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WEIGHTED ESTIMATES FOR VECTOR–VALUED COMMUTATORS OF GENERALIZED FRACTIONAL INTEGRALS

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Abstract. In this paper, the authors study the vector-valued commutator of generalized fractional integral operator $I^k_{\alpha,b,q}$ where the kernel K_α satisfies some conditions associated with the Young functions. The authors prove the two-weight norm inequalities for $I^k_{\alpha,b,q}$ where the weight ω is only a local integrable function. As an application of the main theorems in this paper, the weighted boundedness for vector-valued commutators of fractional integral with a rough kernel is also given.

1. Introduction

In the 1950s, Calderón and Zygmund [6] introduced the classical C-Z theory which plays an important role in harmonic analysis. The classical singular integral $T(f)(x)$ is defined by

$$T(f)(x) = p.v.\int_{\mathbb{R}^n} K(x-y)f(y)dy,$$

where the kernel $K(x)$ satisfies some size and regular conditions. Later, Muckenhoupt [19] introduced the weighted theory for singular integral and fractional integral. In 1972, Coifman [7] established a famous weighted estimate for T. Coifman proved that if $K \in H_\infty$ (see the definition for H_∞ in the next section), then for every Muckenhoupt weight $\omega \in A_\infty$ and every $p \in (0,\infty)$, we have

$$\int_{\mathbb{R}^n} |T(f)(x)|^p \omega(x)dx \leqslant C\int_{\mathbb{R}^n} Mf(x)^p \omega(x)dx. \tag{1}$$

By the classical duality theory and the Coifman's type estimate in (1), Pérez [21] obtained the following two-weight norm inequalities,

$$\int_{\mathbb{R}^n} |T(f)(x)|^p \mu(x)dx \leqslant C\int_{\mathbb{R}^n} |f(x)|^p M^{[p]+1}\mu(x)dx, \tag{2}$$

where $1 < p < \infty$ and μ is only a local integrable function. For a function $b \in \mathrm{BMO}(\mathbb{R}^n)$, we define the commutator of singular integral $T^1_b(f)(x) = b(x)T(f)(x) -$

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$T(bf)(x)$. In 1976, Coifman, Rochberg and Weiss [8] proved that T_b^1 is bounded on $L^p(\mathbb{R}^n)$ $(1<p<\infty)$ space if and only if $b\in \mathrm{BMO}(\mathbb{R}^n)$. When $p=1$, Pérez [22] gave a counterexample that T_b^1 is not of weak type (1,1) and he proved that T_b^1 satisfies a weak $L\log L$ estimate. In 1997, Pérez [25] obtained the following inequality,

$$\int_{\mathbb{R}^n} |T_b^1(f)(x)|^p \mu(x)dx \leqslant C\int_{\mathbb{R}^n} |f(x)|^p M^{[2p]+1}\mu(x)dx, \tag{3}$$

where the weight μ is only a local integrable function. Later by using (3) and the classical C-Z decomposition, Pérez and Pradolini [26] proved the following weighted endpoint estimate for T_b^1.

$$\mu(\{x\in\mathbb{R}^n : |T_b^1(f)(x)>\lambda|\}) \leqslant C\phi_1(\|b\|_{\mathrm{BMO}})\int_{\mathbb{R}^n}\phi_1\Big(\frac{|f(x)|}{\lambda}\Big)M_{L(\log L)^{1+\delta}}\mu(x)dx, \tag{4}$$

where $\phi_1(t)=t(1+\log^+ t)$ and the definition of $M_{L(\log L)^{1+\delta}}\mu(x)$ will be introduced in Section 2.

The Coifman's type estimate has also been studied by many other mathematicians, see [14] or [30] for details. In these papers, the authors proved that if $K\in H_r$ (see the definition for H_r in the next section), then (1) also holds if we replace $Mf(x)$ by another maximal function $M_{r'}(f)(x)=M(|f|^{r'})^{1/r'}(x)$ for some $1\leqslant r<\infty$. In general, we can conclude that if we strengthen the roughness of the kernel, then the corresponding maximal function will become bigger.

However, in 2005, Martell, Pérez and Trujillo-González [18] gave a counterexample that (1) no longer holds in general with $M_r(f)(x)$ for $T(f)(x)$ with any $r\in[1,\infty)$ if the kernel K satisfies the classical Hörmander type condition H_1. So it is intetesting to seek for new maximal functions so that (1) can still hold if we add some new Hörmander type conditions on the kernel K. In 2005, Lorente, Riveros and Torre [17] gave a new class of Hörmander type conditions in the scale of the Orlicz spaces that lies between the intersection of H_1 and H_∞. They gave the analogous results of Coifman's type estimates for $T(f)(x)$. They proved the following theorem.

THEOREM A. [17] *Assume that T is a singular integral operator, bounded on some L^p spaces, $1<p<\infty$, whose kernel satisfies the $L^{\mathscr{A}}$ ($\mathscr{A}$ is a Young function which will appear in Section 2) – Hörmander type condition. If there are numbers $c_{\mathscr{A}}>1$ and $C_{\mathscr{A}}>0$ such that for any x and $R>c_{\mathscr{A}}|x|$,*

$$\sum_{m=1}^{\infty}(2^mR)^n\|(K(x-.)-K(-.))\chi_{2^mR<|y|\leqslant 2^{m+1}R}(.)\|_{\mathscr{A},B(0,2^{m+1}R)}\leqslant C_{\mathscr{A}}. \tag{5}$$

Then, for any $0<p<\infty$ and $\omega\in A_\infty$, there exists a constant C such that

$$\int_{\mathbb{R}^n}|T(f)(x)|^p\omega(x)dx\leqslant C\int_{\mathbb{R}^n}(M_{\bar{\mathscr{A}}}(f)(x))^p\omega(x)dx \tag{6}$$

where the definition of $M_{\bar{\mathscr{A}}}$ will be given in the next section.

In 2008, Lorente, Martell, Riveros and Torre [16] studied the commutator of singular integral operator T_b^1 with the kernel satisfying some conditions of Hörmander

Young type. Recently, Lorente, Martell, Pérez and Riveros [15] proved the weighted norm inequalities for the commutators of singular integral with the kernel K satisfying Hörmander conditions of Young type where the weight is only a local integrable function.

On the other hand, the following fractional integral operator

$$I_\alpha(f)(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy\ \ (0<\alpha<n)$$

has also been studied by many mathematicians. For example, see references [9], [10], [12] and [20] for the fractional integrals with rough kernels. Espically in [24], Pérez proved that (2) still holds if we replace $T(f)(x)$ by $I_\alpha(f)(x)$ and $M^{[p]+1}\mu(x)$ by $M_{\alpha p}(M^{[p]})\mu(x)$.

In 1982, Chanillo [4] considered the commutator of fractional integral operator as follows,

$$I_\alpha^b(f)(x)=b(x)I_\alpha f(x)-I_\alpha(bf)(x),$$

where $b\in\mathrm{BMO}(\mathbb{R}^n)$ and Chanillo proved that the operator I_α^b is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $p>1$ and $1/q=1/p-\alpha/n$. Here we would like to remark that I_α^b is not of $(L^1,L^{n/(n-\alpha),\infty})$ type, readers may see [11] for details.

Recently, Riveros [29] as well as Bernardis, Lorente and Riveros [2] considered the generalized fractional integral $I_{K_\alpha}(f)(x)=\int_{\mathbb{R}^n}K_\alpha(x-y)f(y)dy$ with the kernel K_α satisfying the Hörmander conditions of Young type defined by

$$\sum_{m=1}^{\infty}(2^mR)^{n-\alpha}\|(K_\alpha(x-.)-K_\alpha(-.))\chi_{2^mR<|y|\leqslant 2^{m+1}R}(.)\|_{\mathscr{A},B(0,2^{m+1}R)}\leqslant C_{\mathscr{A}}. \tag{7}$$

They proved that I_{K_α} satisfies the analogous results of (1), (2) and (4). So it is natural to ask whether (1)–(4) still hold if we consider the vector-valued commutator of the generalized fractional integral with the kernel satisfying the Hörmander conditions of Young type. In this paper, we will show that the analogous results of (1)–(4) still hold for the following vector-valued commutators of generalized fractional integral,

$$I_{\alpha,b,q}^k(f)(x)=|I_{\alpha,b}^k(f)(x)|_q=\left(\sum_{j=1}^{\infty}|I_{\alpha,b}^k(f_j)(x)|^q\right)^{1/q},$$

where $I_{\alpha,b}^k(f_j)(x)=\int_{\mathbb{R}^n}K_\alpha(x-y)f_j(y)(b(x)-b(y))^k dy$ and $b\in\mathrm{BMO}(\mathbb{R}^n)$.

We say a kernel $K_\alpha\in H_{\mathscr{A},k,\alpha}$ if K_α satisfies the following condition,

$$\sum_{m=1}^{\infty}(2^mR)^{n-\alpha}m^k\|(K_\alpha(.-y)-K_\alpha(.))\chi_{2^mR<|y|\leqslant 2^{m+1}R}(.)\|_{\mathscr{A},B(0,2^{m+1}R)}\leqslant C_{\mathscr{A}}. \tag{8}$$

If $k=0$, we denote $K_\alpha\in H_{\mathscr{A},\alpha}$.

Before giving the main results of this paper, we introduce the B_p condition.

DEFINITION 1. (B_p condition, [23]) For a Young function B, we say $B \in B_p$ $(1 < p < \infty)$ if there exists a positive constant c such that

$$\int_c^{\infty} \frac{B(t)}{t^p} \frac{dt}{t} < \infty.$$

Now let us state our main results.

THEOREM 1. *Let $\mathscr{A}$, $\mathscr{B}$ and C_k be Young functions such that $\bar{\mathscr{A}}^{-1}(t)\mathscr{B}^{-1}(t) \cdot \bar{C_k}^{-1}(t) \leqslant t$ with $\bar{C_k}^{-1}(t) = e^{1/k}$. If $b \in BMO$ and the kernel $K_\alpha \in H_{\mathscr{A},k,\alpha} \cap H_{\mathscr{B},\alpha}$, then there exists a constant C such that*

$$\int_{\mathbb{R}^n} |I^k_{\alpha,b,q}(f)(x)|^p \omega(x)dx \leqslant C \int_{\mathbb{R}^n} (M_{\alpha,\bar{\mathscr{A}}})(|f|_q)(x)^p \omega(x)dx, \tag{9}$$

where $0 < p < \infty$, $\omega \in A_\infty$ and the definition of $M_{\alpha,\bar{\mathscr{A}}}$ will be introduced in the next section.

By Theorem 1 and the classical duality argument, we can draw the following theorem.

THEOREM 2. *Suppose that there exist Young functions ξ, $\mathscr{A}$ and θ satisfying $\xi \in B_{p'}$ and $\xi^{-1}(t)\theta^{-1}(t) \leqslant \bar{\mathscr{A}}^{-1}(t)$. Furthermore, if $I^k_{\alpha,b,q}$ is a linear operator and its adjoint operator $I^{*,k}_{\alpha,b,q}$ satisfies*

$$\int_{\mathbb{R}^n} |I^{*,k}_{\alpha,b,q}(f)(x)|^p \omega(x)dx \leqslant C \int_{\mathbb{R}^n} (M_{\alpha,\bar{\mathscr{A}}})(|f|_q)(x)^p \omega(x)dx \tag{10}$$

for all $0 < p < \infty$ and every $\omega \in A_\infty$. Then for any non-negative weight μ which is only local integrable on $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} |I^k_{\alpha,b,q}(f)(x)|^p \mu(x)dx \leqslant C \int_{\mathbb{R}^n} |f(x)|_q^p M_{\alpha p,D}\mu(x)dx \tag{11}$$

where $1 < p < \infty$ and $D(t) = \theta(t^{1/p})$.

Finally, we have the weak weighted $L\log L$ estimates for $I^k_{\alpha,b,q}$.

THEOREM 3. *Let $I^k_{\alpha,b,q}(f)(x)$ and its kernel $K_\alpha(x)$ be as in Theorem 1. Suppose that there exists a Young function D satisfying*

$$\int_{\mathbb{R}^n} |I^k_{\alpha,b,q}(f)(x)|^p \omega(x)dx \leqslant C \int_{\mathbb{R}^n} |f(x)|_q^p M_{\alpha p,D}\omega(x)dx \tag{12}$$

for any local integrable function ω, then there exists a constant C such that

$$\begin{aligned}&\omega\{x \in \mathbb{R}^n : |I^k_{\alpha,b,q}(f)(x)| > \lambda\}\\ &\leqslant C \int_{\mathbb{R}^n} \phi_k\Big(\frac{\|b\|^k_{BMO}|f(x)|_q}{\lambda}\Big)(M\omega(x) + M_{\alpha,\bar{\mathscr{A}}}\omega(x) + M_{\alpha p,D}\omega(x))dx,\end{aligned}$$

where $\phi_k(x) = x[log(e+x)]^k$ with $k \in Z^+$.

REMARK 1. As far as we know, Theorem 1 is still new even in the non-vector-valued case. Furthermore, Theorems 1 and 2 improve the main results in [2].

REMARK 2. As an application of our main results in this paper, we get the weighted boundedness for vector-valued commutators of fractional integrals with a rough kernel and we will discuss these facts in Section 6 of this paper.

REMARK 3. In [1], the authors got the weak weighted $L\log L$ estimates for commutators of fractional integrals where the weight ω is only a local integrable function. However, in [1], the authors considered the case when $K_\alpha \equiv 1$. So Theorem 1 in our paper can be regarded as an improvement of Theorem 1.5 in [1] in some sense.

2. Preliminaries

For $\delta > 0$, the Hardy-Littlewood maximal function of order δ and the sharp function of order δ is defined by M_δ and $M_\delta^\sharp$ respectively, that is

$$M_\delta(f)(x) = \sup_{x\in Q}\left(\frac{1}{|Q|}\int_Q |f(y)|^\delta dy\right)^{1/\delta}$$

and

$$M_\delta^\sharp(f)(x) = \sup_{x\in Q}\inf_c\left(\frac{1}{|Q|}\int_Q |f(y)-c|^\delta dy\right)^{1/\delta}.$$

For $\delta = 1$, from [13], we know that

$$M_1^\sharp(f)(x) \approx \sup_{x\in Q}\frac{1}{|Q|}\int_Q |f(y)-f_Q|dy,$$

where $f_Q = \frac{1}{|Q|}\int_Q f(y)dy$ denotes the average of f over Q.

Obviously, M_1 is the classical Hardy-Littlewood maximal function and $M_1^\sharp$ is the Fefferman-Stein's sharp maximal function ([13]). For simplicity, we denote $M_1(f)(x) = M(f)(x)$ and $M_1^\sharp(f)(x) = M^\sharp(f)(x)$.

Now we will give a famous lemma which is related to the sharp maximal function.

LEMMA 1. (Fefferman-Stein's inequalities) [13]

(i) Suppose that $\phi : (0,\infty) \to (0,\infty)$ is a doubling function and $\omega \in A_\infty$. Then there exists a positive constant C depending on the A_∞ constant of ω and the doubling condition of ϕ, such that

$$\sup_{\lambda>0}\phi(\lambda)\omega(\{x\in\mathbb{R}^n : M_\delta(f)(x) > \lambda\}) \leqslant C\sup_{\lambda>0}\phi(\lambda)\omega(\{x\in\mathbb{R}^n : M_\delta^\sharp(f)(x) > \lambda\})$$

for every function such that the left-hand side is finite.

(ii) Suppose that $0 < p < \infty$. Then there exists a positive constant C depending upon the A_∞ constant of ω and p, such that

$$\int_{\mathbb{R}^n} (M_\delta f(x))^p \,\omega(x)dx \leqslant C \int_{\mathbb{R}^n} \left(M^\sharp_\delta f(x)\right)^p \omega(x)dx$$

for every function if the left-hand side is finite.

We say that a kernel K satisfies the classical Hörmander type condition which is simply denoted by $K \in H_1$, if the following inequality holds,

$$\sup_{x\in\mathbb{R}^n} \int_{|y|>2|x|} |K(x-y) - K(-y)|dy < \infty. \tag{13}$$

Furthermore, we say that a kernel K satisfies the L^r $(1 \leqslant r < \infty)$ – Hörmander type condition which is denoted by $K \in H_r$, if there exist numbers $c_r > 1$ and $C_r > 0$, such that for $\forall x \in \mathbb{R}^n$ and $R > c_r|x|$, we have

$$\sum_{m=1}^{\infty} (2^m R)^n \left(\frac{1}{(2^m R)^n} \int_{2^m R<|y|\leqslant 2^{m+1}R} |K(x-y) - K(-y)|^r dy \right)^{1/r} \leqslant C_r \tag{14}$$

with $r < \infty$. If $r = \infty$ (here we denote $K \in H_\infty$), we have

$$\sum_{m=1}^{\infty} (2^m R)^n \sup_{2^m R<|y|\leqslant 2^{m+1}R} |K(x-y) - K(-y)| \leqslant C_\infty. \tag{15}$$

Next we state some basic facts from the theory of Orlicz spaces. For more information about Orlicz spaces, readers can see [28].

Let $\Phi : [0,\infty) \to [0,\infty)$ be a Young function. That is a continuous, convex, increasing function with $\Phi(0) = 0$ and such that $\Phi(t) \to \infty$ as $t \to \infty$. For a function f defined on a cube Q, the mean Luxemburg norm of f is defined by

$$\|f\|_{\Phi,Q} = \inf\{\lambda > 0 : \frac{1}{|Q|} \int_Q \Phi\Big(\frac{|f(y)|}{\lambda}\Big)dt \leqslant 1\}.$$

For the Luxemburg norm, the following generalized Hölder inequality holds,

$$\frac{1}{|Q|} \int_Q f(x)g(x)h(x)dx \leqslant C\|f\|_{A,Q}\|g\|_{B,Q}\|h\|_{C,Q},$$

where A,B,C are Young functions and satisfy $A^{-1}(t)B^{-1}(t)C^{-1}(t) \leqslant t \ \ (t \geqslant 1)$.

At the same time, we have the following generalized Hölder inequality,

$$\|fg\|_{B,Q} \leqslant 2\|f\|_{A,Q}\|g\|_{C,Q},$$

where A,B,C are Young functions and they satisfy the following condition,

$$A^{-1}(t)C^{-1}(t) \leqslant B^{-1}(t) \ \ (\forall t > 0).$$

Associated with each Young function Φ, its complementary function $\bar{\Phi}(s)$ is defined by

$$\bar{\Phi}(s) = \sup_{t>0}\{st - \Phi(t)\}.$$

It is easy to see that $\Phi(t) = t(1+\log^+ t)$ is a Young function and its complementary function $\bar{\Phi}(t) \approx e^t$ (see [28]). In this case, we have

$$\frac{1}{|Q|}\int_Q |f(y)g(y)|dy \leqslant \|f\|_{\Phi,Q}\|g\|_{\bar{\Phi},Q}.$$

When $\Phi(t) = t(1+\log^+ t)$, we write $\|f\|_{L\log L,Q} = \|f\|_{\Phi,Q}$ and $\|f\|_{\exp L,Q} = \|f\|_{\bar{\Phi},Q}$.

Also for each Young function B, we define the following maximal function associated with B,

$$M_B(f)(x) = \sup_{x\in Q}\|f\|_{B,Q}.$$

From [23], we know that if $B \in B_p$, then M_B is bounded on $L^p(\mathbb{R}^n)$ space for $1 < p < \infty$.

Similarly, the fractional maximal function associated with the Young function is defined by

$$M_{\alpha,B}(f)(x) = \sup_{x\in Q}|Q|^{\frac{\alpha}{n}}\|f\|_{B,Q} \ \ (0<\alpha<n).$$

3. Main Lemmas

In this section, we give the main lemmas that will be used throughout this paper.

LEMMA 2. *Let $I_{\alpha,q}$ be as a vector-valued generalized fractional integral with its kernel $K_\alpha \in H_{\mathscr{A},\alpha}$. Suppose that $0<\alpha<n$, $1<q<\infty$ and $0<\delta\leqslant 1$, then there exists a constant C, such that*

$$M_\delta^\sharp(I_{\alpha,q}(f))(x) \leqslant CM_{\alpha,\bar{\mathscr{A}}}(|f|_q)(x). \tag{16}$$

Proof. Let $f = \{f_j\}_1^\infty$ be any smooth vector-valued function. Fix $x\in\mathbb{R}^n$ and let B be a ball centered at x of radius r. Now we decompose $f = f^1 + f^2$, where $f^1 = f\chi_{2B} = \{f_j\chi_{2B}\}_1^\infty$.

By the definition of $M_\delta^\sharp(f)(x)$, it is enough to prove

$$\left(\frac{1}{|Q|}\int_Q ||I_{\alpha,q}(f)(y)|^\delta - c^\delta|dy\right)^{1/\delta} \leqslant CM_{\alpha,\bar{\mathscr{A}}}(|f|_q)(x). \tag{17}$$

In order to prove (17), we choose $c = I_{\alpha,q}(f^2)(x_0)$ and we have the following estimates,

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q ||I_{\alpha,q}(f)(y)|^\delta - c^\delta|dy\right)^{1/\delta}\\
&\leqslant \left(\frac{1}{|Q|}\int_Q |I_{\alpha,q}(f)(y) - I_{\alpha,q}(f^2)(x_0)|^\delta dy\right)^{1/\delta}\\
&\leqslant C\left(\frac{1}{|Q|}\int_Q |I_{\alpha,q}(f^1)(y)|^\delta dy\right)^{1/\delta} + C\left(\frac{1}{|Q|}\int_Q |I_{\alpha,q}(f^2)(y) - I_{\alpha,q}(f^2)(x_0)|^\delta dy\right)^{1/\delta}\\
&= C(I+II).
\end{aligned}$$

For I, by the Kolmogorov's inequality, it is easy to see

$$I \leqslant CM_\alpha(|f|_q)(x_0) \leqslant CM_{\alpha,\bar{\mathscr{A}}}(|f|_q)(x_0). \tag{18}$$

So it suffices to show

$$\left(\frac{1}{|Q|}\int_Q |I_{\alpha,q}(f^2)(y) - I_{\alpha,q}(f^2)(x_0)|^\delta dy\right)^{1/\delta} \leqslant CM_{\alpha,\bar{\mathscr{A}}}(|f|_q)(x_0). \tag{19}$$

To see this, first we note that

$$\left||I_{\alpha,q}(f^2)(x)|^\delta - |I_{\alpha,q}(f^2)(x_0)|^\delta\right| \leqslant |I_{\alpha,q}(f^2)(x) - I_{\alpha,q}(f^2)(x_0)|^\delta$$

and

$$\begin{aligned}
&|I_{\alpha,q}(f^2)(x) - I_{\alpha,q}(f^2)(x_0)|\\
&\leqslant |I_\alpha(f^2)(x) - I_\alpha(f^2)(x_0)|_q = \left(\sum_{j=1}^\infty |I_\alpha f_j^2(x) - I_\alpha f_j^2(x_0)|^q\right)^{1/q}\\
&= \left(\sum_{j=1}^\infty \left|\int_{\mathbb{R}^n/2Q} (K_\alpha(x_0-y) - K_\alpha(x-y))f_j(y)dy\right|^q\right)^{1/q}\\
&= \left(\sum_{j=1}^\infty \left|\sum_{m=1}^\infty \int_{2^mR<|y-x_0|\leqslant 2^{m+1}R} (K_\alpha(x_0-y) - K_\alpha(x-y))f_j(y)dy\right|^q\right)^{1/q}\\
&\leqslant \sum_{m=1}^\infty \int_{2^mR<|y-x_0|\leqslant 2^{m+1}R} (K_\alpha(x_0-y) - K_\alpha(x-y))|\left(\sum_{j=1}^\infty |f_j|^q\right)^{1/q} dy.
\end{aligned}$$

Then by (8) and the generalized Hölder inequality, we can easily get

$$|I_{\alpha,q}(f^2)(x) - I_{\alpha,q}(f^2)(x_0)| \leqslant CM_{\alpha,\bar{\mathscr{A}}}(|f|_q)(x_0).$$

Thus (19) holds and we finish the proof of Lemma 2. □

LEMMA 3. *Let $I^k_{\alpha,b,q}$ be as in Theorem 1 with its kernel $K_\alpha \in H_{\mathscr{A},k,\alpha} \cap H_{\mathscr{B},\alpha}$. Then for $0 < \delta < \varepsilon < 1$, there exists a constant C, such that*

$$M^\sharp_\delta(I^k_{\alpha,b,q}(f))(x) \leqslant C\sum_{j=0}^{k-1} \|b\|^{k-j}_{BMO} M_\varepsilon(I^j_{\alpha,q,b}(f))(x) + C\|b\|^k_{BMO} M_{\alpha,\bar{\mathscr{A}}}(|f|_q)(x). \quad (20)$$

Proof. Without loss of generality, we may assume $\|b\|_{\mathrm{BMO}} = 1$. For any constant λ, we may write

$$I^k_{\alpha,b}(f)(x) = I_\alpha((\lambda - b)^k(f))(x) + \sum_{m=0}^{k-1} C_{k,m}(b(x)-\lambda)^{k-m} I^m_\alpha f(x).$$

By a similar argument as in the proof of Lemma 2, we choose a ball B centered at x of radius $R > 0$. Then we split each f by $f = f^1 + f^2$ where $f^1 = f\chi_{2B} = \{f_j\chi_{2B}\}^\infty_{j=1}$. Let $\bar{B} = 2B$ and $C_B = \{C_j\}^\infty_{j=1}$ which will be chosen later along the proof. Thus we have

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B \left||I^k_{\alpha,b,q}(f)(y)|^\delta - |C_B|^\delta\right|\right)^{1/\delta} \\
&\leqslant C\sum_{m=0}^{k-1}\left(\frac{1}{|B|}\int_B |b(y)-b_{\bar{B}}|^{(k-m)\delta} |I^m_{\alpha,b,q}(f)(y)|^\delta dy\right)^{1/\delta} \\
&\quad + C\left(\frac{1}{|B|}\int_B |I_{\alpha,q}(b_{\bar{B}} - b)^k(f^1)(y)|^\delta dy\right)^{1/\delta} \\
&\quad + C\left(\frac{1}{|B|}\int_B |I_{\alpha,q}(b_{\bar{B}} - b)^k(f^2)(y) - C_B|^\delta dy\right)^{1/\delta} \\
&= C(I + II + III).
\end{aligned}$$

For I, by the Hölder inequality, we get

$$\begin{aligned}
I &\leqslant \sum_{m=0}^{k-1}\left(\frac{1}{|B|}\int_B |b(y)-b_{\bar{B}}|^{(k-m)\delta q'}\right)^{1/\delta q'}\left(\frac{1}{|B|}\int_B |I^m_{\alpha,b,q}(f)(y)|dy\right)^{1/\delta q} \\
&\leqslant C\sum_{m=0}^{k-1}\|b\|^{k-m}_{\mathrm{BMO}} M_\varepsilon(I^m_{\alpha,b,q}(f))(x).
\end{aligned}$$

For II, by the fact that $I_{\alpha,q}$ is of weak type $(1, \frac{n}{n-\alpha})$, then we have the following estimates by the Kolmogorov inequality and the generalized Hölder inequality,

$$\begin{aligned}
II &\leqslant C\frac{\|I_{\alpha,q}((b_{\bar{B}} - b)^k(f^1))\|_{WL^{\frac{n}{n-\alpha}}}}{\|\chi_B\|_{\frac{n-\alpha}{n}}} \\
&\leqslant C\frac{\|(b_{\bar{B}} - b)^k(f^1)\|_{L^1}}{|B|^{1-\alpha/n}} \leqslant CM_{\alpha,\bar{\mathscr{A}}}(f)(x).
\end{aligned}$$

For III, we choose $C_B = \{C_j\}_{j=1}^{\infty} = \{I_\alpha((b_{\bar{B}} - b)^k f_j^2)(x_B)\}_{j=1}^{\infty}$ and let $B_m = 2^{m+1}B$. First, we calculate $|I_{\alpha,q}(b_{\bar{B}} - b)^k f^2(y) - C_B|$ as follows.

$$\begin{aligned}
&|I_{\alpha,q}(b_{\bar{B}} - b)^k f^2(y) - C_B| \\
&\leqslant |I_\alpha((b_{\bar{B}} - b)^k f^2(y) - I_\alpha(b_{\bar{B}} - b)^k f^2)(x_B)|_q \\
&\leqslant \left(\sum_{j=1}^{\infty} | \int_{\mathbb{R}^n \setminus 2B} |b(z) - b_{\bar{B}}|^k |K_\alpha(y-z) - K_\alpha(x_B - z)||f_j(z)|dz|^q \right)^{1/q} \\
&\leqslant \left(\sum_{j=1}^{\infty} | \sum_{m=1}^{+\infty} \int_{2^{m+1}B \setminus 2^m B} |b(z) - b_{\bar{B}}|^k |K_\alpha(y-z) - K_\alpha(x_B - z)||f_j(z)|dz|^q \right)^{1/q} \\
&\leqslant \sum_{m=1}^{+\infty} \int_{2^{m+1}B \setminus 2^m B} |b(z) - b_{\bar{B}}|^k |K_\alpha(y-z) - K_\alpha(x_B - z)|(\sum_{j=1}^{\infty} |f_j(z)|^q)^{1/q} dz \\
&\leqslant \sum_{m=1}^{+\infty} (2^m R)^n \frac{1}{|B_m|} \int_{2^{m+1}B \setminus 2^m B} |b(z) - b_{B_m}|^k |K_\alpha(y-z) - K_\alpha(x_B - z)|(\sum_{j=1}^{\infty} |f_j(z)|^q)^{1/q} dz \\
&\quad + \sum_{m=2}^{\infty} (2^m R)^n m^k \frac{1}{|B_m|} \int_{B_m} |K_\alpha(y-z) - K_\alpha(x_B - z)| \chi_{S_j}(z) |f(z)|_q dz \\
&= C(IV + V).
\end{aligned}$$

For IV, by the generalized Hölder inequality, we obtain

$$\begin{aligned}
IV &\leqslant \sum_{m=1}^{\infty} (2^m R)^n \|(b - b_j)^k\|_{\bar{C}_k, B_j} \||f|_q\|_{\bar{\mathscr{A}}, B_j} \|K_\alpha(y - .) - K_\alpha(x_B - .)\|_{\mathscr{B}, B_j} \\
&\leqslant M_{\alpha, \bar{A}}(|f|_q)(x) \sum_{j=1}^{\infty} (2^m R)^{m-\alpha} \|K_\alpha(y - .) - K_\alpha(x_B - .)\|_{\mathscr{B}, |z| \sim 2R} \\
&\leqslant C M_{\alpha, \bar{\mathscr{A}}}(|f|_q)(x).
\end{aligned}$$

For V, again using the generalized Hölder inequality we can get

$$\begin{aligned}
V &\leqslant \sum_{m=2}^{\infty} (2^m R)^n m^k \|(K_\alpha(y - .) - K_\alpha(x_B - .))\chi_{S_j}\|_{\mathscr{A}, B_j} \||f|_q\|_{\bar{\mathscr{A}}, B_j} \\
&\leqslant C M_{\alpha, \bar{\mathscr{A}}}(|f|_q)(x) \sum_{m=1}^{\infty} (2^m R)^{n-\alpha} m^k \|K_\alpha(. - (x_B - y))) - K_\alpha(.)\|_{\mathscr{A}, |z| \sim 2^m R} \\
&\leqslant C M_{\alpha, \bar{\mathscr{A}}}(|f|_q)(x).
\end{aligned}$$

Combining the estimates above, we finish the proof of Lemma 3. □

4. Proofs of Theorems 1 and 2

In order to prove Theorem 1, by the extrapolation theorem in [5] ((2.3) in [5]), we know that Theorem 1 holds for all $0 < p < \infty$ and all $\omega \in A_\infty$ if and only if (9) holds for some fixed $p_0 \in (0,\infty)$ and all $\omega \in A_\infty$. First we only consider the case that ω and b are all L^∞ functions. By homogeneity, we may assume that $\|b\|_{\mathrm{BMO}} = 1$. Now we proceed by induction.

When $k = 0$, by Lemma 2, Lebesgue differential theorem and the Fefferman-Stein's inequality, we can easily prove Theorem 1 in the case $k = 0$.

Next, we assume that the results hold for all $0 \leqslant j \leqslant k-1$ and we will treat the case $j = k$. For $f \in C_0^\infty(\mathbb{R}^n)$, and without loss of generality we may assume that $\|M_{\alpha,\bar{\mathscr{A}}}(|f|_q)\|_{L^{p_0}_\omega}$ is finite otherwise there is nothing to prove. Therefore, by the Fefferman-Stein's inequality and Lemma 3, we have

$$\begin{aligned}\|I^k_{\alpha,b,q}(f)\|_{L^{p_0}_\omega} &\leqslant \|M_\delta(I^k_{\alpha,b,q}(f))\|_{L^{p_0}_\omega} \leqslant \|M^\sharp_\delta(I^k_{\alpha,b,q}(f))\|_{L^{p_0}_\omega}\\ &\leqslant C\sum_{j=0}^{k-1}\|M_\varepsilon(I^j_{\alpha,b,q}(f))\|_{L^{p_0}_\omega} + C\|M_{\alpha,\bar{\mathscr{A}}}(|f|_q)\|_{L^{p_0}_\omega}.\end{aligned}$$

Next we should give the estimate of $\|M_\varepsilon(I^j_{\alpha,b,q}(f))\|_{L^{p_0}_\omega}$.

Since $\delta < \frac{p_0}{r} < 1$, we can choose $\varepsilon > 1$ such that $\delta < \varepsilon < \frac{p_0}{r} < 1$ and $\omega \in A_{p_0/\varepsilon}$, then we have

$$\|M_\varepsilon(I^j_{\alpha,b,q}(f))\|_{L^{p_0}_\omega} = \|M(|I^j_{\alpha,b,q}(f)|^\varepsilon)\|^{1/\varepsilon}_{L^{p_0/\varepsilon}_\omega} \leqslant C\|I^j_{\alpha,b,q}(f)\|_{L^{p_0}_\omega}.$$

Then by induction hypothesis, we can get

$$\|M_\varepsilon(I^j_{\alpha,b,q}(f))\|_{L^{p_0}_\omega} \leqslant C\|I^j_{\alpha,b,q}(f)\|_{L^{p_0}_\omega} \leqslant C\|M_{\alpha,\bar{\mathscr{A}}}(|f|_q)\|_{L^{p_0}_\omega}.$$

So it remains to prove $\|M_\delta(I^k_{\alpha,b,q}(f))\|_{L^{p_0}_\omega} < \infty$.

For $\omega \in A_\infty$, then there exists $r > 1$ such that $\omega \in A_r$. By the fact that $0 < \delta < p_0/r < 1$, which implies $r < p_0/\delta$, we have $\omega \in A_{p_0/\delta}$. Then by the $L^p(\omega)$ $(\omega \in A_p)$ boundedness of $M(f)(x)$, we can easily check

$$\|M_\delta(I^k_{\alpha,b,q}(|f|))\|_{L^{p_0}_\omega} = \|M(I^k_{\alpha,b,q}(|f|)^\delta)\|^{1/\delta}_{L^{p_0/\delta}_\omega} \leqslant C\|I^k_{\alpha,b,q}(f)\|_{L^{p_0}_\omega}.$$

So it suffices to show

$$\|I^k_{\alpha,b,q}(f)\|_{L^{p_0}_\omega} < \infty.$$

As $\omega \in L^\infty$, the above problem reduces to prove $\|I^k_{\alpha,b,q}(f)\|_{L^{p_0}} < \infty$. By the extrapolation theorem in [5] and Theorem 2.1 in [29], we have the boundedness of $I_{\alpha,q}$ from L^{p_1} to L^{p_0} with $1/p_1 - 1/p_0 = \alpha/n$, thus we have

$$\|I^k_{\alpha,b,q}(f)\|_{L^{p_0}} = \|\sum_{m=0}^{k} C_{m,j}b^{k-m}I_{\alpha,q}(b^m f)\|_{L^{p_0}} \leqslant C\|b\|_{L^\infty}\|f\|_{L^{p_1}} < \infty.$$

In order to remove the restrictions that $\omega \in L^\infty$ and $b \in L^\infty$, we can solve this problem easily if we follow the arguments in [16, p. 1415-1416] and we omit the details here.

So far the proof of Theorem 1 has been finished.

REMARK 4. In [2], the authors proved a similar Coifman's type estimate for the commutators of generalized fractional integral in the non-vector-valued case. However, we cannot use the vector-valued extrapolation theorem ((2.5) in [5]) to get Theorem 1 directly due to the definition of $M_{\alpha,\bar{\mathscr{A}}}(|f|_q)(x)$.

Next, let us give the proof of Theorem 2. By a duality argument, it suffices to show

$$\int_{\mathbb{R}^n} |I^{*,k}_{\alpha,b,q'}(f)(x)|^{p'} (M_{\alpha p,D}\mu(x))^{1-p'} dx \leqslant C \int_{\mathbb{R}^n} |f|_{q'}^{p'} \mu(x)^{1-p'} dx. \tag{21}$$

From [3, p. 1512], we know that $(M_{\alpha p,D}\mu(x))^\delta \in A_1$ for $0 < \delta < 1$. Choosing $r > p'$ and $\delta = (p'-1)/(r-1)$, we have

$$(M_{\alpha p,D}\mu(x))^{1-p'} = \{(M_{\alpha p,D}\mu(x))^{(p'-1)/(r-1)}\}^{1-r} \in A_r \subset A_\infty.$$

Since the B_p condition implies the L^p boundedness of $M_B(f)(x)$, then by Theorem 1, the condition $\xi^{-1}(t)\theta^{-1}(t) \leqslant \bar{\mathscr{A}}^{-1}(t)$ with $D(t) = \theta(t^{1/p})$ and the generalized Hölder inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} |I^{*,k}_{\alpha,b,q}(f)(x)|^{p} (M_{\alpha p,D}\mu(x))^{1-p'} dx \\
&\leqslant \int_{\mathbb{R}^n} (M_{\alpha,\bar{\mathscr{A}}}|f|_{q'}(x))^{p'} (M_{\alpha p,D}\mu(x))^{1-p'} dx \\
&= |Q|^{\frac{\alpha p'}{n}+\frac{\alpha p(1-p')}{n}} \int_{\mathbb{R}^n} (M_{\bar{\mathscr{A}}}|f|_{q'}(x))^{p'} (M_D\mu(x))^{1-p'} dx \\
&\leqslant C \int_{\mathbb{R}^n} M_\xi(|f|_{q'}\mu^{-1/p})(x)^{p'} M_\theta(\mu^{1/p})(x)^{p'} (M_D\mu(x))^{1-p'} dx \\
&= C \int_{\mathbb{R}^n} M_\xi(|f|_{q'}\mu^{-1/p})(x)^{p'} (M_D\mu(x))^{p/p'} (M_D\mu(x))^{1-p'} dx \\
&= C \int_{\mathbb{R}^n} M_\xi(|f|_{q'}\mu^{-1/p})(x)^{p'} dx \leqslant C \int_{\mathbb{R}^n} |f|_{q'}^{p'} \mu(x)^{1-p'} dx.
\end{aligned}$$

So far, we have proved (21) and the proof of Theorem 2 has been finished.

5. Proof of Theorem 3

In order to prove Theorem 3, we proceed by induction. Without loss of generality, we may assume $\|b\|_{\mathrm{BMO}} = 1$.

First, we treat the case $k = 0$. For a fixed $\lambda > 0$, let $\{Q_j\}$ be the standard family of nonoverlapping dyadic cubes satisfying

$$\lambda < \frac{1}{|Q_j|} \int_{Q_j} |f(x)|_q dx \leqslant 2^n \lambda \tag{22}$$

maximal with respect to the left hand side inequality. Furthermore, we denote by $Q_j = Q(z_j, r_j)$ with z_j and r_j be the center and sidelength of each Q_j, respectively. Denote $\Omega = \bigcup_j Q_j$, then we have $|f(x)|_q \leqslant \lambda$ for a.e. $x \in \mathbb{R}^n \setminus \Omega$.

Now we proceed to construct a slightly different version of the classical Calderón-Zygmund decomposition, readers may see [27] for more details. Split f as $f = g + h$, where $g = \{g_i\}_{i=1}^{\infty}$ and each $g_i(x)$ is given by $g_i(x) = f_i(x)$ if $x \in \mathbb{R}^n \setminus \Omega$ and $g_i(x) = (f_i)_{Q_j}$ if $x \in Q_j$, where $(f_i)_{Q_j}$ is defined by the average of f_i on the cube Q_j. Furthermore, we denote

$$h(x) = \{h_i(x)\}_{i=1}^{\infty} = \left\{\sum_j h_{ij}(x)\right\}_{i=1}^{\infty}$$

with $h_{ij}(x) = (f_i(x) - (f_i)_{Q_j})\chi_{Q_j}(x)$.

Let $\tilde{\Omega} = \bigcup\limits_j 2Q_j$, then we have

$$\begin{aligned}
&\omega(\{x \in \mathbb{R}^n : |I_{\alpha,q}(f)(x)| > \lambda\}) \\
&\leqslant \omega(\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |I_{\alpha,q}(g)(x)| > \lambda/2\}) + \omega(\tilde{\Omega}) \\
&\quad + \omega(\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |I_{\alpha,q}(h)(x)| > \lambda/2\})
\end{aligned}$$

To estimate the first term, setting $\omega^*(x) = \omega(x)\chi_{\mathbb{R}^n \setminus \tilde{\Omega}}$ and by Theorem 2, we obtain

$$\begin{aligned}
&\omega(\{y \in \mathbb{R}^n \setminus \tilde{\Omega} : |I_{\alpha,q}(g)(x)| > \lambda/2\}) \\
&\leqslant \frac{C}{\lambda^p} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} |I_{\alpha,q}(g)(x)|^p \omega(x) dx \\
&= \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |I_{\alpha,q}(g)(x)|^p \omega^*(x) dx \\
&\leqslant \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |g(x)|_q^p M_{\alpha p,D} \omega^*(x) dx \\
&\leqslant \frac{C}{\lambda^p} \int_{\mathbb{R}^n \setminus \Omega} |f(x)|_q^p M_{\alpha p,D} \omega^*(x) dx + \frac{C}{\lambda^p} \int_{\Omega} |g(x)|_q^p M_{\alpha p,D} \omega^*(x) dx \\
&= C(I + II).
\end{aligned}$$

The estimate of I is trival since $|f|_q \leqslant \lambda$ for a.e. $x \in \mathbb{R}^n \setminus \Omega$.

Now we only need to treat II, from [15], we know that for any Young function $\mathscr{A}$ and weight v with $M_{\mathscr{A}} v < \infty$ for a.e. $x \in \mathbb{R}^n$ and any cube Q, there is $M_{\mathscr{A}}(v\chi_{\mathbb{R}^n \setminus 2Q})(y) \approx \operatorname{ess\,inf}\limits_{z \in Q} M_{\mathscr{A}}(v\chi_{\mathbb{R}^n \setminus 2Q})(z)$ for a.e. $y \in Q$ and $x \in Q$. Thus we get

$$M_{\alpha p,D} \omega_j(x) \approx ess \inf_{z \in Q_j} M_{\alpha p,D} \omega_j(z). \tag{23}$$

Setting $\omega_j(x) = \omega(x)\chi_{\mathbb{R}^n \setminus Q_j}$, then by (23) and the generalized Minkowski inequality, we obtain

$$\begin{aligned}
II &= \frac{C}{\lambda^p} \int_{\Omega} |g(x)|_q^p M_{\alpha p,D} \omega^*(x) dx \\
&= \frac{C}{\lambda^p} \sum_j \int_{Q_j} |g(x)|_q^p M_{\alpha p,D} \omega^*(x) dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{C}{\lambda^p} \sum_j \int_{Q_j} \left(\sum_{i=1}^{\infty} |(f_i)_{Q_j}|^q \right)^{p/q} M_{\alpha p,D}\omega_j(x)dx \\
&\leqslant \frac{C}{\lambda^p} \sum_j \left(\sum_{i=1}^{\infty} \left| \frac{1}{|Q_j|} \int_{Q_j} f_i(z)dz \right|^q \right)^{p/q} |Q_j| \inf_{y\in Q_j} M_{\alpha p,D}\omega_j(y) \\
&\leqslant \frac{C}{\lambda^p} \sum_j \left(\frac{1}{|Q_j|} \int_{Q_j} |f(z)|_q dz \right)^p |Q_j| \inf_{y\in Q_j} M_{\alpha p,D}\omega_j(y) \\
&\leqslant \frac{C}{\lambda} \sum_j \left(\frac{1}{|Q_j|} \int_{Q_j} |f(z)|_q dz \right) |Q_j| \inf_{y\in Q_j} M_{\alpha p,D}\omega_j(y) \\
&\leqslant \frac{C}{\lambda} \sum_j \int_{Q_j} |f(z)|_q M_{\alpha p,D}(\omega)(z)dz \\
&\leqslant \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(z)|_q M_{\alpha p,D}(\omega)(z)dz.
\end{aligned}
$$

For the second term $\omega(\tilde{\Omega})$, by a standard argument, we have

$$
\begin{aligned}
\omega(\tilde{\Omega}) &\leqslant C \sum_j \frac{\omega(2Q_j)}{|2Q_j|} |2Q_j| \leqslant \frac{C}{\lambda} \sum_j \frac{\omega(2Q_j)}{|2Q_j|} \int_{Q_j} |f(y)|_q dy \\
&\leqslant \frac{C}{\lambda} \sum_j \int_{Q_j} |f(y)|_q M\omega(y)dy \leqslant \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)|_q M\omega(y)dy.
\end{aligned}
$$

For the last term, by the Minkowski inequality, we get

$$
\begin{aligned}
&\omega(\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |I_{\alpha,q}(h)(x)| > \lambda/2\}) \\
&\leqslant \frac{C}{\lambda} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} |I_{\alpha,q}(h)(y)| \omega(y)dy \\
&= \frac{C}{\lambda} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} \left[\sum_{i=1}^{\infty} |I_\alpha(h_i)(y)|^q \right]^{1/q} \omega(y)dy \\
&= \frac{C}{\lambda} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} \left[\sum_{i=1}^{+\infty} \left| \sum_j \int_{Q_j} K_\alpha(y-z) h_{ij}(z)dz \right|^q \right]^{1/q} \omega(y)dy \\
&= \frac{C}{\lambda} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} \left[\sum_{i=1}^{+\infty} \left| \sum_j \int_{Q_j} (K_\alpha(y-z) - K_\alpha(y-z_j)) h_{ij}(z)dz \right|^q \right]^{1/q} \omega(y)dy \\
&\leqslant \frac{C}{\lambda} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} \left(\sum_j \int_{Q_j} \left[\sum_{i=1}^{\infty} |K_\alpha(y-z) - K_\alpha(y-z_j)|^q |h_{ij}(z)|^q \right]^{1/q} dz \right) \omega(y)dy
\end{aligned}
$$

$$\leqslant \frac{C}{\lambda}\sum_j \int_{Q_j}\left(\int_{\mathbb{R}^n\setminus 2Q_j}\left[\sum_{i=1}^{\infty}|K_\alpha(y-z)-K_\alpha(y-z_j)|^q|h_{ij}(z)|^q\right]^{1/q}\omega(y)dy\right)dz$$

$$\leqslant \frac{C}{\lambda}\sum_j \int_{Q_j}\left(\int_{\mathbb{R}^n\setminus 2Q_j}|K_\alpha(y-z)-K_\alpha(y-z_j)|\left[\sum_{i=1}^{\infty}|h_{ij}(z)|^q\right]^{1/q}\omega(y)dy\right)dz.$$

Now we are going to treat the inner term in the last inequality. First, by (8) with $k=0$, we have the following estimates.

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus 2Q_j}|K_\alpha(y-z)-K_\alpha(y-z_j)|\omega(y)dy\\
&\leqslant \sum_{m=1}^{\infty}\int_{|y-z_j|\sim 2^m r_j}|K_\alpha(y-z)-K_\alpha(y-z_j)|\omega(y)dy\\
&\leqslant C\sum_{m=1}^{+\infty}(2^m r_j)^n\|K_\alpha(y-z)-K_\alpha(y-z_j)\|_{\mathscr{A},|y-z_j|\sim 2^m r_j}\|\omega\|_{\bar{\mathscr{A}},|y-z_j|\leqslant 2^{m+1}r_j}\\
&\leqslant C\inf_{Q_j} M_{\alpha,\bar{\mathscr{A}}}\omega(x).
\end{aligned}$$

So by (23) and the generalized Hölder inequality, we have

$$\begin{aligned}
&\omega(\{x\in\mathbb{R}^n\setminus\tilde{\Omega}:|I_{\alpha,q}(h)(x)|>\lambda/2\})\\
&\leqslant \frac{C}{\lambda}\sum_j \int_{Q_j}\left(\int_{\mathbb{R}^n\setminus 2Q_j}|K_\alpha(y-z)-K_\alpha(y-z_j)|\left[\sum_{i=1}^{\infty}|h_{ij}(z)|^q\right]^{1/q}\omega(y)dy\right)dz\\
&\leqslant \frac{C}{\lambda}\sum_j \int_{Q_j}\left[\sum_{i=1}^{\infty}|h_{ij}(z)|^q\right]^{1/q}dz\inf_{Q_j} M_{\alpha,\bar{\mathscr{A}}}\omega(x)\\
&\leqslant \frac{C}{\lambda}\sum_j \int_{Q_j}|f(z)|_q M_{\alpha,\bar{\mathscr{A}}}\omega(z)dz+\frac{C}{\lambda}\sum_j \int_{Q_j}|g(z)|_q dz\inf_{Q_j} M_{\alpha,\bar{\mathscr{A}}}\omega(x)\\
&=III+IV.
\end{aligned}$$

For III, it is obvious that $III\leqslant \frac{C}{\lambda}\int_{\mathbb{R}^n}|f(z)|_q M_{\alpha,\bar{\mathscr{A}}}\omega(z)dz$.

To estimate IV, we have

$$\begin{aligned}
IV &\leqslant \frac{C}{\lambda}\sum_j\left[\sum_{i=1}^{\infty}\left|\frac{1}{|Q_j|}\int_{Q_j}f_i(z)dz\right|^q\right]^{1/q}|Q_j|\inf_{Q_j} M_{\alpha,\bar{\mathscr{A}}}\omega\\
&\leqslant \frac{C}{\lambda}\sum_j\frac{1}{|Q_j|}\int_{Q_j}|f(z)|_q dz|Q_j|\inf_{Q_j} M_{\alpha,\bar{\mathscr{A}}}\omega\\
&\leqslant \frac{C}{\lambda}\sum_j\int_{Q_j}|f(z)|_q M_{\alpha,\bar{\mathscr{A}}}\omega(z)dz\\
&\leqslant \frac{C}{\lambda}\int_{\mathbb{R}^n}|f(z)|_q M_{\alpha,\bar{\mathscr{A}}}\omega(z)dz.
\end{aligned}$$

So far, we have proved Theorem 3 in the case $k=0$.

Next we will consider the case when $k \in Z^+$. Suppose that the theorem is true for all $j<k$, then with the same notation as in the proof of the case $k=0$, we can get

$$\begin{aligned}
&\omega\{x \in \mathbb{R}^n : |I^k_{\alpha,b,q}(f)(x)| > \lambda\}\\
&\leqslant \omega(\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : I^k_{\alpha,b,q}(g)(x) > \lambda/2\}) + \omega(\tilde{\Omega})\\
&\quad + \omega(\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |I^k_{\alpha,b,q}h(x)| > \lambda/2\})\\
&= I + II + III.
\end{aligned}$$

By an argument similar to the proof of the previous case when $k=0$, we have the following estimates for I and II,

$$I \leqslant \frac{C}{\lambda}\int_{\mathbb{R}^n} |f(x)|_q M_{\alpha p,D}(\omega)(x)dx \tag{24}$$

and

$$II \leqslant \frac{C}{\lambda}\int_{\mathbb{R}^n} |f(x)|_q M\omega(x)dx. \tag{25}$$

Now, we plan to give the estimate of III. First we split $I^k_{\alpha,b,q}$ as follows,

$$\begin{aligned}
I^k_{\alpha,b,q}(h)(x) &\leqslant \left(\sum_i \left|\sum_j [(b(x)-(b)_{Q_j})^k I_\alpha(h_{ij})(x)\right|^q\right)^{1/q}\\
&\quad + \left(\sum_i \left|\sum_j I_\alpha((b(x)-(b)_{Q_j})^k h_{ij})(x)\right|^q\right)^{1/q}\\
&\quad + \left(\sum_i \left|\sum_{l=1}^{k-1} C_{k,l} I^l_{\alpha,b}(\sum_j (b-(b)_{Q_j})^{k-l} h_{ij})(x)\right|^q\right)^{1/q}\\
&= A_1(x) + A_2(x) + A_3(x).
\end{aligned}$$

Thus we get

$$III \leqslant III^1 + III^2 + III^3,$$

where $III^i = \omega\{x \in \mathbb{R}^n \setminus \Omega : A_i(x) > \lambda/6\}$.

For III^1, by the generalized Hölder inequality, we have

$$\begin{aligned}
III^1 &= \omega\left\{x \in \mathbb{R}^n \setminus \Omega : \left(\sum_i \left|\sum_j [(b(x)-(b)_{Q_j})^k I_\alpha(h_{ij})(x)\right|^q\right)^{1/q} > \lambda/6\right\}\\
&\leqslant \sum_j \frac{C}{\lambda}\int_{\mathbb{R}^n\setminus 2Q_j} |b(x)-(b)_{Q_j}|^k \left(\sum_i |I_\alpha(h_{ij})(x)|^q\right)^{1/q} \omega(x)dx
\end{aligned}$$

$$\leqslant \sum_j \frac{C}{\lambda} \int_{\mathbb{R}^n\setminus 2Q_j} |b(x)-(b)_{Q_j}|^k \int_{Q_j} |K_\alpha(x-y)-K_\alpha(x-z_j)| \left(\sum_i |h_{ij}(y)|^q\right)^{1/q} \omega(x)dydx$$
$$\leqslant \sum_j \frac{C}{\lambda} \int_{Q_j} \left(\sum_i |h_{ij}(y)|^q\right)^{1/q} \left(\int_{\mathbb{R}^n\setminus 2Q_j} |K_\alpha(x-y)-K_\alpha(x-z_j)||b(x)-(b)_{Q_j}|^k \omega(x)dx\right) dy.$$

Now, we give the estimates of $\int_{\mathbb{R}^n\setminus 2Q_j} |K_\alpha(x-y)-K_\alpha(x-z_j)||b(x)-(b)_{Q_j}|^k\omega(x)dx$. By the generalized Hölder inequality and the fact $K_\alpha \in H_{\mathscr{A},k,\alpha} \cap H_{\mathscr{B},\alpha}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus 2Q_j} |K_\alpha(x-y)-K_\alpha(x-z_j)||b(x)-(b)_{Q_j}|^k\omega(x)dx \\
&= \sum_{m=1}^{\infty} \int_{2^m r_j \leqslant |x-z_j| < 2^{m+1} r_j} |K_\alpha(x-y)-K_\alpha(x-z_j)||b(x)-(b)_{Q_j}|^k\omega(x)dx \\
&\leqslant \sum_{m=1}^{\infty} \int_{2^m r_j \leqslant |x-z_j| < 2^{m+1} r_j} |K_\alpha(x-y)-K_\alpha(x-z_j)||b(x)-(b)_{2^{m+1}Q_j}|^k\omega(x)dx \\
&\quad + \sum_{m=1}^{\infty} \int_{2^m r_j \leqslant |x-z_j| < 2^{m+1} r_j} |K_\alpha(x-y)-K_\alpha(x-z_j)||(b)_{2^{m+1}Q_j}-(b)_{Q_j}|^k\omega(x)dx \\
&\leqslant \sum_{m=1}^{\infty} (2^m r_j)^n \|K_\alpha(.-y)-K_\alpha(.-z_j)\|_{\mathscr{B},|x-z_j|<2^m r_j} \||b-(b)_{2^{m+1}Q_j}|^k\|_{\bar{C}_k,2^{m+1}Q_j} \|\omega\|_{\bar{\mathscr{A}},2^{m+1}Q_j} \\
&\quad + \sum_{m=1}^{\infty} (2^m r_j)^n m^k \|K_\alpha(.-y)-K_\alpha(.-z_j)\|_{\mathscr{A},|x-z_j|<2^m r_j} \|\omega\|_{\bar{\mathscr{A}},2^{m+1}Q_j} \\
&\leqslant \operatorname{ess\,inf} M_{\alpha,\bar{\mathscr{A}}}(\omega)(x).
\end{aligned}$$

So by an argument similar to the previous case when $k=0$, we can easily get

$$III^1 \leqslant C\int_{\mathbb{R}^n} |f(y)|_q M_{\alpha,\bar{\mathscr{A}}}\omega(x)dx.$$

For III^2, noting the following fact

$$(\sum_i |h_{ij}|^q)^{1/q} \leqslant |f|_q\chi_{Q_j} + |g|_q\chi_{Q_j}, \tag{26}$$

then by the induction hypothesis, we have

$$\begin{aligned}
III^2 &= \left\{x\in\mathbb{R}^n\setminus\tilde{\Omega} : \left(\sum_i |\sum_j I_\alpha(b(x)-(b)_{Q_j})^k h_{ij}(x)|^q\right)^{1/q} > \lambda/6\right\} \\
&\leqslant \frac{C}{\lambda}\int_{\mathbb{R}^n} \left[\sum_i |\sum_j |b(x)-(b)_{Q_j}|h_{ij}(x)|^q\right]^{1/q} \\
&\qquad \times \left(M\omega(x) + M_{\alpha,\bar{\mathscr{A}}}\omega(x) + M_{\alpha p,D}\omega(x)\right)\chi_{\mathbb{R}^n\setminus 2Q_j}dx
\end{aligned}$$

$$\leqslant \frac{C}{\lambda}\sum_j\int_{Q_j}|b(x)-(b)_{Q_j}|\left(\sum_i|h_{ij}(x)|^q\right)^{1/q}$$
$$\times\left(M(\chi_{\mathbb{R}^n\setminus 2Q_j}\omega)(x)+M_{\alpha,\bar{\mathscr{A}}}(\chi_{\mathbb{R}^n\setminus 2Q_j}\omega)(x)M_{\alpha p,D}(\chi_{\mathbb{R}^n\setminus 2Q_j}\omega)(x)\right)dx$$
$$\leqslant \frac{C}{\lambda}\sum_j\inf_{Q_j}\left[M(\chi_{\mathbb{R}^n\setminus 2Q_j}\omega)(x)+M_{\alpha,\bar{\mathscr{A}}}(\chi_{\mathbb{R}^n\setminus 2Q_j}\omega)(x)M_{\alpha p,D}(\chi_{\mathbb{R}^n\setminus 2Q_j}\omega)(x)\right]$$
$$\times\left(\int_{Q_j}|b(x)-(b)_{Q_j}||f(x)|_q\chi_{Q_j}dx+\int_{Q_j}|b(x)-(b)_{Q_j}||g(x)|_q\chi_{Q_j}dx\right)$$
$$=V_1+V_2.$$

For V_2, by the definition of BMO, then a similar argument as in the case $k=0$ leads to

$$V_2\leqslant\frac{C}{\lambda}\int_{\mathbb{R}^n}|f(x)|_q\left(M\omega(x)+M_{\alpha,\bar{\mathscr{A}}}\omega(x)+M_{\alpha p,D}\omega(x)\right)dx.$$

For V_1, by the generalized Hölder inequality, we have

$$V_1\leqslant\frac{C}{\lambda}\sum_j\inf_{Q_j}\left[M(\chi_{\mathbb{R}^n\setminus 2Q_j}\omega)(x)+M_{\alpha,\bar{\mathscr{A}}}(\chi_{\mathbb{R}^n\setminus 2Q_j}\omega)(x)+M_{\alpha p,D}(\chi_{\mathbb{R}^n\setminus 2Q_j}\omega)(x)\right]$$
$$\times|Q_j|\||f|_q\|_{Q_j,L(\log L)^k}.$$

Now we should note the following fact,

$$\frac{1}{\lambda}|Q_j|\||f|_q\|_{Q_j,L(\log L)^k}\leqslant 2\int_{Q_j}\phi_k(\frac{|f|_q}{\lambda})(x)dx.$$

Thus we have

$$V_1\leqslant C\int_{\mathbb{R}^n}\phi_k(\frac{|f|_q}{\lambda})(x)(M_{\alpha,\bar{\mathscr{A}}}\omega(x)+M_{\alpha p,D}\omega(x)+M\omega(x))dx.$$

Finally, to estimate III^3, we may decompose III^3 as follows,

$$III^3=\omega\left(x\in\mathbb{R}^n\setminus\Omega:\left(\sum_i\left|\sum_{l=1}^{k-1}C_{k,l}I^l_{\alpha,b}(\sum_j(b-(b)_{Q_j})^{k-1}h_{ij})(x)]\right|^q\right)^{1/q}>\lambda/6\right)$$
$$\leqslant\omega\left(x\in\mathbb{R}^n\setminus\Omega:\sum_{l=1}^{k-1}C_{kl}I^l_{\alpha,b}(\sum_j[(b-b_{Q_j})^{k-l}(\sum_i|h_{ij}|^q)^{1/q}])>\lambda/6\right)$$
$$\leqslant\sum_{l=1}^{k-1}\omega\left(x\in\mathbb{R}^n\setminus\Omega:I^l_{\alpha,b}(\sum_j(b-b_{Q_j})^{k-l}(\sum_i|h_{ij}|^q)^{1/q})>\lambda/6\right)$$
$$\leqslant\sum_{l=1}^{k-1}\omega\left(x\in\mathbb{R}^n\setminus\Omega:I^l_{\alpha,b}(\sum_j(b-b_{Q_j})^{k-l}|f|_q\chi_{Q_j})>\lambda/12\right)$$
$$+\sum_{l=1}^{k-1}\omega\left(x\in\mathbb{R}^n\setminus\Omega:I^l_{\alpha,b}(\sum_j(b-b_{Q_j})^{k-l}|g|_q\chi_{Q_j})>\lambda/12\right)$$

$$= III^{31} + III^{32}.$$

For III^{31}, as $\omega^*(x) = \omega(x)\chi_{\mathbb{R}^n\setminus\tilde{\Omega}}$, then by induction hypothesis and (23), we have

$$III^{31} = \sum_{l=1}^{k-1} \omega\left(x \in \mathbb{R}^n \setminus \Omega : I^l_{\alpha,b}\left(\sum_j (b-b_{Q_j})^{k-l}|f|_q\chi_{Q_j}\right)(x) > \lambda/12\right)$$

$$\leqslant \sum_{l=1}^{k-1} \int_{\mathbb{R}^n} \phi_l\left(\frac{\sum_j |b-b_{Q_j}|^{k-l}|f|_q\chi_{Q_j}}{\lambda}\right)(x)\left(M_{\alpha,\mathscr{A}}\omega^*(x) + M_{\alpha p}\omega^*(x) + M\omega^*(x)\right)dx$$

$$\leqslant \sum_{l=1}^{k-1}\sum_j \int_{Q_j} \phi_l\left(\frac{|f|_q}{\lambda}|b-b_{Q_j}|^{k-l}\right)(x)(M_{\alpha,\mathscr{A}}\omega_j(x) + M_{\alpha p,D}\omega_j(x) + M\omega_j(x))dx$$

$$\leqslant \sum_{l=1}^{k-1}\sum_j \left(\inf_{Q_j} M_{\alpha,\mathscr{A}}\omega_j(x) + \inf_{Q_j} M_{\alpha p,D}\omega_j(x) + \inf_{Q_j} M\omega_j(x)\right)\int_{Q_j} \phi_l\left(\frac{|f|_q}{\lambda}|b-b_{Q_j}|^{k-l}\right)(x)dx.$$

By an argument similar in [1, p. 472], we can easily get

$$\int_{Q_j} \phi_l\left(\frac{|f|_q}{\lambda}|b-b_{Q_j}|^{k-l}\right)(x)dx \leqslant C\int_{Q_j} \phi_k\left(\frac{|f|_q}{\lambda}\right)(x)dx. \tag{27}$$

Thus we obtain

$$III^{31} \leqslant C\int_{\mathbb{R}^n} \phi_k\left(\frac{|f|_q}{\lambda}\right)(x)\left(M\omega(x) + M_{\alpha,\bar{\mathscr{A}}}\omega(x) + M_{\alpha p,D}\omega(x)\right)dx.$$

To estimate III^{32}, by the Jensen inequality, (27) and the similar argument as in the case $k = 0$, we obtain

$$III^{32} \leqslant C\sum_j \int_{Q_j} \phi_l\left(\frac{|f|_q}{\lambda}|b-b_{Q_j}|^{k-l}\right)(x)M_{\alpha,\bar{\mathscr{A}}}\omega(x)dx \leqslant C\int_{\mathbb{R}^n} \phi_k\left(\frac{|f|_q}{\lambda}\right)(x)M_{\alpha,\bar{\mathscr{A}}}\omega(x)dx.$$

Combining the above estimates, we can draw that the proof of Theorem 3 has been finished.

6. Applications

In this section, we will show that by Theorems 1-3, we can easily get the weighted boundedness of vector-valued commutator of fractional integral with a rough kernel. Before giving the main results in this section, let us give some definitions and notations.

Denote by $\mathbb{S}^{n-1}$ be the unit sphere on $\mathbb{R}^n$. For any $x \neq 0$, we write $x' = x/|x|$. Assume that $\Omega \in L^1(\mathbb{S}^{n-1})$ is a homogeneous of degree 0 function on $\mathbb{S}^{n-1}$. For $0 < \alpha < n$, let $\mathscr{A}$ be a Young function such that $\mathscr{B}(t) = \mathscr{A}(t^{\frac{n-\alpha}{n}})$ is also a Young function. Suppose that Ω belongs to $\in L^{\mathscr{A}}(\mathbb{S}^{n-1})$ and Ω satisfies the following $L^{\mathscr{A}}(\mathbb{S}^{n-1})$-Dini condition, i.e.,

$$\int_0^1 \omega_{\mathscr{A}}(t)\frac{dt}{t} < \infty,$$

where

$$\omega_{\mathscr{A}}(t) = \sup_{|y|\leqslant t} \|\Omega(.+y) - \Omega(.)\|_{\mathscr{A},\mathbb{S}^{n-1}}.$$

Now we will study the vector-valued commutator of fractional integral operator with a rough kernel as follows,

$$I^{\Omega,k}_{\alpha,b,q}(f)(x) = |I^{\Omega,k}_{\alpha,b}(f)(x)|_q = \left(\sum_{j=1}^{\infty} |I^{\Omega,k}_{\alpha,b}(f_j)(x)|^q\right)^{1/q},$$

where $I^{\Omega,k}_{\alpha,b}(f)(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)(b(x)-b(y))^k dy$. By the discussions in [15] and [29], we can easily conclude that the kernel $\frac{\Omega(y)}{|y|^{n-\alpha}} \in H_{\mathscr{A},k,\alpha} \cap H_{\mathscr{B},\alpha}$. Thus from Theorems 1-3, we get the following results and we omit the proofs here.

THEOREM 4. *Assume that $\Omega \in L^r(\mathbb{S}^{n-1})$ is homogeneous of degree 0 on $\mathbb{S}^{n-1}$. If Ω satisfies the L^r-Dini condition, we have*

(a) If $0 < p < \infty$ and $\omega \in A_\infty$, then

$$\int_{\mathbb{R}^n} |I^{\Omega,k}_{\alpha,b,q}(f)(x)|^p \omega(x)dx \leqslant C \int_{\mathbb{R}^n} (M_{\alpha,r'}|f|_q(x))^p \omega(x)dx.$$

(b) If $1 < p < r$ and μ is a weight which is only local integrable, then

$$\int_{\mathbb{R}^n} |I^{\Omega,k}_{\alpha,b,q}(f)(x)|^p \mu(x)dx \leqslant C \int_{\mathbb{R}^n} |f(x)|^p_q M_{\alpha p,D}\mu(x)dx.$$

(c) If $1 < p < r$ and μ is a weight which is only local integrable, then for any $\lambda > 0$,

$$\mu\{x \in \mathbb{R}^n : |I^{\Omega,k}_{\alpha,b,q}(f)(x) > \lambda\} \leqslant \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)|_q (M_{r'}\mu(x) + M_{\alpha p,D}\mu(x))dx.$$

In the above cases, $D(t) = t^{(r/p)'}(1+\log^+ t)^{(r/p)'(p-1)+\varepsilon}$ and ε is a positive number which is small enough.

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