{2\alpha}{1+\alpha} B_X + u_i\right),$$

where, for each $i \in [m]$,

$$u_i := \frac{1}{1+\alpha} x_i \in S_X.$$

PROPOSITION 12. *$\beta_2^{\alpha}(B_X)$ is non-increasing with respect to α in $(0, +\infty)$.*

Proof. Let α_1 and α_2 be two numbers in $(0, +\infty)$ satisfying $\alpha_1 < \alpha_2$. Then

$$\frac{2\alpha_1}{1+\alpha_1} < \frac{2\alpha_2}{1+\alpha_2}.$$

Let $m = \beta_2^{\alpha_1}(B_X)$. Then there exists a generalized α_1-blocking configuration $\{x_i + \alpha_1 B_X : \quad i \in [m]\}$ of B_X. By Corollary 11,

$$\left\{\frac{1}{1+\alpha_1} x_i : \quad i \in [m]\right\} \subset S_X$$

and

$$S_X \subset \bigcup_{i=1}^{m} \operatorname{int}\left(\frac{2\alpha_1}{1+\alpha_1} B_X + \frac{1}{1+\alpha_1} x_i\right) \subset \bigcup_{i=1}^{m} \operatorname{int}\left(\frac{2\alpha_2}{1+\alpha_2} B_X + \frac{1}{1+\alpha_1} x_i\right).$$

It follows that

$$\left\{\frac{1+\alpha_2}{1+\alpha_1}x_i+\alpha_2 B_X:\quad i\in[m]\right\}$$

is a generalized α_2-blocking configuration. Therefore $\beta_2^{\alpha_2}(B_X)\leqslant\beta_2^{\alpha_1}(B_X)$. □

The following Theorem is our main result. Note that $1\leqslant RC(X)\leqslant\sqrt{2}$ (cf. Theorem 21).

THEOREM 13. *Let X and Y be two isomorphic Minkowski spaces. If*

$$\delta:=d(X,Y)<\frac{1+\sqrt{1+2RC(X)}}{2},$$

then $c(B_X)\leqslant\beta_2^{\alpha}(B_Y)$, where

$$\alpha=\alpha(X,\delta):=\frac{2\delta^2}{4\delta^2-2\delta-RC(X)}-1.$$

Proof. Without loss of generality, we may assume that $X=(\mathbb{R}^n,\|\cdot\|_X)$ and $Y=(\mathbb{R}^n,\|\cdot\|_Y)$, and that

$$B_Y\subseteq B_X\subseteq\delta B_Y.$$

It follows that

$$\frac{1}{\delta}\|x\|_Y\leqslant\|x\|_X\leqslant\|x\|_Y$$

holds for each vector $x\in\mathbb{R}^n$.

Let $m=\beta_2^{\alpha}(B_Y)$. By Corollary 11, there exist vectors $u_1,u_2,\cdots,u_m$ in S_Y such that

$$S_Y\subset\bigcup_{i=1}^{m}\operatorname{int}\left(\frac{2\alpha}{1+\alpha}B_Y+u_i\right).$$

For each vector $x\in\mathbb{R}^n\setminus\{o\}$, set

$$T(x)=\frac{x}{\|x\|_X}.$$

Let i be an arbitrary integer in $[m]$ and $v_i=T(u_i)$. For an arbitrary point

$$w\in\operatorname{int}\left(\frac{2\alpha}{1+\alpha}B_Y+u_i\right)\cap S_Y,$$

set $z = T(w)$. It follows that

$$\begin{aligned}
&\|v_i - z\|_X \\
&= \left\| \frac{u_i}{\|u_i\|_X} - \frac{w}{\|w\|_X} \right\|_X \\
&= \frac{1}{\|u_i\|_X \cdot \|w\|_X} \| \|w\|_X \, u_i - \|u_i\|_X \, w\|_X \\
&= \frac{1}{\|u_i\|_X \cdot \|w\|_X} \| \|w\|_X \, u_i - \|u_i\|_X \, u_i + \|u_i\|_X \, u_i - \|u_i\|_X \, w\|_X \\
&\leqslant \delta^2 \left(|\|w\|_X - \|u_i\|_X| \, \|u_i\|_X + \|u_i\|_X \, \|u_i - w\|_X \right) \\
&\leqslant \delta^2 \left(|\|w\|_X - \|w\|_Y + \|w\|_Y - \|u_i\|_Y + \|u_i\|_Y - \|u_i\|_X| + \|u_i - w\|_Y \right) \\
&\leqslant \delta^2 \left(2 - \frac{2}{\delta} + \|u_i - w\|_Y \right) \\
&< \delta^2 \left(2 - \frac{2}{\delta} + \frac{2\alpha}{1+\alpha} \right) \\
&= RC(X).
\end{aligned}$$

This implies that the image of the set

$$\operatorname{int}\left(\frac{2\alpha}{1+\alpha} B_Y + u_i \right)$$

under T is contained in

$$\operatorname{int}(RC(X) \cdot B_X + v_i).$$

Thus

$$S_X \subset \bigcup_{i=1}^{m} \operatorname{int}(RC(X) \cdot B_X + v_i).$$

From Theorem 8 it follows that $c(B_X) \leqslant m = \beta_2^{\alpha}(B_Y)$. □

REMARK 14. Recall that the *Hausdorff metric* between two convex bodies K and K' in $\mathbb{R}^n$ is given by

$$\begin{aligned}
\delta^H(K, K') &= \max\{\max_{x\in K} \min_{y\in K'} \|x - y\|, \max_{y\in K'} \min_{x\in K} \|x - y\|\} \\
&= \inf\{\delta \geqslant 0 : \; K \subseteq K' + \delta B^n, K' \subseteq K + \delta B^n\},
\end{aligned}$$

where B^n is the unit ball of $\mathbb{R}^n$. It is clear that when $\delta^H(K, K')$ is small, then $d(K, K')$ will be close to 1, but the converse is not true.

Solving Hadwiger's covering conjecture is difficult also because the functional $c(K)$ is upper semi-continuous with respect to the Hausdorff metric. More precisely, for any convex body K' sufficiently close to K in the Hausdorff metric, the inequality $c(K') \leqslant c(K)$ holds (cf. [8, Theorem 34.9]). Thus confirming Hadwiger's covering conjecture for a set dense in $\mathscr{K}^n$ (see, e.g., [6] and [7], where Hadwiger's covering conjecture is confirmed for belt bodies, a class of convex bodies which is dense in $\mathscr{K}^n$) does not imply that this conjecture is true in general.

In contrast, Theorem 13 shows that, when Y is sufficiently close to X, then $c(B_X)$ can be controlled by $\beta_2^\alpha(Y)$.

If $RC(X) > 1$, then

$$\frac{1+\sqrt{1+3RC(X)}}{3} > 1.$$

Thus, when

$$1 \leqslant \delta := d(X,Y) < \frac{1+\sqrt{1+3RC(X)}}{3},$$

we have

$$\alpha = \alpha(X,\delta) > 1.$$

When X is linearly isometric to Y, then $\delta = d(X,Y) = 1$ and $\beta_2^\alpha(B_Y) = \beta_2^\alpha(B_X)$.

DEFINITION 15. Put

$$\beta'(B_X) = \beta_2^\alpha(B_X),$$

where

$$\alpha = \frac{2}{2-RC(X)} - 1 \geqslant 1.$$

We have the following corollary.

COROLLARY 16.

$$1+n \leqslant c(B_X) \leqslant \beta'(B_X) \leqslant \beta_2^1(B_X) \leqslant \beta(B_X).$$

The *generalized kissing number* $N_\alpha(K)$ of $K \in \mathscr{K}^n$ is the maximal number of non-overlapping translates of αK which can touch K at its boundary. Clearly, we have the following inequality:

$$\beta_2^\alpha(B_X) \leqslant N_\alpha(B_X).$$

By Theorem 1 in [9] we obtain

COROLLARY 17.

$$\beta'(B_X) \leqslant N_\alpha(B_X) \leqslant \frac{(1+2\alpha)^n - 1}{\alpha^n},$$

where

$$\alpha = \frac{2}{2-RC(X)} - 1.$$

There exist spaces such that $\beta'(B_X) = n+1$. See the following example.

EXAMPLE 1. If X is Euclidean, then $RC(X) = \sqrt{2}$. Thus $\beta'(B_X) = \beta_2^{\alpha}(B_X)$, where

$$\alpha = \frac{2}{2-\sqrt{2}} - 1.$$

In this case, for each unit vector u,

$$S_X \cap \mathrm{int}\left(\frac{2\alpha}{1+\alpha} B_X + u\right)$$

is precisely the portion of S_X that is illuminated by the direction $-u$. Since $c(K) = n+1$ holds for each convex body $K \in \mathscr{K}^n$ having smooth boundary (cf., for example, Theorem 35.2 in [8]), we have

$$\beta'(B_X) = c(B_X) = n+1.$$

EXAMPLE 2. Let $X = l_\infty^n$. Then $RC(X) = 1$. In this case, for each unit vector u, the set $\mathrm{int}(B_X + u)$ cannot contain two vertices of B_X. Thus $\beta'(B_X) \geqslant 2^n$. Since $\beta(B_X) = 2^n$, it follows that $\beta'(B_X) = 2^n$.

Thus, if Conjecture 2 is true, then the following conjecture is also true.

CONJECTURE 3. *For each space* $X = (\mathbb{R}^n, \|\cdot\|)$,

$$n+1 \leqslant \beta'(B_X) \leqslant 2^n,$$

and $\beta'(B_X) = 2^n$ *if and only if* X *is isometric to* l_∞^n.

Clearly, the value of $\beta'(B_X)$ is closely related to the value of $RC(X)$. In the next section, we will study this constant in more detail.

4. The constant $RC(X)$

In the first subsection we present some old and new results that will be used to estimate $RC(X)$.

4.1. Radial projections of bisectors and related constants

The *bisector* $B(p,q)$ of the line segment having endpoints $p \neq q$ in X is given by

$$B(p,q) := \{z \in X : \quad \|z-p\| = \|z-q\|\}.$$

The *radial projection* $P(x)$ of the bisector $B(-x,x)$ is defined by

$$P(x) := \left\{\frac{z}{\|z\|} : \quad z \in B(-x,x) \setminus \{o\}\right\}.$$

We have the following simple proposition concerning the relation between $P(x)$ and $\mathrm{Sbd}_{B_X}(x)$.

PROPOSITION 18. *For each* $x \in S_X$,

$$\mathrm{Sbd}_{B_X}(x) \subseteq \mathrm{cl}P(x). \tag{2}$$

Proof. By Theorem 2.6 in [23], for any y in S_X, $y \in \mathrm{cl}P(x)$ whenever $y \perp_B x$, from which (2) follows. □

PROPOSITION 19. *For each unit vector* $x \in X$, S_X *is the union of three disjoint sets, namely,* $P(x)$,

$$N(x) := \{z \in S_X : \quad \|\alpha z - x\| < \|\alpha z + x\|, \forall \alpha > 0\},$$

and

$$N(-x) := \{z \in S_X : \quad \|\alpha z - x\| > \|\alpha z + x\|, \forall \alpha > 0\}.$$

Moreover, $N(x)$ *and* $N(-x)$ *are two path-connected sets contained in* $\mathrm{cl}(\mathrm{IC}_X(-x))$ *and* $\mathrm{cl}(\mathrm{IC}_X(x))$, *respectively.*

Proof. It is clear that $P(x)$, $N(x)$, and $N(-x)$ are disjoint. Now suppose that u is a point in $S_X \setminus P(x)$. Then $\|u-x\| \neq \|u+x\|$. Suppose that $\|u-x\| < \|u+x\|$. If there exists a positive number α such that $\|\alpha u - x\| > \|\alpha u + x\|$, then, since the function

$$f(\alpha) = \|\alpha u - x\| - \|\alpha u + x\|$$

is continuous, there exists an $\alpha_0 > 0$ such that $\|\alpha_0 u - x\| = \|\alpha_0 u + x\|$. This implies that $u \in P(x)$, a contradiction. Thus $u \in N(x)$. Similarly, if $\|u-x\| > \|u+x\|$ then $u \in N(-x)$.

Next we show that $N(x)$ and $N(-x)$ are path-connected. Since $z \in N(x)$ if and only if $-z \in N(-x)$, it suffices to consider the case of $N(x)$. Let y be an arbitrary point in $N(x) \setminus \{x\}$. For each $\lambda \in (0,1)$, let

$$z := z(\lambda) = \frac{\lambda x + (1-\lambda)y}{\|\lambda x + (1-\lambda)y\|}. \tag{3}$$

For each number $\alpha > 0$, by the monotonicity lemma (cf. [22, Proposition 31]) we have

$$\|\alpha z - x\| \leqslant \|\alpha y - x\| < \|\alpha y + x\| \leqslant \|\alpha z + x\|.$$

Thus $z \in N(x)$. It follows that $\mathrm{arc}(x,y)$, which is a curve connecting y and x, is contained in $N(x)$. This implies that $N(x)$ is path-connected.

In the rest of the proof we show that $N(x) \subseteq \mathrm{cl}(\mathrm{IC}_X(-x))$ and the inclusion $N(-x) \subseteq \mathrm{cl}(\mathrm{IC}_X(x))$ can be proved in a similar way.

Let y be an arbitrary point in $N(x) \setminus \{x\}$. For each number $\lambda \in (0,1)$, let $z(\lambda)$ be defined as in (3). Since $y \in N(x)$,

$$\begin{aligned}
\|\lambda x + (1-\lambda)y\| &= \lambda \left\| x + \frac{1-\lambda}{\lambda} y \right\| \\
&> \frac{\lambda}{2} \left(\left\| x + \frac{1-\lambda}{\lambda} y \right\| + \left\| -x + \frac{1-\lambda}{\lambda} y \right\| \right) \\
&\geqslant 1 - \lambda.
\end{aligned}$$

Thus

$$\left\| z(\lambda) - \frac{\lambda x}{\|\lambda x + (1-\lambda)y\|} \right\| = \left\| \frac{(1-\lambda)y}{\|\lambda x + (1-\lambda)y\|} \right\| < 1,$$

which shows that $z(\lambda) \in \mathrm{IC}_X(-x)$. Hence

$$y = \lim_{\lambda \to 0} z(\lambda) \in \mathrm{cl}(\mathrm{IC}_X(-x)).$$

Therefore, $N(x) \subseteq \mathrm{cl}(\mathrm{IC}_X(-x))$. □

For a unit vector x, we denote by $\mathrm{relint}(N(x))$ the relative interior of $N(x)$ with respect to S_X. From the proof of the Proposition 19, we can deduce the following corollary.

COROLLARY 20. *For each unit vector x and each point $y \in \mathrm{relint}(N(x))$, $y \in \mathrm{IC}_X(-x)$.*

The so called *critical number* $c(X)$ of X (cf. [23]) is defined as

$$c(X) := \inf\{d(x, P(x)) : \quad x \in S_X\}.$$

The *James non-square constant* $J(X)$ and the *Schäffer non-square constant* $S(X)$ of X are defined by

$$J(X) := \sup\{\min\{\|x+y\|, \|x-y\|\} : \quad x, y \in S_X\}$$

and

$$S(X) := \inf\{\max\{\|x+y\|, \|x-y\|\} : \quad x, y \in S_X\},$$

respectively. The following facts are well known in the geometry of Banach spaces (cf. [13] and [16]):

$$1 \leqslant S(X) \leqslant \sqrt{2} \leqslant J(X) \leqslant 2 \text{ and } S(X) \cdot J(X) = 2.$$

For $\varepsilon \in [0,2]$, the *Gurarii modulus of convexity* $\beta_X(\varepsilon)$ (cf. [24]) is defined by

$$\beta_X(\varepsilon) := \inf\left\{ 1 - \inf_{\lambda \in [0,1]} \|\lambda x + (1-\lambda)y\| : \quad x, y \in S_X, \|x-y\| = \varepsilon \right\}.$$

We remark that the *M-curvature* defined in Definition 1 in [12] is equal to $\beta_X(1)$.

4.2. Lower and upper bounds on $RC(X)$

THEOREM 21. *For each Minkowski space X,*

$$1 \leqslant c(X) \leqslant RC(X) \leqslant S(X) \leqslant \sqrt{2}.$$

Moreover, $RC(X) = 1$ if and only if S_X contains a segment whose length is at least 1; $RC(X) = \sqrt{2}$ if and only if X is Euclidean.

Proof. Theorem 3.1 in [23] shows the inequality

$$1 \leqslant c(X) \leqslant S(X) \leqslant \sqrt{2}.$$

Therefore we only need to prove that

$$c(X) \leqslant RC(X) \leqslant S(X).$$

Let x be an arbitrary unit vector. Then, for each number $\varepsilon \in (0, d(x,P(x)))$, we have

$$((d(x,P(x)) - \varepsilon)B_X + x) \cap S_X \subseteq \operatorname{relint}(N(x)) \subseteq \mathrm{IC}_X(-x),$$

which implies that

$$d(x,P(x)) \leqslant r'(x, \mathrm{IC}_X(-x)).$$

Therefore $c(X) \leqslant RC(X)$.

Let x and y be an arbitrary pair of unit vectors in X. If x and y are linearly dependent, then

$$\max\{\|x+y\|, \|x-y\|\} = 2 \geqslant RC(X).$$

Now suppose that x and y are linearly independent. Let z be a unit vector in the two-dimensional subspace L spanned by x and y such that $z \perp_B x$ and that the two points y and z lie in the same open half-plane of L bounded by $\{\lambda x : \quad \lambda \in \mathbb{R}\}$. If $y = z$, then

$$\begin{aligned}\max\{\|x+y\|, \|x-y\|\} &\geqslant \min\{\|x+z\|, \|x-z\|\} \\ &\geqslant \min\{r'(x, \mathrm{IC}_X(-x)), r'(-x, \mathrm{IC}_X(x))\} \geqslant RC(X).\end{aligned}$$

Otherwise we may assume that, without loss of generality,

$$z \in \{-\alpha x + \beta y : \quad \alpha, \beta > 0\}.$$

Then, by the monotonicity lemma (again, cf. [22, Proposition 31]),

$$\begin{aligned}\max\{\|x+y\|, \|x-y\|\} &\geqslant \|x+y\| \\ &\geqslant \|z+x\| \\ &\geqslant \min\{\|x+z\|, \|x-z\|\} \\ &\geqslant \min\{r'(x, \mathrm{IC}_X(-x)), r'(-x, \mathrm{IC}_X(x))\} \geqslant RC(X).\end{aligned}$$

It follows that

$$RC(X) \leqslant S(X).$$

If $RC(X) = 1$, then $c(X) = 1$. From Theorem 3.1 in [23] it follows that S_X contains a segment whose length is not less than 1. Conversely, suppose that S_X contains a segment $[u,v] \subset S_X$, where u and v are two unit vectors satisfying $\|u-v\| \geqslant 1$. Let $z = \frac{u-v}{\|u-v\|}$. Then $u \perp_B z$ and $[u, u-z] \subseteq [u,v] \subset S_X$. Proposition 6 shows that $r'(z, \mathrm{IC}_X(-z)) = 1$. Therefore $RC(X) = 1$.

If X is Euclidean, then $c(X) = S(X) = \sqrt{2}$, from which it follows that $RC(X) = \sqrt{2}$.

Conversely, suppose that $RC(X) = \sqrt{2}$. Then, by Proposition 6, $\|x-y\| \geqslant \sqrt{2}$ holds for each pair of unit vectors x and y satisfying $x \perp_B y$. By Theorem 2 in [11], X is Euclidean. □

LEMMA 22. *Let p, q, s, and t be the vertices of a planar convex quadrilateral $pqst$. If $\|p-t\| = \|q-s\|$ and there exists a number $\gamma > 1$ such that $s-t = \gamma(q-p)$, then, for each relative interior point z of $[p,q]$, we have*

$$\min\{\|z-s\|, \|z-t\|\} \geqslant \|q-s\|. \tag{4}$$

Proof. There exists a number $\lambda \in (0,1)$ such that $z = \lambda p + (1-\lambda)q$. Let

$$x = s + \lambda(p-q) \quad \text{and} \quad y = t + (1-\lambda)(q-p).$$

Then

$$\|z-x\| = \|z-y\| = \|p-t\| = \|q-s\| \quad \text{and} \quad [x,y] \subset [s,t].$$

Since the function

$$f(\alpha) = \|z - (\alpha x + (1-\alpha)y)\|$$

is convex and $f(1) = f(0)$, inequality (4) follows. □

THEOREM 23.

$$RC(X) \geqslant \frac{1}{1-\beta_X(1)}.$$

Proof. The case when $\beta_X(1) = 0$ is trivial. In the following we assume that $\beta_X(1) > 0$. In this case, S_X does not contain a segment whose length is at least 1.

Let x be an arbitrary unit vector. We show that

$$r'(x, \mathrm{IC}_X(-x)) \geqslant \frac{1}{1-\beta_X(1)}.$$

We only need to show that, for each unit vector y satisfying $y \perp_B x$, we have

$$\|x-y\| \geqslant \frac{1}{1-\beta_X(1)}. \tag{5}$$

For each $\alpha \in [0,1]$, denote by $l(\alpha)$ the length of the intersection of B_X and the line $\alpha y + \langle -x, x\rangle$. Note that $l(1)$ might be 0. Then $l(\alpha)$ is continuous and non-increasing, which is also due to the monotonicity lemma (cf. [22, Proposition 31] once more). Since S_X does not contain a segment whose length is at least 1, $l(1) < 1$. Then there exists an $\alpha_0 \in (0,1)$ such that $l(1-\alpha_0) = 1$. Let u and v be the endpoints of the segment

$$(1-\alpha_0)y + \langle -x, x\rangle \cap B_X.$$

We may assume that $u - v = x$. Then

$$\|v+x\| = \|u-v\| = \|x-u\| = 1$$

and

$$\alpha_0 = 1 - \inf_{\alpha\in[0,1]} \|\alpha u + (1-\alpha)v\| \geqslant \beta_X(1).$$

Set $z=u+v$. Then, for each point $w\in([z,u]\cup[z,v])\setminus\{u,v\}$, $\|w\|>1$ (otherwise, by Lemma 5 in [22], S_X would contain a segment whose length is 1). It follows that y lies in the relative interior of the convex hull of $\{z,u,v\}$. Thus there exist two points p and q such that $p\in[z,x]$, $q\in[z,-x]$, $y\in[p,q]$, and that

$$\frac{p-q}{\|p-q\|}=x.$$

From Lemma 22 it follows that

$$\|y-x\|\geqslant\|p-x\|=\frac{\|y\|}{\|(1-\alpha_0)y\|}=\frac{1}{1-\alpha_0}\geqslant\frac{1}{1-\beta_X(1)}.$$

The proof is complete. □

EXAMPLE 3. It is known that (cf. [24, Corollary 3.1] and [16, Example 9]), when $p\geqslant 2$,

$$\beta_{l_p^n}(1)=1-\left(1-\left(\frac{1}{2}\right)^p\right)^{\frac{1}{p}}\quad\text{and}\quad S(l_p^n)=2^{\frac{1}{p}}.$$

Therefore,

$$\left(1-\left(\frac{1}{2}\right)^p\right)^{-\frac{1}{p}}\leqslant RC(l_p^n)\leqslant 2^{\frac{1}{p}}.$$

Moreover,

$$\begin{aligned}\lim_{p\to\infty}\frac{2^{\frac{1}{p}}}{\left(1-\left(\frac{1}{2}\right)^p\right)^{-\frac{1}{p}}}&=\lim_{p\to\infty}2^{\frac{1}{p}}\cdot\left(1-\left(\frac{1}{2}\right)^p\right)^{\frac{1}{p}}\\&=\lim_{p\to\infty}\left(2-2^{1-p}\right)^{\frac{1}{p}}=1.\end{aligned}$$

Therefore, for spaces sufficiently close to l_∞^n, the estimation in Theorem 23 is sharp.

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