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A NOTE ON STRONG LAW OF LARGE NUMBERS FOR DEPENDENT RANDOM SEQUENCE

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Abstract. This note is devoted to establish a general strong law of large numbers for dependent random variables. As corollaries, we generalized some known results.

1. Introduction

The most classical results in the literature on the SLLN problem for a sequence of independent identically distributed random variables are apparently the Kolmogorov and Marcinkiewicz's SLLN. Recently, in reference [1], Jajte gave a strong law of large numbers (SLLN) for a large class of means for independent and identically distributed (i.i.d.) random variables.

THEOREM 1. (Jajte, 2003) *Let $g(\cdot)$, be a positive, increasing function and $h(\cdot)$ a positive function such that $\phi(y) \equiv g(y)h(y)$ satisfies the following conditions:*

(1) For some $d \geqslant 0$, $\phi(\cdot)$ is strictly increasing on $[d,+\infty)$ with range $[0,+\infty)$.

(2) There exist C and a positive integer k_0 such that $\phi(y+1)/\phi(y) \leqslant C$, $y \geqslant k_0$.

(3) There exist constants a and b such that $\phi^2(s)\int_s^\infty \frac{1}{\phi^2(x)}dx \leqslant as+b$, $s > d$.

Then, for i.i.d. random variables $\{X_n, n \in \mathbb{N}\}$

$$\frac{1}{g(n)}\sum_{k=1}^{n}\frac{X_k - X_k\mathbf{1}_{(|X_k|\leqslant\phi(k))}}{h(k)} \to 0 \quad a.s. \quad \textit{if and only if } E[\phi^{-1}(|X|)] < \infty, \tag{1.1}$$

where ϕ^{-1} is the inverse of function ϕ.

Inspired by Jajte's idea, in this paper, we consider the problem of arbitrary random variables and their limiting behavior from a new prospective. Throughout this paper, let $\mathbb{N}$ denote the set of positive integers, $\{X, X_n, \mathscr{F}_n, n \in \mathbb{N}\}$ be a stochastic sequence defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, i.e., the sequence of σ-fields $\{\mathscr{F}_n, n \in \mathbb{N}\}$ in $\mathscr{F}$ is increasing in n, and $\{\mathscr{F}_n\}$ are adapted to random variables $\{X_n\}$. Throughout this paper $\mathscr{F}_0$ will denote the trivial σ field $\{\Phi, \Omega\}$.

We begin by introducing the terminology and two lemmas.

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DEFINITION 1. Let $\{X_n, n \in \mathbb{N}\}$ be a sequence of random variables and is said to be: stochastically dominated by a random variable X (we write $\{X_n, n \in \mathbb{N}\} \prec X$) if there exists a constant $D > 0$, for almost every $\omega \in \Omega$, such that

$$\sup_{n \in \mathbb{N}} P\{|X_n| > t\} \leqslant DP\{|X| > t\} \; for \; all \; t > 0. \tag{1.2}$$

LEMMA 1. (Chow and Teicher, 1988) *Let $\{X_n, \mathscr{F}_n, n \in \mathbb{N}\}$ be an $L_p (1 \leqslant p \leqslant 2)$ martingale difference sequence, if $\sum_{n=1}^{\infty} E(|X|_n^p | \mathscr{F}_{n-1}) < \infty$, then $\sum_{n=1}^{\infty} X_n$ a.s. converges.*

LEMMA 2. *Let $\{X, X_n, n \in \mathbb{N}\}$ be a sequence of random variables. If $\{X_n\} \prec X$, then for all $t > 0$*

$$\mathbb{E}|X_n|^2 \mathbf{1}_{(|X_n| \leqslant t)} \leqslant D[t^2 \mathbb{P}(|X| > t) + \mathbb{E}X^2 \mathbf{1}_{(|X| \leqslant t)}] \tag{1.3}$$

Proof. By the integral equality

$$2\int_0^t s\mathbb{P}(|X_n| > s)ds = t^2 \mathbb{P}(|X_n| > t) + \mathbb{E}|X_n|^2 \mathbf{1}_{(|X_n| \leqslant t)},$$

it follows that

$$\begin{aligned}
&\mathbb{E}|X_n|^2 \mathbf{1}_{(|X_n| \leqslant t)} \\
\leqslant & 2\int_0^t s\mathbb{P}(|X_n| > s)ds \\
\leqslant & 2\int_0^t s\mathbb{P}(|X| > s)ds \\
= & D[t^2 \mathbb{P}(|X| > t) + \mathbb{E}X^2 \mathbf{1}_{(|X| \leqslant t)}] \quad \square
\end{aligned}$$

2. Strong law of large numbers

THEOREM 2. *Let $g(\cdot)$, $h(\cdot)$ and $\phi(\cdot)$ be as in Theorem 1, and let $\{X, X_n, \mathscr{F}_n, n \in \mathbb{N}\}$ be a sequence of random variables defined as before. Assume that $\{X_n, n \in \mathbb{N}\} \prec X$. If $E[\phi^{-1}(|X|)] < \infty$, then*

$$\lim_n \frac{1}{g(n)} \sum_{k=1}^{n} \frac{X_k - E(X_k \mathbf{1}_{(|X_k| \leqslant \phi(k))} | \mathscr{F}_{k-1})}{h(k)} = 0 \;\; a.s. \tag{2.1}$$

Proof. To prove (2.1) by applying the Kronecker lemma, it suffices to show that

$$\text{the series } \sum_{n=1}^{\infty} \frac{X_n - E(X_n \mathbf{1}_{(|X_n| \leqslant \phi(n))} | \mathscr{F}_{n-1})}{\phi(n)} \text{ converges } a.s. \tag{2.2}$$

Let $Y_n = \frac{X_n}{\phi(n)}\mathbf{1}_{(|X_n|\leqslant\phi(n))}$, $Z_n = \frac{X_n}{\phi(n)}\mathbf{1}_{(|X_n|>\phi(n))}$, then $\frac{X_n}{\phi(n)} = Y_n + Z_n$, $n\in\mathbb{N}$. Note that $\{X_n\}\prec X$ and condition $E[\phi^{-1}(|X|)]<\infty$, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}P(|X_n|>\phi(n))\\ \leqslant&D\sum_{n=1}^{\infty}P(|X|>\phi(n))\\ =&D\sum_{n=1}^{\infty}P(\phi^{-1}(|X|)>n)\\ \leqslant&DE[\phi^{-1}(|X|)]<\infty,\end{aligned}$$

which shows that $P(\{|X_n|>\phi(n)\}, i.o.) = 0$, and hence $\sum_{k=1}^{\infty}Z_n<\infty$ *a.s.*

Let $W_n = Y_n - E(Y_n|\mathscr{F}_{n-1})$, then $(W_n,\mathscr{F}_n, n\in\mathbb{N})$ is a martingale difference sequence.

Since

$$\begin{aligned}&\sum_{k=1}^{\infty}E(Y_k^2)=\sum_{k=1}^{\infty}\frac{E[X_k^2\mathbf{1}_{(|X_k|\leqslant\phi(k))}]}{\phi^2(k)}\\ \leqslant&D\sum_{k=1}^{\infty}[E\mathbf{1}_{(|X|>\phi(k))}+\frac{E(X^2\mathbf{1}_{(|X|\leqslant\phi(k))})}{\phi^2(k)}]\ \text{(by lemma 2)}\\ \leqslant&D\sum_{k=1}^{\infty}P(|X|>\phi(k))+D[k_0+C^2\sum_{k=k_0+1}^{\infty}\frac{E[X^2\mathbf{1}_{(|X|\leqslant\phi(k))}]}{\phi^2(k+1)}]\\ \leqslant&DE[\phi^{-1}(|X|)]+Dk_0+DC^2E[X^2\int_{\phi^{-1}(|X|)}^{\infty}\frac{1}{\phi^2(x)}dx]\\ \leqslant&DE[\phi^{-1}(|X|)]+Dk_0+DC^2aE[\phi^{-1}(|X|)]+DC^2b<\infty\end{aligned}$$

Note that

$$E[\sum_{n=1}^{\infty}E(W_n^2|\mathscr{F}_{n-1})]\leqslant E[\sum_{n=1}^{\infty}E(Y_n^2|\mathscr{F}_{n-1})]=\sum_{n=1}^{\infty}EY_n^2<\infty. \tag{2.3}$$

which implies that $\sum_{n=1}^{\infty}E(W_n^2|\mathscr{F}_{n-1})<\infty$ *a.s.*, hence by lemma 1, we have $\sum_{n=1}^{\infty}W_n$ *a.s.* convergence

Note that $\frac{X_n - E(X_n\mathbf{1}_{(|X_n|\leqslant\phi(n))}|\mathscr{F}_{n-1})}{\phi(n)} = Z_n + W_n$, these complete the Theorem 2. □

Theorem 2 also includes some particular cases of means, we can establish the following:

COROLLARY 1. *Under the conditions of Theorem 2, we have*

$$\lim_n\frac{1}{\log n}\sum_{k=1}^{n}\frac{X_k - E(X_k\mathbf{1}_{(|X_k|\leqslant\phi(k))}|\mathscr{F}_{k-1})}{k}=0\ \ a.s. \tag{2.4}$$

Proof. Let $h(y)=y$, $g(y)=\log y$ i.e. $\phi(y)=y\log y$. In this case $\phi^{-1}(y)\sim\frac{y}{\log y}$ as $y\to\infty$, therefore $E(|X|^{\alpha})\leqslant E[\phi^{-1}(|X|)]\leqslant E(|X|)$, for $0<\alpha<1$. □

COROLLARY 2. *Under the conditions of Theorem 2, if* $E|X|^\alpha < \infty$ $(1 \leqslant \alpha \leqslant 2)$, *we have*

$$\lim_n \frac{1}{n^{1/\alpha}} \sum_{k=1}^{n} [X_k - E(X_k \mathbf{1}_{(|X_k| \leqslant \phi(k))} | \mathscr{F}_{k-1})] = 0 \ \ a.s. \tag{2.5}$$

Proof. Let $h(y) = 1$, $g(y) = y^\alpha$ i.e. $\phi(y) = y^\alpha$. In this case $\phi^{-1}(y) = y^{1/\alpha}$. □

COROLLARY 3. *Let* $\{X, X_n, n \in \mathbb{N}\}$ *be a sequence of random variables with* $\{X_n\} \prec X$. *Further, let* $\mathscr{F}_n = \sigma(X_1, \cdots, X_n)$ *and* $\mathscr{F}_{-n} = \{\Omega, \phi\}$, $n \geqslant 0$, ϕ_n *be as in Theorem 1. If* $E[\phi^{-1}(|X|)] < \infty$, *then for any* $m \geqslant 1$

$$\lim_n \frac{1}{g(n)} \sum_{k=1}^{n} \frac{X_k - E(X_k \mathbf{1}_{(|X_k| \leqslant \phi(k))} | \mathscr{F}_{k-m})}{h(k)} = 0 \ \ a.s. \tag{2.6}$$

Proof. Since $\{X_{nm+l}, \mathscr{F}_{nm+l}, n \geqslant 1\}$ is a stochastic sequence and $\{X_{nm+l}\} \prec X$, by Theorem 2, we have for $l = 0, 1, \cdots, m-1$ that

$$\sum_{n=1}^{\infty} \frac{X_{nm+l} - E(X_{nm+l} \mathbf{1}_{(|X_{nm+l}| \leqslant \phi(nm+l))} | \mathscr{F}_{(n-1)m+l})}{\phi(nm+l)} \text{ converges } a.s. \tag{2.7}$$

therefore we have

$$\begin{aligned} &\sum_{n=m}^{\infty} \frac{X_n - E(X_n \mathbf{1}_{(|X_n| \leqslant \phi(n))} | \mathscr{F}_{n-m})}{\phi(n)} \\ &= \sum_{l=0}^{m-1} \sum_{n=1}^{\infty} \frac{X_{nm+l} - E(X_{nm+l} \mathbf{1}_{(|X_{nm+l}| \leqslant \phi(nm+l))} | \mathscr{F}_{(n-1)m+l})}{\phi(nm+l)} \text{ converges } a.s. \quad \square \end{aligned} \tag{2.8}$$

COROLLARY 4. *Let* $\{X_n, n \in \mathbb{N}\}$ *be a sequence of m-dependent random variables. Further, let* $\mathscr{F}_n = \sigma(X_1, \cdots, X_n)$ *and* $\mathscr{F}_{-n} = \{\Omega, \phi\}$, $n \geqslant 0$, ϕ_n *be as in Theorem 1. If there exists a random variable* X *such that* $\{X_n\} \prec X$ *and* $E[\phi^{-1}(|X|)] < \infty$, *then*

$$\lim_n \frac{1}{g(n)} \sum_{k=1}^{n} \frac{X_k - E(X_k \mathbf{1}_{(|X_k| \leqslant \phi(k))})}{h(k)} = 0 \ \ a.s. \tag{2.9}$$

Proof. Note that $\{X_n, n \in \mathbb{N}\}$ is a sequence of m-dependent random variables, then $E(X_n | \mathscr{F}_{n-m}) = EX_n$, the corollary follows from Corollary 3. □

DEFINITION 2. Let $\{X_n, n \in \mathbb{N}\}$ be a sequence of random variables, and let $\mathscr{F}_n^m = \sigma(X_n, \cdots, X_m)$. We say that the sequence $\{X_n, n \in \mathbb{N}\}$ is *-mixing if there exist a positive integer M and a nondecreasing function $\varphi(n)$ defined on integers $n \geqslant M$ with $\lim_n \varphi(n) = 0$, such that for $n > M$, $A \in \mathscr{F}_0^m$ and $B \in \mathscr{F}_{m+n}^\infty$ the relation

$$|P(A \cap B) - P(A)P(B)| \leqslant \varphi(n) P(A) P(B),$$

holds for any integer $m \geqslant 1$.

It has been proved (see [5]) that the *-mixing condition is equivalent to the condition

$$|P(B|\mathscr{F}_0^m) - P(B)| \leqslant \varphi(n)P(B) \ \ a.s.$$

for $B \in \mathscr{F}_{m+n}^{\infty}$ and $m \geqslant 1$, implies

$$|E(X_{n+m}|\mathscr{F}_0^m) - EX_{n+m}| \leqslant \varphi(n)E|X_{n+m}| \ \ a.s. \tag{2.10}$$

THEOREM 3. *Let $\{X, X_n, n \in \mathbb{N}\}$ be a sequence of *-mixing random variables with $\{X_n\} \prec X$. Further, let $\mathscr{F}_n = \sigma(X_1, \cdots, X_n)$ and $\mathscr{F}_{-n} = \{\Omega, \phi\}$, $n \geqslant 0$, ϕ_n be as in Theorem 1. Assume $E|X_k \mathbf{1}_{(|X_k| \leqslant \phi(k))}| \leqslant K < \infty$ for every $k \geqslant 1$. If $E[\phi^{-1}(|X|)] < \infty$, then*

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} [X_k - EX_k \mathbf{1}_{(|X_k| \leqslant \phi(k))}] = 0 \ \ a.s. \tag{2.11}$$

Proof. By Corollary 3 of Theorem 2, we have for each $m \geqslant 1$

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} [X_k - E(X_k \mathbf{1}_{(|X_k| \leqslant \phi(k))}|\mathscr{F}_{k-m})] = 0 \ \ a.s. \tag{2.12}$$

Since $\{X_n, n \in \mathbb{N}\}$ is *-mixing, by (2.10) and (2.12), we obtain

$$\begin{aligned}
&|\frac{1}{n} \sum_{k=1}^{n} [X_k - EX_k \mathbf{1}_{(|X_k| \leqslant \phi(k))}]| \\
\leqslant &|\frac{1}{n} \sum_{k=1}^{n} [X_k - E(X_k \mathbf{1}_{(|X_k| \leqslant \phi(k))}|\mathscr{F}_{k-m})]| \\
&+ \frac{1}{n} \sum_{k=1}^{n} |[E(X_k \mathbf{1}_{(|X_k| \leqslant \phi(k))}|\mathscr{F}_{k-m}) - EX_k \mathbf{1}_{(|X_k| \leqslant \phi(k))}]| \\
\leqslant &|\frac{1}{n} \sum_{k=1}^{n} [X_k - E(X_k \mathbf{1}_{(|X_k| \leqslant \phi(k))}|\mathscr{F}_{k-m})]| + \frac{\varphi(m)}{n} \sum_{k=1}^{n} E|X_k \mathbf{1}_{(|X_k| \leqslant \phi(k))}| \\
\leqslant &|\frac{1}{n} \sum_{k=1}^{n} [X_k - E(X_k \mathbf{1}_{(|X_k| \leqslant \phi(k))}|\mathscr{F}_{k-m})]| + \varphi(m)K \to 0 \ \ a.s. \text{ (as } n \to \infty).
\end{aligned}$$

thus, using the Kroneker lemma, (2.11) follows. □

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