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INEQUALITIES AND ASYMPTOTIC EXPANSIONS RELATED TO GLAISHER–KINKELIN CONSTANT

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Abstract. We present sharp inequalities and asymptotic expansions related to Glaisher-Kinkelin constant. Also, we present sharp inequality and asymptotic expansion for sum $\sum_{k=1}^{n} k\ln\left(1+\frac{1}{k}\right)$. This solves an open problem proposed in 2009 by Mihály Bencze.

1. Introduction

The Glaisher-Kinkelin constant $A = 1.28242712\cdots$ is defined by

$$\lim_{n\to\infty} n^{-n^2/2-n/2-1/12} e^{n^2/4} \prod_{k=1}^{n} k^k = A \tag{1.1}$$

(see [8, 9, 14]), as well as

$$\lim_{n\to\infty} \frac{G(n+1)}{n^{n^2/2-1/12}(2\pi)^{n/2}e^{-3n^2/4}} = \frac{e^{1/12}}{A}, \tag{1.2}$$

where $G(n)$ is the Barnes G-function [2].

The Glaisher-Kinkelin constant A has closed-form representations

$$\begin{aligned} A &= e^{\frac{1}{12}-\zeta'(-1)} \\ &= (2\pi)^{1/12}\left[e^{\gamma\pi^2/6-\zeta'(2)}\right]^{1/(2\pi^2)} \end{aligned}$$

(see [5, p. 129, Eq. (3.22)]), where $\zeta'(z)$ is the derivative of the Riemann zeta function $\zeta(z)$ (see [6]).

The Glaisher-Kinkelin constant A appears in a number of sums and integrals, especially those involving gamma functions and zeta functions. Finch introduced this constant A in a section of his book [7, pp. 135–138].

Define the sequence $(A_n)_{n\in\mathbb{N}}$ by

$$A_n := n^{-n^2/2-n/2-1/12} e^{n^2/4} \prod_{k=1}^{n} k^k. \tag{1.3}$$

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Very recently, Chen [4, Theorem 1] established the asymptotic expansion of the sequence $(\ln A_n)_{n\in\mathbb{N}}$:

$$\begin{aligned}\ln A_n &= \sum_{k=1}^{n} k\ln k - \left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)\ln n + \frac{n^2}{4}\\ &\sim \ln A - \sum_{k=1}^{\infty} \frac{B_{2k+2}}{2k(2k+1)(2k+2)}\frac{1}{n^{2k}},\end{aligned} \tag{1.4}$$

where B_k $(k \in \mathbb{N}_0 := \mathbb{N}\cup\{0\},\ \mathbb{N} := \{1,2,3,\ldots\})$ are the n-th Bernoulli numbers defined by the following generating function (see, for example, [12, Section 1.6] and [13, Section 1.7]):

$$\frac{z}{e^z-1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} \qquad (|z| < 2\pi). \tag{1.5}$$

The asymptotic representation (1.4) can be rewritten as

$$1^1 2^2 \cdots n^n \sim A\cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} e^{-\frac{n^2}{4}} \exp\left(\sum_{j=1}^{\infty}\frac{a_j}{n^j}\right), \tag{1.6}$$

where

$$a_j = -\frac{B_{j+2}}{j(j+1)(j+2)} \qquad (j\in\mathbb{N}). \tag{1.7}$$

Namely,

$$\begin{aligned}1^1 2^2 \cdots n^n \sim A\cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} e^{-\frac{n^2}{4}} \exp\Bigg(&\frac{1}{720n^2} - \frac{1}{5040n^4} + \frac{1}{10080n^6}\\ &- \frac{1}{9504n^8} + \frac{691}{3603600n^{10}} - \frac{1}{1872n^{12}} + \cdots\Bigg).\end{aligned} \tag{1.8}$$

Mortici [11] established (1.6) and gave the following system for successively determining the coefficients a_j:

$$\begin{aligned}a_1 - \binom{k-1}{1}a_2 + \cdots + (-1)^k\binom{k-1}{k-2}a_{k-1}&\\ = \frac{1}{2k+2} - \frac{1}{2k+4} - \frac{1}{12k}& \qquad (k\in\mathbb{N}\setminus\{1\}).\end{aligned} \tag{1.9}$$

By using $e^x = \sum_{j=0}^{\infty}\frac{x^j}{j!}$, we deduce that

$$1^1 2^2 \cdots n^n \sim A\cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} e^{-\frac{n^2}{4}} \left(1 + \frac{1}{720n^2} - \frac{1433}{7257600n^4} + \cdots\right). \tag{1.10}$$

Our first aim in this paper is to give two general asymptotic expansions for $1^1 2^2 \cdots n^n$ which includes (1.10) as their special case (see Theorems 2.1 and 2.2).

In 2009, Bencze [3, p. 451] posed the following open problem: (i) Prove that

$$n-\frac{1}{2}-\frac{1}{2}\ln n<\sum_{k=1}^{n}k\ln\left(1+\frac{1}{k}\right)<n+2-\frac{1}{n}-\frac{1}{2}\ln(n+1). \tag{1.11}$$

(ii) Determine $\alpha,\beta\in\mathbb{R}$ such that

$$\sum_{k=1}^{n}k\ln\left(1+\frac{1}{k}\right)=n+\alpha+\beta\ln n+O\left(\frac{1}{n}\right). \tag{1.12}$$

Our second aim in this paper is to answer this question (see Theorem 2.3).

Our last aim in this work is to present the asymptotic expansion and sharp inequality for $\sum_{k=1}^{n}k^2\ln\left(1+\frac{1}{k}\right)$, which are related to the the Glaisher-Kinkelin constant (see Theorem 2.4).

2. Main results

By using a fundamental theorem of algebra and the Newton formulas, we prove Theorem 2.1 which includes (1.10) as its special case.

THEOREM 2.1. *Let r be a given nonzero real number. The following asymptotic expression holds:*

$$1^12^2\cdots n^n\sim A\cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}}e^{-\frac{n^2}{4}}\left(1+\sum_{j=1}^{\infty}\frac{b_j}{n^j}\right)^{1/r}, \tag{2.1}$$

where the coefficients $b_j=b_j(r)\,(j\in\mathbb{N})$ are given by

$$\begin{aligned}b_j=&\sum_{k_1+2k_2+\cdots+jk_j=j}\frac{(-r)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!}\\&\cdot\left(\frac{B_3}{1\cdot2\cdot3}\right)^{k_1}\left(\frac{B_4}{2\cdot3\cdot4}\right)^{k_2}\cdots\left(\frac{B_{j+2}}{j(j+1)(j+2)}\right)^{k_j},\end{aligned} \tag{2.2}$$

the summation being taken over all combinations of nonnegative integers k_j satisfying the equation

$$k_1+2k_2+\cdots+jk_j=j.$$

Proof. To determine $b_j\,(j\in\mathbb{N})$, we first express (2.1) as follows:

$$r\ln\left(\frac{A_n}{A}\right)=\ln\left(1+\sum_{j=1}^{m}\frac{b_j}{n^j}\right)+O(n^{-m-1}).$$

By using the fundamental theorem of algebra, we see that there exist unique complex numbers $x_1,\ldots,x_m$ such that

$$1+\frac{b_1}{n}+\cdots+\frac{b_m}{n^m}=\left(1+\frac{x_1}{n}\right)\cdots\left(1+\frac{x_m}{n}\right). \tag{2.3}$$

By applying the following series expansion:

$$\ln\left(1+\frac{z}{n}\right)=\sum_{j=1}^{m}\frac{(-1)^{j-1}z^j}{jn^j}+O(n^{-m-1}) \qquad (|z|<1;\ n\to\infty), \tag{2.4}$$

we obtain

$$\ln\left(1+\frac{b_1}{n}+\cdots+\frac{b_m}{n^m}\right)=\sum_{j=1}^{m}\frac{(-1)^{j-1}S_j}{jn^j}+O(n^{-m-1}) \qquad (n\to\infty), \tag{2.5}$$

where

$$S_j=x_1^j+\cdots+x_m^j \qquad (j=1,\ldots,m).$$

It follows from (1.6) that

$$r\ln\left(\frac{A_n}{A}\right)=\sum_{j=1}^{m}\frac{-rB_{j+2}}{j(j+1)(j+2)}\frac{1}{n^j}+O(n^{-m-1}), \tag{2.6}$$

From (2.5) and (2.6), we obtain

$$S_j=S_j(r)=(-1)^j\frac{rB_{j+2}}{(j+1)(j+2)} \qquad (j=1,2,\ldots,m). \tag{2.7}$$

That is,

$$\begin{cases} x_1+\cdots+x_m=S_1,\\ x_1^2+\cdots+x_m^2=S_2,\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ x_1^m+\cdots+x_m^m=S_m. \end{cases} \tag{2.8}$$

Let

$$P_m(x)=x^m+c_1x^{m-1}+\cdots+c_{m-1}x+c_m$$

be a polynomial with zeros: $x_1,\ldots,x_m$ which satisfy the system of equations (2.8). So we have

$$P_m(x)=(x-x_1)\cdots(x-x_m).$$

The Newton formulas (see, for example, [10] and references therein) give the connection between the coefficients c_j and the power sums S_j:

$$S_j+S_{j-1}c_1+S_{j-2}c_2+\cdots+S_1c_{j-1}+jc_j=0 \qquad (j=1,\ldots,m).$$

It is known (see [10]) that c_j can be expressed in terms of S_j:

$$b_j=\sum_{k_1+2k_2+\cdots+jk_j=j}\frac{(-1)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!}\left(\frac{S_1}{1}\right)^{k_1}\left(\frac{S_2}{2}\right)^{k_2}\cdots\left(\frac{S_j}{j}\right)^{k_j}.$$

Clearly,

$$\frac{(-1)^m}{n^m}P_m(-n)=\left(1+\frac{x_1}{n}\right)\cdots\left(1+\frac{x_m}{n}\right).$$

We thus have

$$1+\frac{(-1)c_1}{n}+\frac{(-1)^2c_2}{n^2}+\cdots+\frac{(-1)^m c_m}{n^m}=\left(1+\frac{x_1}{n}\right)\cdots\left(1+\frac{x_m}{n}\right). \tag{2.9}$$

By (2.3) and (2.9), the coefficients b_j are given by

$$\begin{aligned} b_j &= (-1)^j c_j \\ &= (-1)^j \sum_{k_1+2k_2+\cdots+jk_j=j} \frac{(-1)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!}\left(\frac{S_1}{1}\right)^{k_1}\left(\frac{S_2}{2}\right)^{k_2}\cdots\left(\frac{S_j}{j}\right)^{k_j}, \end{aligned}$$

where S_j are given in (2.7). That is

$$\begin{aligned} b_j = \sum_{k_1+2k_2+\cdots+jk_j=j} &\frac{(-r)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!} \\ &\cdot\left(\frac{B_3}{1\cdot2\cdot3}\right)^{k_1}\left(\frac{B_4}{2\cdot3\cdot4}\right)^{k_2}\cdots\left(\frac{B_{j+2}}{j(j+1)(j+2)}\right)^{k_j} \qquad (j\in\mathbb{N}). \end{aligned}$$

This completes the proof of Theorem 2.1. □

By using another proving method, we prove Theorem 2.2 which includes Theorem 2.1 as its special case.

THEOREM 2.2. *Let r be a given nonzero real number and $\ell \geqslant 0$ be a given integer. The following asymptotic expression holds:*

$$1^1 2^2\cdots n^n \sim A\cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}}e^{-\frac{n^2}{4}}\left(1+\sum_{j=1}^{\infty}\frac{d_j}{n^j}\right)^{n^\ell/r}, \tag{2.10}$$

where the coefficients $d_j=d_j(\ell,r)\,(j\in\mathbb{N})$ are given by

$$\begin{aligned} d_j = d_j(\ell,r) = \sum_{(1+\ell)k_1+(2+\ell)k_2+\cdots+(j+\ell)k_j=j} &\frac{(-r)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!} \\ &\cdot\left(\frac{B_3}{1\cdot2\cdot3}\right)^{k_1}\left(\frac{B_4}{2\cdot3\cdot4}\right)^{k_2}\cdots\left(\frac{B_{j+2}}{j(j+1)(j+2)}\right)^{k_j}. \end{aligned} \tag{2.11}$$

summed over all nonnegative integers k_j satisfying the equation

$$(1+\ell)k_1+(2+\ell)k_2+\cdots+(j+\ell)k_j=j.$$

Proof. To determine $d_j\,(j\in\mathbb{N})$, we first express (2.10) as follows:

$$\left(\frac{A_n}{A}\right)^{r/n^\ell}=1+\sum_{j=1}^{m}\frac{d_j}{n^j}+O(n^{-m-1}). \tag{2.12}$$

Write (1.6) as

$$\ln\left(\frac{A_n}{A}\right)=\sum_{k=1}^{m}\frac{-B_{k+2}}{k(k+1)(k+2)n^k}+\mathscr{R}_m(n)\qquad(n\to\infty),$$

where $\mathscr{R}_m(n)=O(n^{-m-1})$. Further, we have

$$\begin{aligned}\left(\frac{A_n}{A}\right)^{r/n^\ell}&=e^{r\mathscr{R}(x)/x^\ell}e^{\sum_{k=1}^{m}\frac{-rB_{k+2}}{k(k+1)(k+2)n^{k+\ell}}}\\
&=e^{r\mathscr{R}(x)/x^\ell}\prod_{k=1}^{m}\left[1+\left(\frac{-rB_{k+2}}{k(k+1)(k+2)n^{k+\ell}}\right)+\frac{1}{2!}\left(\frac{-rB_{k+2}}{k(k+1)(k+2)n^{k+\ell}}\right)^2+\cdots\right]\\
&=e^{r\mathscr{R}(x)/x^\ell}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}\frac{1}{k_1!k_2!\cdots k_m!}\\
&\quad\cdot\left(\frac{-rB_3}{1\cdot2\cdot3\cdot n^{1+\ell}}\right)^{k_1}\left(\frac{-rB_4}{2\cdot3\cdot4\cdot n^{2+\ell}}\right)^{k_2}\cdots\left(\frac{-rB_{m+2}}{m(m+1)(m+2)n^{m+\ell}}\right)^{k_m}\\
&=e^{r\mathscr{R}(x)/x^\ell}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}\frac{(-r)^{k_1+k_2+\cdots+k_m}}{k_1!k_2!\cdots k_m!}\\
&\quad\cdot\left(\frac{B_3}{1\cdot2\cdot3}\right)^{k_1}\left(\frac{B_4}{2\cdot3\cdot4}\right)^{k_2}\cdots\left(\frac{B_{m+2}}{m(m+1)(m+2)}\right)^{k_m}\\
&\quad\cdot\frac{1}{x^{(1+\ell)k_1+(2+\ell)k_2+\cdots+(m+\ell)k_m}}.\end{aligned}\tag{2.13}$$

Equating the coefficients by the equal powers of x in (2.12) and (2.13), we see that

$$\begin{aligned}d_j=\sum_{(1+\ell)k_1+(2+\ell)k_2+\cdots+(j+\ell)k_j=j}&\frac{(-r)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_m!}\\
&\cdot\left(\frac{B_3}{1\cdot2\cdot3}\right)^{k_1}\left(\frac{B_4}{2\cdot3\cdot4}\right)^{k_2}\cdots\left(\frac{B_{j+2}}{j(j+1)(j+2)}\right)^{k_j}.\end{aligned}$$

This completes the proof of Theorem 2.2. □

Setting $(\ell,r)=(0,1)$ in (2.10), yields (1.10). Here, from (2.10), we give several explicit expressions:

$$1^12^2\cdots n^n\sim A\cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}}e^{-\frac{n^2}{4}}\left(1+\frac{1}{360n^2}-\frac{713}{1814400n^4}+\cdots\right)^{1/2};\tag{2.14}$$

$$1^12^2\cdots n^n\sim A\cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}}e^{-\frac{n^2}{4}}\left(1+\frac{1}{720n^3}-\frac{1}{5040n^5}+\cdots\right)^{n};\tag{2.15}$$

$$1^1 2^2 \cdots n^n \sim A \cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} e^{-\frac{n^2}{4}} \left(1+\frac{1}{720n^4}-\frac{1}{5040n^6}+\cdots\right)^{n^2}. \tag{2.16}$$

Theorem 2.3 answers the open problem of Bencze.

THEOREM 2.3. (i) *The following asymptotic expansion holds:*

$$\sum_{k=1}^{n} k\ln\left(1+\frac{1}{k}\right) \sim n+1-\ln(\sqrt{2\pi})-\frac{1}{2}\ln n + \sum_{j=2}^{\infty}\left(\frac{(-1)^{j-1}}{j}-\frac{B_j}{j(j-1)}\right)\frac{1}{n^{j-1}} \qquad (n\to\infty), \tag{2.17}$$

namely,

$$\sum_{k=1}^{n} k\ln\left(1+\frac{1}{k}\right) \sim n+1-\ln(\sqrt{2\pi})-\frac{1}{2}\ln n-\frac{7}{12n}+\frac{1}{3n^2}-\frac{89}{360n^3}+\cdots. \tag{2.18}$$

(ii) *For* $n\in\mathbb{N}$, *we have*

$$\ln 2-1+n-\frac{1}{2}\ln n \leqslant \sum_{k=1}^{n} k\ln\left(1+\frac{1}{k}\right) < 1-\ln(\sqrt{2\pi})+n-\frac{1}{2}\ln n. \tag{2.19}$$

The constants $\ln 2-1$ *and* $1-\ln(\sqrt{2\pi})$ *are the best possible.*

Proof. Elementary calculations show that

$$\begin{aligned}\sum_{k=1}^{n} k\ln\left(1+\frac{1}{k}\right) &= \sum_{k=1}^{n} k\ln(k+1)-\sum_{k=1}^{n} k\ln k\\ &= \sum_{k=2}^{n+1} k\ln k-\sum_{k=2}^{n+1}\ln k-\sum_{k=1}^{n} k\ln k\\ &= n\ln(n+1)-\ln\Gamma(n+1)\\ &= n\ln n+n\ln\left(1+\frac{1}{n}\right)-\ln\Gamma(n+1),\end{aligned} \tag{2.20}$$

where Γ denotes the gamma function. It is well-known that

$$\ln\Gamma(n+1)=\left(n+\frac{1}{2}\right)\ln n-n+\ln\sqrt{2\pi}+\sum_{m=1}^{\infty}\frac{B_{2m}}{2m(2m-1)n^{2m-1}} \qquad (n\to\infty) \tag{2.21}$$

(see [1, p. 257, Equation (6.1.40)]) and

$$\ln\left(1+\frac{1}{n}\right)=\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{jn^j} \qquad (n\geqslant 1). \tag{2.22}$$

Substituting from (2.21) and (2.22) into (2.20), yields the desired formula (2.17).

The upper and lower bounds are both obtained by considering the sequence $(x_n)_{n\in\mathbb{N}}$ defined by

$$x_n = \sum_{k=1}^{n} k\ln\left(1+\frac{1}{k}\right) - \left(n - \frac{1}{2}\ln n\right).$$

Elementary calculations show that

$$x_{n+1} - x_n = (n+1)\ln\left(1+\frac{1}{n+1}\right) + \frac{1}{2}\ln\left(1+\frac{1}{n}\right) - 1 =: f(n).$$

Differentiation yields

$$f'(x) = \ln\left(\frac{x+2}{x+1}\right) - \frac{2x^2+3x+2}{2(x+1)(x+2)x};$$

$$f''(x) = \frac{7x^2+12x+4}{2(x+1)^2(x+2)^2x^2} > 0 \qquad (x \geqslant 1).$$

Hence,

$$f'(x) < \lim_{x\to\infty} f'(x) = 0 \qquad (x \geqslant 1),$$

which implies

$$x_{n+1} - x_n = f(n) > \lim_{n\to\infty} f(n) = 0 \qquad (n \in \mathbb{N}).$$

Therefore, the sequence $(x_n)_{n\geqslant 1}$ is strictly increasing, and we have

$$\ln 2 - 1 = x_1 \leqslant x_n < \lim_{n\to\infty} x_n = 1 - \ln(\sqrt{2\pi}) \qquad (n \in \mathbb{N}).$$

This completes the proof of Theorem 2.3. □

REMARK. The inequalities (2.19) are sharper than inequalities (1.11).

Motivated by the open problem of Bencze, we establish the asymptotic expansion and sharp inequality for $\sum_{k=1}^{n} k^2\ln\left(1+\frac{1}{k}\right)$, which are related to the the Glaisher-Kinkelin constant.

THEOREM 2.4. (i) *The following asymptotic expansion holds:*

$$\sum_{k=1}^{n} k^2\ln\left(1+\frac{1}{k}\right) \sim n^2\ln(n+1) - \left(n^2 - \frac{1}{3}\right)\ln n + \frac{n(n-2)}{2} + \ln(\sqrt{2\pi}/A^2) + \sum_{j=1}^{\infty}\left(B_{j+1} + \frac{2B_{j+2}}{j+2}\right)\frac{1}{j(j+1)n^j} \qquad (n\to\infty), \tag{2.23}$$

namely,

$$\sum_{k=1}^{n} k^2\ln\left(1+\frac{1}{k}\right) \sim n^2\ln(n+1) - \left(n^2 - \frac{1}{3}\right)\ln n + \frac{n(n-2)}{2} + \ln(\sqrt{2\pi}/A^2) + \frac{1}{12n} - \frac{1}{360n^2} + \cdots \qquad (n\to\infty). \tag{2.24}$$

(ii) *For $n\in\mathbb{N}$, we have*

$$\frac{5}{12}\leqslant\sum_{k=1}^{n}k^2\ln\left(1+\frac{1}{k}\right)-\left[n^2\ln(n+1)-\left(n^2-\frac{1}{3}\right)\ln n+\frac{n(n-2)}{2}+\frac{1}{12n}\right]<\ln(\sqrt{2\pi}/A^2). \tag{2.25}$$

The constants $\frac{5}{12}$ and $\ln(\sqrt{2\pi}/A^2)$ are the best possible.

Proof. Elementary calculations show that

$$\begin{aligned}\sum_{k=1}^{n}k^2\ln\left(1+\frac{1}{k}\right)&=\sum_{k=1}^{n}k^2\ln(k+1)-\sum_{k=1}^{n}k^2\ln k\\&=\sum_{k=2}^{n+1}(k-1)^2\ln k-\sum_{k=1}^{n}k^2\ln k\\&=n^2\ln(n+1)-2\sum_{k=1}^{n}k\ln k+\ln\Gamma(n+1).\end{aligned}\tag{2.26}$$

The asymptotic formula (1.6) can be rewritten as

$$\sum_{k=1}^{n}k\ln k\sim\ln A+\left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)\ln n-\frac{n^2}{4}-\sum_{j=1}^{\infty}\frac{B_{j+2}}{j(j+1)(j+2)}\frac{1}{n^j}. \tag{2.27}$$

Substituting from (2.21) and (2.27) into (2.26), yields the desired formula (2.23).

Consider the sequence $(y_n)_{n\in\mathbb{N}}$ defined by

$$y_n=\sum_{k=1}^{n}k^2\ln\left(1+\frac{1}{k}\right)-\left[n^2\ln(n+1)-\left(n^2-\frac{1}{3}\right)\ln n+\frac{n(n-2)}{2}+\frac{1}{12n}\right].$$

Elementary calculations show that

$$y_{n+1}-y_n=\frac{12n^4+12n^3-4n^2-4n}{12n(n+1)}g(n),$$

where

$$g(x)=\ln\left(1+\frac{1}{x}\right)-\frac{12x^3+6x^2-6x-1}{12x^4+12x^3-4x^2-4x}.$$

Differentiation yields

$$g'(x)=-\frac{x^2+2x-1}{4x^2(3x^3+3x^2-x-1)^2}<0\qquad(x\geqslant 1).$$

Therefore, $g(x)$ is strictly decreasing for $x\geqslant 1$, and we have

$$g(n)>\lim_{n\to\infty}g(n)=0\quad\text{and so}\quad y_{n+1}>y_n\qquad(n\in\mathbb{N}).$$

Therefore, the sequence $(y_n)_{n\geqslant 1}$ is strictly increasing, and we have

$$\frac{5}{12} = y_1 \leqslant y_n < \lim_{n\to\infty} y_n = \ln(\sqrt{2\pi}/A^2) \qquad (n\in\mathbb{N}).$$

This completes the proof of Theorem 2.4. □

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