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STRONG APPROXIMATION OF SOME ADDITIVE FUNCTIONALS OF SYMMETRIC STABLE PROCESS

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Abstract. This paper deals with some additive functionals based on the local time of symmetric stable process. In concrete, we obtain some L_p-inequalities of the local time and the fractional derivative of the local time of symmetric stable process of index $1 < \alpha \leqslant 2$. As an application, we generalize the well known Barlow-Yor [4] inequality, which we use to give a strong approximation version, (almost surely estimate), of occupation times problem of this process. Our results generalize those obtained by Csaki et *al.* [7] for Brownian motion, and Ait Ouahra and Ouali [2] for symmetric stable process of index $1 < \alpha \leqslant 2$ in L_p-norm.

1. Introduction

The strong approximations of Brownian additive functionals has been studied by Csaki et *al.* [7] as counterparts of limit theorem for additive functionals which feature the fractional derivative of Brownian local time. We first recall their result.

THEOREM 1. *Let f be a Borel function on $\mathbb{R}$ such that $\int_{\mathbb{R}} |x|^k |f(x)| dx < \infty$, for some $k > 0$. Then for any $0 < \gamma < \frac{3}{2}$ (with $\gamma \neq 1$) and all sufficiently small $\varepsilon > 0$, when t goes to infinity, we have*

$$\int_0^t D^{\gamma-1} f(B_s) ds = \frac{I(f)}{\Gamma(1-\gamma)} D^{\gamma-1} l_t(0) + o(t^{1-\frac{\gamma}{2}-\varepsilon}), \qquad a.s.$$

where $I(f) = \int_{\mathbb{R}} f(x) dx$, l_t^x is the local time of the Brownian motion B and $D^{\gamma-1}$ is a fractional derivative of order $\gamma - 1$, (see definition below).

This theorem and the law of the iterated logarithm, (LIL for brevity), of Csaki et *al.* [8], proved for $D^{\gamma-1} L_t(0)$, together imply that there exists a constant $0 < c(\gamma) < \infty$, depending only on γ, such that

$$\limsup_{t \to \infty} \frac{\int_0^t D^{\gamma-1} f(B_s) ds}{t^{1-\frac{\gamma}{2}} (\log \log t)^{\frac{\gamma}{2}}} = c(\gamma) \qquad a.s.$$

On the other hand, Ait Ouahra and Ouali [2] have established the following result, in L_p-norm, for some self similar process, namely symmetric stable process X of index $1 < \alpha \leqslant 2$ and fractional Brownian motion with Hurst parameter $0 < H < 1$.

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THEOREM 2. *Let f be a Borel function on $\mathbb{R}$ such that $\int_{\mathbb{R}}|x|^k|f(x)|dx<\infty$, for some $k>0$. Then for any $0\leqslant\gamma<\frac{\alpha-1}{2}$ and all sufficiently small $\varepsilon>0$ and $p\geqslant 1$, when t goes to infinity, we have*

$$\|\int_0^t D^\gamma f(X_s)ds\|_{2p}=\frac{I(f)}{\Gamma(1-\gamma)}\|D^\gamma L_t^{\cdot}(0)\|_{2p}+o(t^{\frac{\alpha-1}{\alpha}-\frac{\gamma}{\alpha}-\varepsilon}),$$

where $\|.\|_{2p}=(\mathbb{E}_0|.|^{2p})^{\frac{1}{2p}}$ and L_t^x the local time of symmetric stable process X.

REMARK 1. The same estimation in Theorem 2 can be obtained for the self similar process called Sub-fractional Brownian motion, (sfBm for brevity), of Hurst paramter $0<H<1$, (see definition of sfBm in Bojdecky et *al.* [5]).

Our purpose in this paper is to extend the result of Csaki et *al.* [7], to symmetric stable process of index $1<\alpha\leqslant 2$. We will prove the following theorem.

THEOREM 3. *Let f be a Borel function on $\mathbb{R}$ such that $\int_{\mathbb{R}}|x|^k|f(x)|dx<\infty$, for some $k>0$. Then for any $0<\gamma<\frac{\alpha-1}{2}$ and all sufficiently small $\varepsilon>0$, when t goes to infinity, we have*

$$\int_0^t D^\gamma f(X_s)ds=\frac{I(f)}{\Gamma(-\gamma)}D^\gamma L_t^{\cdot}(0)+o(t^{\frac{\alpha-1}{\alpha}-\frac{\gamma}{\alpha}-\varepsilon}),\qquad a.s.$$

The remainder of this paper is organized as follows: In the next section, we establish some L_p-inequalities for some additive functionals of symmetric stable process. Finally, in the last section, we give the prove of Theorem 3.

Most of the estimates in this paper contain unspecified positive constants. We use the same symbol C for these constants, even when they vary from one line to the next.

2. Some inequalities for some additive functionals of symmetric stable process

Throughout this paper, $X=(\Omega,\mathscr{F},\mathscr{F}_t,\theta_t,X_t,P^x)$ will denote the canonical realization of a real valued symmetric stable process of index $1<\alpha\leqslant 2$, with $X_0=0$. The sample paths of X_t are right-continuous with left limits a.s. (càdlàg for brevity), and has stationary independent increments with characteristic function

$$\mathbb{E}_0\exp(i\lambda X_t)=\exp(-t|\lambda|^\alpha),\qquad \forall t\geqslant 0,\lambda\in\mathbb{R}.$$

$\mathbb{E}_0$ denotes the expectation with respect to the distribution P^0 of the process starting from 0. And $(\theta_t):\Omega\to\Omega$ are the translation operators defined by $(\theta_t(\omega))(s)=\omega(t+s)$.

Notice that for $\alpha=2$, X is a Brownian motion.

It is known from Barlow [3] and Boylan [6] that the local time $\{L_t^x\ ;\ t\geqslant 0,x\in\mathbb{R}\}$ of X exists and is jointly continuous in t and x with compact support and satisfies the scaling property

$$(L_t^x,t\geqslant 0,x\in\mathbb{R},P^y)\ \underline{\underline{d}}\ \left(\lambda^{-\frac{\alpha-1}{\alpha}}L_{\lambda t}^{x\lambda^{\frac{1}{\alpha}}},t\geqslant 0,x\in\mathbb{R},P^{y\lambda^{\frac{1}{\alpha}}}\right)\ \forall\lambda>0,\qquad (1)$$

where $\underline{\underline{d}}$ means the equality in the sense of the finite-dimensional distributions.

In addition, L_t^x is an additive functional

$$L_t^x(\omega) = L_s^x(\omega) + L_{t-s}^x(\theta_s(\omega)), \tag{2}$$

and satisfies the occupation density formula

$$\int_0^t f(X_s(\omega))ds = \int_{\mathbb{R}} f(x)L_t^x(\omega)dx, \tag{3}$$

for any bounded or nonnegative Borel function f.

Moreover, by Lemma 3.3 in Marcus and Rosen [10] and Theorem 1 in Ait Ouahra and Eddahbi [1] and the Kolmogorov criterion, for all $T > 0$ fixed, we have almost surely: $\forall\ 0 < \beta < \frac{\alpha-1}{\alpha}$, $\exists\ 0 < C < \infty$ such that $\forall\ 0 \leqslant t,s \leqslant T$, $|x| \leqslant M$, where M is a constant

$$|L_t^x - L_s^x| \leqslant C|t-s|^{\beta}. \tag{4}$$

$\forall\ 0 < \beta_1 < \frac{\alpha-1}{2\alpha}$, $\forall\ 0 < \beta_2 < \frac{\alpha-1}{2}$, $\exists\ 0 < C < \infty$ such that $\forall\ 0 \leqslant t,s \leqslant T$, $|x|,|y| \leqslant M$, where M is a constant

$$|L_t^x - L_t^y - L_s^x + L_s^y| \leqslant C|t-s|^{\beta_1}|x-y|^{\beta_2}. \tag{5}$$

For $0 < \gamma < \frac{\alpha-1}{2}$, we define the fractional derivative of L_t^x as follows:

$$H_t^x := D^{\gamma}L_t^{\cdot}(x) = \frac{1}{\Gamma(-\gamma)}\int_0^t \frac{ds}{(X_s - x)^{1+\gamma}},$$

where $y^{\gamma} := |y|^{\gamma}sgn(y)$.

According to Fitzsimmons and Getoor [9], H_t^x is an additive functional satisfying the scaling property

$$(H_t^x, t \geqslant 0, x \in \mathbb{R}, P^y)\ \underline{\underline{d}}\ \left(\lambda^{-\frac{\alpha-1}{\alpha}-\frac{\gamma}{\alpha}} H_{\lambda t}^{x\lambda^{\frac{1}{\alpha}}}, t \geqslant 0, x \in \mathbb{R}, P^{y\lambda^{\frac{1}{\alpha}}}\right)\ \forall \lambda > 0. \tag{6}$$

On the other hand, Ait Ouahra and Eddahbi [1] showed in Lemma 1 that, for all $T > 0$ fixed, we have almost surely $\forall\ 0 < \beta < \frac{\alpha-1}{\alpha} - \frac{\gamma}{\alpha}$, $\exists\ 0 < C < \infty$ such that $\forall\ 0 \leqslant t,s \leqslant T$, $|x| \leqslant M$, where M is a constant

$$|H_t^x - H_s^x| \leqslant C|t-s|^{\beta}. \tag{7}$$

We refer the reader for a complete survey on the fractional derivative to Samko et *al.* [11] and the references therein.

The next lemma follows easily from the scaling properties (1) and (6).

LEMMA 1. *For all* $1 < \alpha \leqslant 2$, $0 < \nu < \frac{\alpha-1}{2}$ *and* $p \geqslant 1$, *we have*

$$\left(\sup_{x\in\mathbb{R}} L_t^x, t \geqslant 0, P^0\right) \stackrel{d}{=} \left(t^{\frac{\alpha-1}{\alpha}} \sup_{x\in\mathbb{R}} L_1^x, t \geqslant 0, P^0\right) \tag{8}$$

$$\left(\|L_t^{\cdot}\|_{p,\mathbb{R}}, t \geqslant 0, P^0\right) \stackrel{d}{=} \left(t^{\frac{\alpha-1}{\alpha}+\frac{1}{p\alpha}} \|L_1^{\cdot}\|_{p,\mathbb{R}}, t \geqslant 0, P^0\right) \tag{9}$$

$$\left(\sup_{x\in\mathbb{R}} H_t^x, t \geqslant 0, P^0\right) \stackrel{d}{=} \left(t^{\frac{\alpha-1}{\alpha}-\frac{\gamma}{\alpha}} \sup_{x\in\mathbb{R}} H_1^x, t \geqslant 0, P^0\right) \tag{10}$$

$$\left(\sup_{0\leqslant s\leqslant t} \|H_s^{\cdot}\|_{p,\mathbb{R}}, t \geqslant 0, P^0\right) \stackrel{d}{=} \left(t^{\frac{\alpha-1}{\alpha}-\frac{\gamma}{\alpha}+\frac{1}{p\alpha}} \|H_1^{\cdot}\|_{p,\mathbb{R}}, t \geqslant 0, P^0\right) \tag{11}$$

$$\left(\sup_{0\leqslant s\leqslant t} \sup_{x\neq y} \frac{|L_s^x - L_s^y|}{|x-y|^\nu}, t \geqslant 0, P^0\right) \stackrel{d}{=} \left(t^{\frac{\alpha-1}{\alpha}-\frac{\nu}{\alpha}} \sup_{x\neq y} \frac{|L_1^x - L_1^y|}{|x-y|^\nu}, t \geqslant 0, P^0\right) \tag{12}$$

where $\|.\|_{p,\mathbb{R}} = (\int_{\mathbb{R}} |.|^p)^{\frac{1}{p}}$.

In order to establish our results, we need the following lemma, (see Fitzsimmons and Getoor [9]).

LEMMA 2. *Let* $(A_t)_{t\geqslant 0}$ *be a continuous increasing* $(\mathscr{F}_t)$*-adapted real valued process with* $A_0 = 0$. *Assume that:*

(i) $A_t \leqslant A_s + K.A_{t-s}\circ\theta_s$, $\forall\ t,s \geqslant 0$, *for some constant* $K > 0$.

(ii) There exists a constant $q > 0$ *such that*

$$\lim_{z\to\infty} \sup_{\lambda>0, y\in\mathbb{R}} P^y(A_\lambda > \lambda^{\frac{1}{q}} z) = 0.$$

Then for each $p > 0$, *there exists a constant* $0 < C < \infty$ *such that*

$$\|A_t\|_p \leqslant C t^{\frac{1}{q}}, \qquad \forall\ t \geqslant 0,$$

where $\|.\|_p = (\mathbb{E}_0|.|^p)^{\frac{1}{p}}$.

The first application of Lemma 2 is the following result.

LEMMA 3. *For each* $p, p' \geqslant 1$, *there is a constant* $0 < C < \infty$ *such that*

$$\|\sup_{x\in\mathbb{R}} L_t^x\|_p \leqslant C t^{\frac{\alpha-1}{\alpha}} \tag{13}$$

$$\|\sup_{x\in\mathbb{R}} |L_t^x - L_s^x|\|_p \leqslant C|t-s|^{\frac{\alpha-1}{\alpha}} \tag{14}$$

$$\|\|L_t^{\cdot}\|_{p',\mathbb{R}}\|_p \leqslant C t^{\frac{\alpha-1}{\alpha}+\frac{1}{p'\alpha}} \tag{15}$$

$$\|\sup_{0\leqslant s\leqslant t} \|H_s^{\cdot}\|_{p',\mathbb{R}}\|_p \leqslant C|t-s|^{\frac{\alpha-1}{\alpha}-\frac{\gamma}{\alpha}+\frac{1}{p'\alpha}} \tag{16}$$

Proof. (13): We apply Lemma 2 with

$$A_t = \sup_{x\in\mathbb{R}} L_t^x, \qquad q = \Big(\frac{\alpha-1}{\alpha}\Big)^{-1}.$$

Clearly, A_t is increasing and satisfies (i) in Lemma 2 by (2).

Using (3), we get $L_t^x(\tau_y(\omega)) = L_t^{x-y}(\omega)$ where $\tau_y:\Omega\to\Omega$ is the translation $\omega \to \omega(.)+y$. Since X is spatially homogeneous *i.e.* ($\tau_y(P^0) = P^y$), it follows that the P^y distribution of (A_t) does not depend on y. Thus, by applying (8) and the fact that $0 < A_1 < \infty$ by (4), we get

$$\lim_{z\to\infty} \sup_{\lambda>0, y\in\mathbb{R}} P^y(A_\lambda > \lambda^{\frac{1}{q}} z) = \lim_{z\to\infty}\sup_{\lambda>0} P^0(A_\lambda > \lambda^{\frac{1}{q}} z) = \lim_{z\to\infty} P^0(A_1 > z) = 0.$$

Which completes the proof of (13).

Now, we verify (14). From the Markov property of X in s and (13), we have

$$\begin{aligned}
\mathbb{E}_0(\sup_{x\in\mathbb{R}} |L_t^x - L_s^x|^p) &= \mathbb{E}_0(\sup_{x\in\mathbb{R}} |L_{t-s}^x \circ \theta_s|^p) \\
&= \mathbb{E}_0(\sup_{x\in\mathbb{R}} |L_{t-s}^x|^p \circ \theta_s) \\
&= \mathbb{E}_0(\mathbb{E}_0(\sup_{x\in\mathbb{R}} |L_{t-s}^x|^p \circ \theta_s / \mathscr{F}_s) \\
&= \int P^0(X_s \in dy)\mathbb{E}_0(\sup_{x\in\mathbb{R}} |L_{t-s}^{x-y}|^p) \\
&= \int P^0(X_s \in dy)\mathbb{E}_0(\sup_{x\in\mathbb{R}} |L_{t-s}^{x}|^p) \\
&\leqslant C|t-s|^{\frac{\alpha-1}{\alpha}}.
\end{aligned}$$

This gives the desired estimate.

Next, we apply Lemma 2 with

$$A_t = \|L_t^{\cdot}\|_{p',\mathbb{R}}, \qquad q = \Big(\frac{\alpha-1}{\alpha} + \frac{1}{p'\alpha}\Big)^{-1}.$$

Clearly, A_t is increasing and satisfies (i) in Lemma 2.

The P^y distribution of (A_t) does not depend on y by the translation invariance of the norm $\|.\|_{p',\mathbb{R}}$. The finiteness of A_1 follows from (4) and the fact that L_t^x has a compact support. Thus by a scaling property (9), we get

$$\lim_{z\to\infty} \sup_{\lambda>0, y\in\mathbb{R}} P^y(A_\lambda > \lambda^{\frac{1}{q}} z) = \lim_{z\to\infty} P^0(A_1 > z) = 0.$$

Finally, using the same method as above, we obtain (16). □

A more interesting application of Lemma 2 is the following inequality which generalize the well known Barlow-Yor inequality to symmetric stable process of index $1 < \alpha \leqslant 2$. (See Barlow and Yor [4] for the Brownian motion case).

LEMMA 4. *For all $t > 0$, $0 < \nu < \frac{\alpha-1}{2}$ and $p \geqslant 1$, there exists a constant $0 < C < \infty$, such that*

$$\left\| \sup_{0\leqslant s\leqslant t} \sup_{x\neq y} \frac{|L_s^x - L_s^y|}{|x-y|^{\nu}} \right\|_p \leqslant C t^{\frac{\alpha-1}{\alpha} - \frac{\nu}{\alpha}}.$$

Proof. We apply Lemma 2 with

$$A_t = \sup_{0\leqslant s\leqslant t} \sup_{x\neq y} \frac{|L_s^x - L_s^y|}{|x-y|^{\nu}}, \qquad q = \left(\frac{\alpha-1}{\alpha} - \frac{\nu}{\alpha}\right)^{-1}.$$

Clearly, (A_t) is increasing and satisfies (i) in Lemma 2.

In fact, for all $0 \leqslant s \leqslant t$, we have

$$\begin{aligned} A_t &= \sup_{0\leqslant r\leqslant t} \sup_{x\neq y} \frac{|L_r^x - L_r^y|}{|x-y|^{\nu}} \\ &\leqslant \max\left\{ \sup_{0\leqslant r\leqslant s} \sup_{x\neq y} \frac{|L_r^x - L_r^y|}{|x-y|^{\nu}} \; ; \; \sup_{s\leqslant r\leqslant t} \sup_{x\neq y} \frac{|L_r^x - L_r^y|}{|x-y|^{\nu}} \right\} \\ &= \max\{A_s \; ; \; A_{t-s}\circ\theta_s\} \\ &\leqslant A_s + A_{t-s}\circ\theta_s. \end{aligned}$$

Moreover,

$$\begin{aligned} A_{t-s}\circ\theta_s &= \sup_{0\leqslant r\leqslant t-s} \sup_{x\neq y} \frac{|L_r^x\circ\theta_s - L_r^y\circ\theta_s|}{|x-y|^{\nu}} \\ &= \sup_{s\leqslant r+s\leqslant t} \sup_{x\neq y} \frac{|L_r^x\circ\theta_s - L_r^y\circ\theta_s|}{|x-y|^{\nu}} \\ &= \sup_{s\leqslant r+s\leqslant t} \sup_{x\neq y} \frac{|L_{r+s}^x - L_s^x - L_{r+s}^y + L_s^y|}{|x-y|^{\nu}} \\ &= \sup_{s\leqslant r\leqslant t} \sup_{x\neq y} \frac{|L_r^x - L_s^x - L_r^y + L_s^y|}{|x-y|^{\nu}}. \end{aligned}$$

It follows from (5) that

$$A_t - A_s \leqslant A_{t-s}\circ\theta_s \leqslant C \sup_{s\leqslant r\leqslant t} |r-s|^{\beta_1} \leqslant C|t-s|^{\beta_1},$$

for any $0 < \beta_1 < \frac{\alpha-1}{2\alpha}$. Which implies that A_t is continuous.

Since $L_t^x(\tau_y(\omega)) = L_t^{x-y}(\omega)$ and X is spatially homogenous, it follows that the P^y distribution of A_t does not depend on y. Thus, by (12) we have

$$\lim_{z\to\infty} \sup_{\lambda>0, y\in\mathbb{R}} P^y(A_\lambda > \lambda^{\frac{1}{q}} z) = \lim_{z\to\infty} \sup_{\lambda>0} P^0(A_\lambda > \lambda^{\frac{1}{q}} z) = \lim_{z\to\infty} P^0\left(\sup_{x\neq y} \frac{|L_1^x - L_1^y|}{|x-y|^{\nu}} > z \right) = 0.$$

Finally, by (5) and the fact that $L_0^x = 0$, we get

$$\sup_{x\neq y} \frac{|L_s^x - L_s^y|}{|x-y|^\nu} < \infty, \qquad a.s.$$

This completes the proof of Lemma (4). □

The proof of Theorem 3 is based on the following

COROLLARY 1. *For any* $\nu \in]0, \frac{\alpha-1}{2}[$ *and* $\varepsilon > 0$, *when* $t \to \infty$,

$$\sup_{x\neq y} \frac{|L_s^x - L_s^y|}{|x-y|^\nu} = o(t^{\frac{\alpha-1}{\alpha}-\frac{\nu}{\alpha}+\varepsilon}), \qquad a.s. \tag{17}$$

Proof. Using Tchebychev's inequality and Lemma 4 with $p = \frac{2}{\varepsilon}$, for any $n \geqslant 1$,

$$P^0\left(\sup_{0\leqslant s\leqslant n}\sup_{x\neq y} \frac{|L_s^x - L_s^y|}{|x-y|^\nu} > n^{\frac{\alpha-1}{\alpha}-\frac{\nu}{\alpha}+\varepsilon}\right) \leqslant C(\nu,\varepsilon)n^{-2}.$$

Then, by the Borel-Cantelli lemma, we get as $n \to +\infty$, almost surely,

$$\sup_{0\leqslant s\leqslant n}\sup_{x\neq y} \frac{|L_s^x - L_s^y|}{|x-y|^\nu} = \mathscr{O}(n^{\frac{\alpha-1}{\alpha}-\frac{\nu}{\alpha}+\varepsilon}).$$

Since A_t is increasing and since ε can be arbitrarily small, we have proved the corollary. □

3. Proof of Theorem 3

The idea of the proof is inspired from that used in Csaki et *al.* [7] for the Brownian motion case. For this, we need the following lemmas.

LEMMA 5. *For any* $0 < \gamma < \delta < \frac{\alpha-1}{2}$ *and* $\varepsilon > 0$, *when* $t \to \infty$,

$$\sup_{x\in\mathbb{R}} \left| \int_0^1 \frac{L_t^{x+y} - L_t^{x-y}}{y^{1+\gamma}} dy \right| = o(t^{\frac{\alpha-1}{\alpha}-\frac{\delta}{\alpha}+\varepsilon}), \qquad \text{a.s.} \tag{18}$$

$$\sup_{|x|\leqslant t^a} \left| \int_1^\infty \frac{L_t^{x+y} - L_t^{y}}{y^{1+\gamma}} dy \right| = o(t^{\frac{\alpha-1}{\alpha}-\frac{\delta}{\alpha}+a\delta+\varepsilon}), \qquad \text{a.s.} \tag{19}$$

for some $a > 0$.

Proof. We have almost surely,

$$\sup_{x\in\mathbb{R}} \left| \int_0^1 \frac{L_t^{x+y} - L_t^{x-y}}{y^{1+\gamma}} dy \right| \leqslant \sup_{x\in\mathbb{R}}\sup_{0<y\leqslant 1} \frac{|L_t^{x+y} - L_t^{x-y}|}{y^\delta} \int_0^1 \frac{dy}{y^{1+\gamma-\delta}} = o(t^{\frac{\alpha-1}{\alpha}-\frac{\delta}{\alpha}+\varepsilon}),$$

where we have used in the last equality (17) and the fact that $\delta > \gamma$.

Next, by (17), almost surely,

$$\sup_{|x|\leqslant t^a}\left|\int_1^\infty \frac{L_t^{x+y}-L_t^y}{y^{1+\gamma}}dy\right| \leqslant \sup_{|x|\leqslant t^a}|x|^\delta \sup_{y\in\mathbb{R}}\frac{|L_t^{x+y}-L_t^y|}{|x|^\delta}\int_1^\infty \frac{dy}{y^{1+\gamma}} = o(t^{\frac{\alpha-1}{\alpha}-\frac{\delta}{\alpha}+a\delta+\varepsilon}). \quad \square$$

LEMMA 6. *For any* $0<\gamma<\frac{\alpha-1}{2}$ *and* $\varepsilon>0$, *when* $t\to\infty$,

$$\sup_{x\in\mathbb{R}}\left|\int_0^t \frac{ds}{(X_s-x)^{1+\gamma}}\right| = o(t^{\frac{\alpha-1}{\alpha}-\frac{\gamma}{\alpha}+\varepsilon}), \qquad \text{a.s.} \tag{20}$$

Proof. This lemma follows immediately by the same arguments used in the proof of Corollary 1. More precisely, we apply Lemma 2 with

$$A_t = \sup_{0\leqslant s\leqslant t}\sup_{x\in\mathbb{R}}|H_t^x|, \qquad q=\left(\frac{\alpha-1}{\alpha}-\frac{\gamma}{\alpha}\right)^{-1}. \quad \square$$

We are now able to prove Theorem 3.

Proof of Theorem 3. By Fubini's theorem, we have

$$\begin{aligned} I(t) &= \int_0^t D^\gamma f(X_s)ds - \frac{I(f)}{\Gamma(-\gamma)}D^\gamma L_t^\cdot(0) \\ &= \frac{1}{\Gamma(-\gamma)}\int_{\mathbb{R}}\left(\int_0^t \frac{ds}{(X_s-x)^{1+\gamma}} - \int_0^t \frac{ds}{X_s^{1+\gamma}}\right)f(x)dx \\ &= \frac{1}{\Gamma(-\gamma)}(I_1(t)+I_1(t)), \end{aligned}$$

where

$$I_1(t) = \int_{|x|>t^a}\left(\int_0^t \frac{ds}{(X_s-x)^{1+\gamma}} - \int_0^t \frac{ds}{X_s^{1+\gamma}}\right)f(x)dx,$$

$$I_2(t) = \int_{|x|\leqslant t^a}\left(\int_0^t \frac{ds}{(X_s-x)^{1+\gamma}} - \int_0^t \frac{ds}{X_s^{1+\gamma}}\right)f(x)dx,$$

for some $0<a\leqslant\frac{1}{\alpha}$.

Let us deal with the first term $I_1(t)$. By (20), we get

$$\begin{aligned} I_1(t) &\leqslant \sup_{|x|>t^a}\left|\int_0^t \frac{ds}{(X_s-x)^{1+\gamma}} - \int_0^t \frac{ds}{(X_s)^{1+\gamma}}\right|\int_{|x|>t^a}|x|^{-k}|x|^k|f(x)|dx \\ &\leqslant o(t^{\frac{\alpha-1}{\alpha}-\frac{\gamma}{\alpha}-ak+\varepsilon})\int_{|x|>t^a}|x|^k|f(x)|dx \\ &= o(t^{\frac{\alpha-1}{\alpha}-\frac{\gamma}{\alpha}-ak+\varepsilon}). \qquad a.s. \end{aligned}$$

Now, we deal with $I_2(t)$. Using (3) and the fact that f is integrable, we obtain

$$I_2(t) \leqslant \sup_{|x|\leqslant t^a}\left|\int_0^t \frac{ds}{(X_s-x)^{1+\gamma}} - \int_0^t \frac{ds}{(X_s)^{1+\gamma}}\right| \int_{|x|\leqslant t^a} |f(x)|dx$$
$$\leqslant C \sup_{|x|\leqslant t^a}\left|\int_0^\infty \frac{L_t^{x+y}-L_t^{x-y}-L_t^{y}-L_t^{-y}}{y^{1+\gamma}}\right|$$
$$\leqslant C \sup_{|x|\leqslant t^a}\left|\int_0^1 \frac{L_t^{x+y}-L_t^{x-y}-L_t^{y}-L_t^{-y}}{y^{1+\gamma}}\right| + C \sup_{|x|\leqslant t^a}\left|\int_1^\infty \frac{L_t^{x+y}-L_t^{x-y}-L_t^{y}-L_t^{-y}}{y^{1+\gamma}}\right|,$$

which, in view of (18) and (19), implies

$$I_2(t) = o(t^{\frac{\alpha-1}{\alpha}-\frac{\delta}{\alpha}+\varepsilon}) + o(t^{\frac{\alpha-1}{\alpha}-\frac{\delta}{\alpha}+a\delta+\varepsilon}) = o(t^{\frac{\alpha-1}{\alpha}-\frac{\delta}{\alpha}+a\delta+\varepsilon}). \qquad a.s.$$

Then

$$I(t) = o(t^{\frac{\alpha-1}{\alpha}-\frac{\delta}{\alpha}+a\delta+\varepsilon}) + o(t^{\frac{\alpha-1}{\alpha}-\frac{\gamma}{\alpha}-ak+\varepsilon}). \qquad a.s.$$

Choosing

$$a = \frac{\delta-\gamma}{\alpha(\delta+k)},$$

it is clear that $0 < a \leqslant \frac{1}{\alpha}$. It follows that

$$I(t) = o(t^{b+\varepsilon}), \qquad a.s.$$

with

$$b = \frac{\alpha-1}{\alpha} - \frac{\gamma}{\alpha} - k\frac{\delta-\nu}{\alpha(\delta+k)} < \frac{\alpha-1}{\alpha} - \frac{\gamma}{\alpha}.$$

Then for all sufficiently small $\varepsilon > 0$, when $t \to \infty$,

$$I(t) = o(t^{\frac{\alpha-1}{\alpha}-\frac{\gamma}{\alpha}-\varepsilon}), \qquad a.s.$$

The theorem is proved. □

REMARK 2.

1. It would be interesting to prove the LIL for $D^\gamma L_t^{\cdot}(0)$ in case of symmetric stable process of index $1 < \alpha \leqslant 2$. This allows to deduce the LIL of the functional $\int_0^t D^\gamma f(X_s)ds$.

2. The L_p- estimate of Ait Ouahra and Ouali [2] is proved for fractional Brownian motion of Hurst parameter $0 < H < 1$ which is a non Markovian process. The question which arises is if we can also extend previous results to this process.

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