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STRONG APPROXIMATION OF ALMOST PERIODIC FUNCTIONS

WŁODZIMIERZ ŁENSKI AND BOGDAN SZAL

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Abstract. We consider summability methods generated by the class $GM({}_2\beta)$. We generalize some related results of P. Pych-Taberska [Studia Math. XCVI (1990), 91–103] on strong approximation of almost periodic functions by their Fourier series and S. M. Mazhar and V.Totik [J. Approx. Theory, 60(1990), 174–182] on approximation of periodic functions by matrix means of their Fourier series.

1. Introduction

Let S^p $(1<p\leqslant\infty)$ be the class of all almost periodic functions in the sense of Stepanov with the norm

$$\|f\|_{S^p} := \begin{cases} \sup\limits_u \left\{ \frac{1}{\pi}\int_u^{u+\pi} |f(t)|^p\,dt \right\}^{1/p} & \text{when } 1<p<\infty, \\ \sup\limits_u |f(u)| & \text{when } p=\infty. \end{cases}$$

Denote yet by $C_{2\pi}$ the class of all 2π-periodic functions continuous over $Q=[-\pi,\pi]$ with the norm

$$\|f\|_{C_{2\pi}} := \sup_{t\in Q} |f(t)|\,.$$

Suppose that the Fourier series of $f\in S^p$ has the form

$$Sf(x)=\sum_{\nu=-\infty}^{\infty} A_\nu(f)\,e^{i\lambda_\nu x}, \quad \text{where } A_\nu(f)=\lim_{L\to\infty}\frac{1}{L}\int_0^L f(t)e^{-i\lambda_\nu t}dt,$$

with the partial sums

$$S_{\gamma_k}f(x)=\sum_{|\lambda_\nu|\leqslant\gamma_k} A_\nu(f)\,e^{i\lambda_\nu x}$$

and that $0=\lambda_0<\lambda_\nu<\lambda_{\nu+1}$ if $\nu\in\mathbb{N}=\{1,2,3...\}$, $\lim\lambda_\nu=\infty$, $\lambda_{-\nu}=-\lambda_\nu$ $|A_\nu|+|A_{-\nu}|>0$. Let $\Omega_{\alpha,p}$, with some fixed positive α , be the set of functions of class S^p bounded on $\mathbb{R}=(-\infty,\infty)$ whose Fourier exponents satisfy the condition

$$\lambda_{\nu+1}-\lambda_\nu\geqslant\alpha \quad (\nu\in\mathbb{N}\cup\{0\}).$$

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In case $f\in\Omega_{\alpha,p}$

$$S_{\lambda_k}f(x)=\int_0^{\infty}\{f(x+t)+f(x-t)\}\Psi_{\lambda_k,\lambda_k+\alpha}(t)\,dt,$$

where

$$\Psi_{\lambda,\eta}(t)=\frac{2\sin\frac{(\eta-\lambda)t}{2}\sin\frac{(\eta+\lambda)t}{2}}{\pi(\eta-\lambda)t^2}\quad(0<\lambda<\eta,\ \ |t|>0).$$

Let $A:=(a_{n,k})$ be an infinite matrix of real nonnegative numbers such that

$$\sum_{k=0}^{\infty}a_{n,k}=1,\text{ where }n=0,1,2,\dots,\tag{1}$$

and let the $A-$transformation of $(S_{\gamma_k}f)$ be given by

$$T_{n,A,\gamma}f(x):=\sum_{k=0}^{\infty}a_{n,k}S_{\gamma_k}f(x)\quad(n=0,1,2,\dots).$$

Let us consider the strong mean

$$T^q_{n,A,\gamma}f(x)=\left\{\sum_{k=0}^{\infty}a_{n,k}\left|S_{\gamma_k}f(x)-f(x)\right|^q\right\}^{1/q}\quad(q>0).\tag{2}$$

If $f\in C_{2\pi}$, then as usually

$$Sf(x)=\frac{a_o(f)}{2}+\sum_{\nu=1}^{\infty}(a_\nu(f)\cos kx+b_\nu(f)\sin kx)$$

and instead of $S_{\gamma_k}f$ we will consider the partial sums

$$S_kf(x)=\frac{a_o(f)}{2}+\sum_{\nu=1}^{k}(a_\nu(f)\cos kx+b_\nu(f)\sin kx).$$

Thus, instead of $T_{n,A,\gamma}f$ and $T^q_{n,A,\gamma}f$ we will consider the quantities $T_{n,A}f$ and $T^q_{n,A}f$ defined by the formulas

$$T_{n,A}f(x):=\sum_{k=0}^{\infty}a_{n,k}S_kf(x)\quad(n=0,1,2,\dots)\tag{3}$$

and

$$T^q_{n,A}f(x)=\left\{\sum_{k=0}^{\infty}a_{n,k}|S_kf(x)-f(x)|^q\right\}^{1/q}\quad(q>0),\tag{4}$$

respectively. As measures of approximation by the quantities (2), (3) and (4) we use the best approximation of f by trigonometric polynomials t_k of order at most k or by entire functions g_σ of exponential type σ bounded on the real axis, shortly $g_\sigma \in B_\sigma$ and the modulus of continuity of f, defined by the formulas

$$E_k(f)_{C_{2\pi}} = \inf_{t_k} \|f - t_k\|_{C_{2\pi}}$$

or

$$E_\sigma(f)_{S^p} = \inf_{g_\sigma} \|f - g_\sigma\|_{S^p}$$

and

$$\omega f(\delta)_X = \sup_{|t|\leqslant\delta} \|f(\cdot + t) - f(\cdot)\|_X, \quad X = C_{2\pi} \text{ or } X = S^p,$$

respectively.

In [10] S. M. Mazhar and V. Totik proved the following theorem:

THEOREM 1. *Let $f \in C_{2\pi}$. Suppose $A := (a_{n,k})$ satisfies (1), $lim_{n\to\infty} a_{n,0} = 0$ and*

$$a_{n,k} \geqslant a_{n,k+1} \quad k = 0,1,2,\ldots \quad n = 0,1,2,\ldots,$$

then

$$\left\|T_{n,A} f(x) - f\right\|_{C_{2\pi}} \leqslant K \sum_{k=0}^{\infty} a_{n,k} \omega f\left(\frac{1}{k+1}\right)_{C_{2\pi}}.$$

Recently, L. Leindler [5] defined a new class of sequences named as sequences of rest bounded variation, briefly denoted by $RBVS$, i.e.,

$$RBVS = \left\{ a := (a_n) \in \mathbb{C} : \sum_{k=m}^{\infty} |a_k - a_{k+1}| \leqslant K(a)|a_m| \text{ for all } m \in \mathbb{N} \right\}, \tag{5}$$

where here and throughout the paper $K(a)$ always indicates a constant only depending on a.

Denote by MS the class of nonincreasing sequences. Then it is obvious that

$$MS \subset RBVS.$$

In [6] L. Leindler considered the class of mean rest bounded variation sequences $MRBVS$, where

$$MRBVS = \{a := (a_n) \in \mathbb{C} :$$

$$\left. \sum_{k=m}^{\infty} |a_k - a_{k+1}| \leqslant K(a) \frac{1}{m} \sum_{k\geqslant m/2}^{m} |a_k| \text{ for all } m \in \mathbb{N} \right\}. \tag{6}$$

It is clear that

$$RBVS \subseteq MRBVS.$$

In [13] the second author proved that $RBVS \neq MRBVS$. Moreover, the above theorem was generalized for the class $MRBVS$ in [12] .

Further, the class of general monotone coefficients, GM, is defined as follows (see [14]):

$$GM = \left\{ a := (a_n) \in \mathbb{C} : \sum_{k=m}^{2m-1} |a_k - a_{k+1}| \leqslant K(a)\,|a_m| \text{ for all } m \in \mathbb{N} \right\}. \tag{7}$$

It is clear

$$RBVS \subset GM.$$

In [7, 14, 15, 16] was defined the class of $\beta-$general monotone sequences as follows:

DEFINITION 1. Let $\beta := (\beta_n)$ be a nonnegative sequence. The sequence of complex numbers $a := (a_n)$ is said to be $\beta-$general monotone, or $a \in GM(\beta)$, if the relation

$$\sum_{k=m}^{2m-1} |a_k - a_{k+1}| \leqslant K(a)\,\beta_m \tag{8}$$

holds for all m.

In the paper [16] Tikhonov considered, among others, the following examples of the sequences β_n:

(1) ${}_1\beta_n = |a_n|$,

(2) ${}_2\beta_n = \sum_{k=[n/c]}^{[cn]} \frac{|a_k|}{k}$ for some $c > 1$.

It is clear that $GM({}_1\beta) = GM$. Moreover (see [16, Remark 2.1])

$$GM({}_1\beta + {}_2\beta) \equiv GM({}_2\beta).$$

Consequently, we assume that the sequence $(K(\alpha_n))_{n=0}^{\infty}$ is bounded, that is, that there exists a constant K such that

$$0 \leqslant K(\alpha_n) \leqslant K$$

holds for all n, where $K(\alpha_n)$ denote the sequence of constants appearing in the inequalities (5)-(8) for the sequences $\alpha_n := (a_{nk})_{k=0}^{\infty}$.

Now we can give the conditions to be used later on. We assume that for all n

$$\sum_{k=m}^{2m-1} \left| a_{n,k} - a_{n,k+1} \right| \leqslant K \sum_{k=[m/c]}^{[cm]} \frac{a_{n,k}}{k} \tag{9}$$

holds if $\alpha_n = \left(a_{n,k}\right)_{k=0}^{\infty}$ belongs to $GM({}_2\beta)$, for $n = 1, 2, \ldots$

In this paper we consider the class $GM({}_2\beta)$ in estimate of the quantity $\left\| T_{n,A,\gamma}^{q} f \right\|_{S^p}$. Precisely, we extend the result of S. M. Mazhar and V. Totik (see [10, Theorem 1]) and generalize the following result of P. Pych-Taberska (see [11, Theorem 5]):

THEOREM 2. *If* $f \in \Omega_{\alpha,\infty}$, $\alpha > 0$ *and* $q \geqslant 2$, *then*

$$\left\| T^{q}_{n,A,\gamma} f \right\|_{S^{\infty}} \ll \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \left[\omega f \left(\frac{\pi}{k+1} \right)_{S^{\infty}} \right]^{q} \right\}^{1/q} + \frac{\|f\|_{S^{\infty}}}{(n+1)^{1/q}},$$

for $n = 0,1,2,\ldots$, *where* $\gamma = (\gamma_k)$ *is a sequence with* $\gamma_k = \frac{\alpha k}{2}$, $a_{n,k} = \frac{1}{n+1}$ *when* $k \leqslant n$ *and* $a_{n,k} = 0$ *otherwise.*

We shall write $I_1 \ll I_2$ if there exists a positive constant K, sometimes depended on some parameters, such that $I_1 \leqslant K I_2$.

2. Statement of the results

We start with two propositions.

PROPOSITION 1. *If* $f \in \Omega_{\alpha,p}$, $\alpha > 0$, $n = O(r_n)$ *and* $q > 0$, *then*

$$\left\| \left\{ \frac{1}{r_n} \sum_{k=n-r_n}^{n-1} \left| S_{\frac{\alpha k}{2}} f - f \right|^{q} \right\}^{1/q} \right\|_{S^{p}} \ll \|f\|_{S^{p}},$$

for $n = 0,1,2,\ldots$

PROPOSITION 2. *If* $f \in \Omega_{\alpha,p}$, $\alpha > 0$, $n = O(r_n)$ *and* $q > 0$, *then*

$$\left\| \left\{ \frac{1}{r_n} \sum_{k=n-r_n}^{n-1} \left| S_{\frac{\alpha k}{2}} f - f \right|^{q} \right\}^{1/q} \right\|_{S^{p}} \ll E_{\frac{\alpha(n-r_n)}{2}}(f)_{S^{p}},$$

for $n = 0,1,2,\ldots$

In the special case $p = \infty$ and $f \in C_{2\pi}$ Proposition 2 reduce to the fundamental result of L. Leindler (see [8, Theorem 1]).

Our main results are following

THEOREM 3. *If* $f \in \Omega_{\alpha,p}$, $\alpha > 0$, $p \geqslant q$, $\left(a_{n,k}\right)_{k=0}^{\infty} \in GM({}_{2}\beta)$ *for all* n, *(1) and* $\lim_{n\to\infty} a_{n,0} = 0$ *hold, then*

$$\left\| T^{q}_{n,A,\gamma} f \right\|_{S^{p}} \ll \left\{ \sum_{k=0}^{\infty} a_{n,k} E^{q}_{\frac{\alpha k}{2^{[c]}+1}}(f)_{S^{p}} \right\}^{1/q}, \qquad (10)$$

for some $c > 1$ *and* $n = 0,1,2,\ldots$, *where* $\gamma = (\gamma_k)$ *is a sequence with* $\gamma_k = \frac{\alpha k}{2}$.

THEOREM 4. *If* $f \in \Omega_{\alpha,p}$, $\alpha > 0$, $p \geqslant q$, $\left(a_{n,k}\right)_{k=0}^{\infty} \in GM\left({}_2\beta\right)$ *for all* n, *(1) and* $\lim_{n\to\infty} a_{n,0} = 0$ *hold, then*

$$\left\|T_{n,A,\gamma}^{q} f\right\|_{S^p} \ll \left\{\sum_{k=0}^{\infty} a_{n,k} \omega^q f\left(\frac{\pi}{k+1}\right)_{S^p}\right\}^{1/q},$$

for $n = 0,1,2,\ldots$, *where* $\gamma = (\gamma_k)$ *is a sequence with* $\gamma_k = \frac{\alpha k}{2}$.

THEOREM 5. *If we additionally suppose that* $\left(a_{n,k}\right)_{k=0}^{\infty} \in MS$ *then*

$$\left\|T_{n,A,\gamma}^{q} f\right\|_{S^p} \ll \left\{\sum_{k=0}^{\infty} a_{n,k} E_{\frac{\alpha k}{2}}^{p} (f)_{S^p}\right\}^{1/q},$$

for $n = 0,1,2,\ldots$, *where* $\gamma = (\gamma_k)$ *is a sequence with* $\gamma_k = \frac{\alpha k}{2}$.

REMARK 1. Taking $a_{n,k} = \frac{1}{n+1}$ when $k \leqslant n$ and $a_{n,k} = 0$ otherwise, in the case $p = \infty$ we obtain the better estimate than this one from [11, Theorem 5].

3. Proofs of the results

3.1. Proof of Proposition 1

Denote by $S_k^* f$ the sums of the form

$$S_{\frac{\alpha k}{2}} f(x) = \sum_{|\lambda_\nu| \leqslant \frac{\alpha k}{2}} A_\nu(f) e^{i\lambda_\nu x}$$

such that the interval $\left(\frac{\alpha k}{2}, \frac{\alpha(k+1)}{2}\right)$ does not contain any λ_ν. Applying Lemma 1.10.2 of [9] we easily verify that

$$S_k^* f(x) - f(x) = \int_0^{\infty} \varphi_x(t) \Psi_k(t)\, dt,$$

where $\varphi_x(t) := f(x+t) + f(x-t) - 2f(x)$ and $\Psi_k(t) = \Psi_{\frac{\alpha k}{2}, \frac{\alpha(k+1)}{2}}(t)$ i.e.,

$$\Psi_k(t) = \frac{4 \sin\frac{\alpha t}{4} \sin\frac{\alpha(2k+1)t}{4}}{\pi \alpha t^2}$$

(see also [3], p. 41). Evidently, if the interval $\left(\frac{\alpha k}{2}, \frac{\alpha(k+1)}{2}\right)$ contains a Fourier exponent λ_ν, then

$$S_{\frac{\alpha k}{2}} f(x) = S_{k+1}^* f(x) - \left(A_\nu(f) e^{i\lambda_\nu x} + A_{-\nu}(f) e^{-i\lambda_\nu x}\right).$$

Since

$$\left\{\sum_{\nu=-\infty}^{\infty} \left|A_\nu(f)\right|^q\right\}^{1/q} \leqslant \|f\|_{B^p} \quad \text{for } 1 < p \leqslant 2 \text{ and } q = \frac{p}{p-1} \text{ ([1, p. 78])}$$

and

$$\|f\|_{B^p} \leqslant \|f\|_{S^p} \quad \text{for } p \geqslant 1 \text{ ([2, p. 7])},$$

where $\|\cdot\|_{B^p}$ is the Besicovitch norm, we have

$$|A_{\pm\nu}(f)| \leqslant \|f\|_{S^p} \quad \text{for } p > 1,$$

whence the deviation

$$\frac{1}{r_n}\sum_{k=n-r_n}^{n-1}\left|S_{\frac{\alpha k}{2}}f(x)-f(x)\right|^q$$

can be estimated from above by

$$\frac{1}{r_n}\sum_{k=n-r_n}^{n-1}\left|\int_0^\infty \varphi_x(t)\Psi_{k+\kappa}(t)\,dt\right|^q+\frac{1}{r_n}\sum_{k=n-r_n}^{n-1}\left(\|f\|_{S^p}\right)^q,$$

where κ equals 0 or 1. Putting $h=2\pi/(\alpha n)$ we obtain

$$\begin{aligned}\int_0^\infty \varphi_x(t)\Psi_{k+\kappa}(t)\,dt &= \left(\int_0^h+\int_h^{nh}+\int_{nh}^\infty\right)\varphi_x(t)\Psi_{k+\kappa}(t)\,dt\\ &= I_1(k)+I_2(k)+I_3(k).\end{aligned}$$

By elementary calculations we get

$$|I_1(k)| \leqslant \frac{(2k+3)\alpha}{4\pi}\int_0^h|\varphi_x(t)|\,dt \ll \frac{1}{h}\int_0^h|\varphi_x(t)|\,dt$$

and

$$|I_3(k)| \leqslant \int_{nh}^\infty|\varphi_x(t)\Psi_{k+\kappa}(t)|\,dt \ll \int_{nh}^\infty\frac{|\varphi_x(t)|}{t^2}dt.$$

Therefore

$$\frac{1}{r_n}\sum_{k=n-r_n}^{n-1}\left[|I_1(k)|+|I_3(k)|\right]^q \ll \left[\frac{1}{h}\int_0^h|\varphi_x(t)|\,dt+\int_{nh}^\infty\frac{|\varphi_x(t)|}{t^2}dt\right]^q.$$

Consequently, we have to estimate the quantity $\frac{1}{r_n}\sum_{k=n-r_n}^{n-1}|I_2(k)|^q$. The inequality of Hausdorf-Young [17, Chap. XII, Th. 3.3 II] yields (cf. [11, p. 102])

$$\frac{1}{r_n}\sum_{k=n-r_n}^{n-1}|I_2(k)|^q \ll \frac{1}{n}\sum_{k=1}^{n}|I_2(k)|^q \ll \frac{1}{n}\left[\int_h^{nh}\frac{|\varphi_x(t)|^{q'}}{t^{q'}}dt\right]^{q/q'},$$

where $q'=\frac{q}{q-1}$ and $q\geqslant 2$.

By monotonicity of $\left\{\frac{1}{r_n}\sum_{k=n-r_n}^{n-1}\left|S_{\frac{\alpha k}{2}}f-f\right|^{v}\right\}^{1/v}$ with respect to $v>0$,

$$\left\|\left\{\frac{1}{r_n}\sum_{k=n-r_n}^{n-1}\left|S_{\frac{\alpha k}{2}}f-f\right|^{v}\right\}^{1/v}\right\|_{S^p} \leqslant \left\|\left\{\frac{1}{r_n}\sum_{k=n-r_n}^{n-1}\left|S_{\frac{\alpha k}{2}}f-f\right|^{q}\right\}^{1/q}\right\|_{S^p}$$

$$\ll \frac{1}{h}\int_0^h \|\varphi_{\cdot}(t)\|_{S^p}\,dt+\int_{nh}^{\infty}\frac{\|\varphi_{\cdot}(t)\|_{S^p}}{t^2}dt+\left\{\frac{1}{n}\int_h^{nh}\frac{\|\varphi_{\cdot}(t)\|_{S^p}^{q'}}{t^{q'}}dt\right\}^{1/q'}+\|f\|_{S^p}$$

$$\ll \frac{1}{h}\int_0^h \|f\|_{S^p}\,dt+\int_{nh}^{\infty}\frac{\|f\|_{S^p}}{t^2}dt+\left\{\frac{1}{n}\int_h^{nh}\frac{\|f\|_{S^p}^{q'}}{t^{q'}}dt\right\}^{1/q'}+\|f\|_{S^p}$$

$$\ll \|f\|_{S^p}\left[2+\int_{nh}^{\infty}\frac{1}{t^2}dt+\left(\frac{1}{n}\int_h^{nh}\frac{1}{t^{q'}}dt\right)^{1/q'}\right]\ll \|f\|_{S^p},$$

for any $v\in(0,q]$ such that $q'\leqslant p$.

Thus the desired result follows. □

3.2. Proof of Proposition 2

The proof is standard and the estimate follows from that of Proposition 1. Namely, taking $g_\sigma\in B_\sigma$ such that $E_\sigma(f)_{S^p}=\|f-g_\sigma\|_{S^p}$ we obtain

$$\begin{aligned}&\left\|\left\{\frac{1}{r_n}\sum_{k=n-r_n}^{n-1}\left|S_{\frac{\alpha k}{2}}f-f\right|^{q}\right\}^{1/q}\right\|_{S^p}\\ &=\left\|\left\{\frac{1}{r_n}\sum_{k=n-r_n}^{n-1}\left|S_{\frac{\alpha k}{2}}f-g_\sigma-(f-g_\sigma)\right|^{q}\right\}^{1/q}\right\|_{S^p}\\ &=\left\|\left\{\frac{1}{r_n}\sum_{k=n-r_n}^{n-1}\left|S_{\frac{\alpha k}{2}}(f-g_\sigma)-(f-g_\sigma)\right|^{q}\right\}^{1/q}\right\|_{S^p}\\ &\ll \|f-g_\sigma\|_{S^p},\end{aligned}$$

with $\sigma=\frac{\alpha(n-r_n)}{2}$, and thus our result follows. □

3.3. Proof of Theorem 3

Let

$$\left\|T_{n,A,\gamma}^{q}f\right\|_{S^p}=\left\|\left\{\sum_{k=0}^{2^{[c]}-1}a_{n,k}\left|S_{\frac{\alpha k}{2}}f-f\right|^{q}+\sum_{k=2^{[c]}}^{\infty}a_{n,k}\left|S_{\frac{\alpha k}{2}}f-f\right|^{q}\right\}^{1/q}\right\|_{S^p}$$

$$\leqslant \left\| \left\{ \sum_{k=0}^{2^{[c]}-1} a_{n,k} \left| S_{\frac{\alpha k}{2}} f - f \right|^q \right\}^{1/q} \right\|_{S^p} + \left\| \left\{ \sum_{m=[c]}^{\infty} \sum_{k=2^m}^{2^{m+1}-1} a_{n,k} \left| S_{\frac{\alpha k}{2}} f - f \right|^q \right\}^{1/q} \right\|_{S^p}$$

$$= I_1 + I_2.$$

for some $c > 1$. Using Proposition 2 we obtain, for $p \geqslant q$,

$$\begin{aligned} I_1 &\leqslant \left\| \left\{ \sum_{k=0}^{2^{[c]}-1} a_{n,k} \frac{k/2+1}{k/2+1} \sum_{l=k/2}^{k} \left| S_{\frac{\alpha l}{2}} f - f \right|^q \right\}^{1/q} \right\|_{S^p} \\ &\leqslant \left\{ 2^{[c]} \sum_{k=0}^{2^{[c]}-1} a_{n,k} \left\| \left[\frac{1}{k/2+1} \sum_{l=k/2}^{k} \left| S_{\frac{\alpha l}{2}} f - f \right|^q \right]^{1/q} \right\|_{S^p}^q \right\}^{1/q} \\ &\ll \left\{ \sum_{k=0}^{2^{[c]}-1} a_{n,k} E^q_{\frac{\alpha k}{4}}(f)_{S^p} \right\}^{1/q}. \end{aligned}$$

By partial summation, our Proposition 2 gives

$$\begin{aligned} I_2 &= \left\| \left\{ \sum_{m=[c]}^{\infty} \left[\sum_{k=2^m}^{2^{m+1}-2} \left(a_{n,k} - a_{n,k+1}\right) \sum_{l=2^m}^{k} \left| S_{\frac{\alpha l}{2}} f - f \right|^q \right. \right. \right. \\ &\quad \left. \left. \left. + a_{n,2^{m+1}-1} \sum_{l=2^m}^{2^{m+1}-1} \left| S_{\frac{\alpha l}{2}} f - f \right|^q \right] \right\}^{1/q} \right\|_{S^p} \\ &\leqslant \left\{ \sum_{m=[c]}^{\infty} \left[\sum_{k=2^m}^{2^{m+1}-2} \left| a_{n,k} - a_{n,k+1} \right| \left\| \left(\sum_{l=2^m}^{k} \left| S_{\frac{\alpha l}{2}} f - f \right|^q \right)^{1/q} \right\|_{S^p}^q \right. \right. \\ &\quad \left. \left. + a_{n,2^{m+1}-1} \left\| \left(\sum_{l=2^m}^{2^{m+1}-1} \left| S_{\frac{\alpha l}{2}} f - f \right|^q \right)^{1/q} \right\|_{S^p}^q \right] \right\}^{1/q} \\ &\ll \left\{ \sum_{m=[c]}^{\infty} \left[2^m \sum_{k=2^m}^{2^{m+1}-2} \left| a_{n,k} - a_{n,k+1} \right| E^q_{\frac{\alpha 2^m}{2}}(f)_{S^p} \right. \right. \\ &\quad \left. \left. + 2^m a_{n,2^{m+1}-1} E^q_{\frac{\alpha 2^m}{2}}(f)_{S^p} \right] \right\}^{1/q} \\ &\ll \left\{ \sum_{m=[c]}^{\infty} 2^m E^q_{\frac{\alpha 2^m}{2}}(f)_{S^p} \left[\sum_{k=2^m}^{2^{m+1}-2} \left| a_{n,k} - a_{n,k+1} \right| + a_{n,2^{m+1}-1} \right] \right\}^{1/q}, \end{aligned}$$

for $p \geqslant q$.

Since (9) holds, we have

$$\begin{aligned}
&a_{n,s+1}-a_{n,r}\\
&\leqslant |a_{n,r}-a_{n,s+1}| \leqslant \sum_{k=r}^{s}|a_{n,k}-a_{n,k+1}|\\
&\leqslant \sum_{k=2^m}^{2^{m+1}-2}|a_{n,k}-a_{n,k+1}| \ll \sum_{k=[2^m/c]}^{[c2^m]}\frac{a_{n,k}}{k} \quad \left(2\leqslant 2^m\leqslant r\leqslant s\leqslant 2^{m+1}-2\right),
\end{aligned}$$

whence

$$a_{n,s+1} \ll a_{n,r}+\sum_{k=[2^m/c]}^{[c2^m]}\frac{a_{n,k}}{k} \ \ \left(2\leqslant 2^m\leqslant r\leqslant s\leqslant 2^{m+1}-2\right).$$

Consequently,

$$\begin{aligned}
2^m a_{n,2^{m+1}-1} &= \frac{2^m}{2^m-1}\sum_{r=2^m}^{2^{m+1}-2}a_{n,2^{m+1}-1}\\
&\ll \sum_{r=2^m}^{2^{m+1}-2}\left(a_{n,r}+\sum_{k=[2^m/c]}^{[c2^m]}\frac{a_{n,k}}{k}\right)\\
&\ll \sum_{r=2^m}^{2^{m+1}-1}a_{n,r}+2^m\sum_{k=[2^m/c]}^{[c2^m]}\frac{a_{n,k}}{k}
\end{aligned}$$

and therefore

$$I_2 \ll \left\{\sum_{m=[c]}^{\infty}\left[2^m E^q_{\frac{\alpha 2^m}{2}}(f)_{S^p}\sum_{k=[2^m/c]}^{[c2^m]}\frac{a_{n,k}}{k}+E^q_{\frac{\alpha 2^m}{2}}(f)_{S^p}\sum_{k=2^m}^{2^{m+1}-1}a_{n,k}\right]\right\}^{1/q}.$$

Using typical transformations we get

$$\begin{aligned}
I_2 &\ll \left\{\sum_{m=[c]}^{\infty}\left[2^m E^q_{\frac{\alpha 2^m}{2}}(f)_{S^p}\sum_{k=2^{m-[c]}}^{2^{m+[c]}}\frac{a_{n,k}}{k}+E^q_{\frac{\alpha 2^m}{2}}(f)_{S^p}\sum_{k=2^m}^{2^{m+1}}a_{n,k}\right]\right\}^{1/q}\\
&\ll \left\{\sum_{m=[c]}^{\infty}E^q_{\frac{\alpha 2^m}{2}}(f)_{S^p}\sum_{k=2^{m-[c]}}^{2^{m+[c]}}a_{n,k}\right\}^{1/q}\\
&= \left\{\sum_{m=[c]}^{\infty}E^q_{\frac{\alpha 2^m}{2}}(f)_{S^p}\sum_{k=2^{m-[c]}}^{2^m-1}a_{n,k}+\sum_{m=[c]}^{\infty}E^q_{\frac{\alpha 2^m}{2}}(f)_{S^p}\sum_{k=2^m}^{2^{m+[c]}}a_{n,k}\right\}^{1/q}\\
&\ll \left\{\sum_{m=[c]}^{\infty}\sum_{k=2^{m-[c]}}^{2^m-1}a_{n,k}E^q_{\frac{\alpha k}{2}}(f)_{S^p}+\sum_{m=[c]}^{\infty}\sum_{k=2^m}^{2^{m+[c]}}a_{n,k}E^q_{\frac{\alpha k}{2^{[c]+1}}}(f)_{S^p}\right\}^{1/q}
\end{aligned}$$

$$\ll \left\{ \sum_{m=[c]}^{\infty} \sum_{k=2^{m-[c]}}^{2^m-1} a_{n,k} E^q_{\frac{\alpha k}{2}}(f)_{S^p} + \sum_{m=[c]}^{\infty} \sum_{k=2^m}^{2^{m+[c]}-1} a_{n,k} E^q_{\frac{\alpha k}{2^{[c]+1}}}(f)_{S^p} + \sum_{m=[c]}^{\infty} E^q_{\frac{\alpha 2^m}{2}}(f)_{S^p} a_{n,2^{m+[c]}} \right\}^{1/q}$$

$$= \left\{ \sum_{m=[c]}^{\infty} \sum_{r=1}^{[c]} \sum_{k=2^{m-r}}^{2^{m-r+1}-1} a_{n,k} E^q_{\frac{\alpha k}{2}}(f)_{S^p} + \sum_{m=[c]}^{\infty} \sum_{r=0}^{[c]-1} \sum_{k=2^{m+r}}^{2^{m+r+1}-1} a_{n,k} E^q_{\frac{\alpha k}{2^{[c]+1}}}(f)_{S^p} + \sum_{m=[c]}^{\infty} E^q_{\frac{\alpha 2^m}{2}}(f)_{S^p} a_{n,2^{m+[c]}} \right\}^{1/q}$$

$$\leqslant \left\{ \sum_{r=1}^{[c]} \sum_{k=2^{[c]-r}}^{\infty} a_{n,k} E^q_{\frac{\alpha k}{2}}(f)_{S^p} + \sum_{r=0}^{[c]-1} \sum_{k=2^{[c]+r}}^{\infty} a_{n,k} E^q_{\frac{\alpha k}{2^{[c]+1}}}(f)_{S^p} + \sum_{k=2^{2[c]}}^{\infty} a_{n,k} E^q_{\frac{\alpha 2^m}{2}}(f)_{S^p} \right\}^{1/q} \ll \left\{ \sum_{k=0}^{\infty} a_{n,k} E^q_{\frac{\alpha k}{2^{[c]+1}}}(f)_{S^p} \right\}^{1/q}.$$

Thus we obtain the desired result. □

3.4. Proof of Theorem 4

The proof follows by the Jackson type theorem

$$E_\sigma(f)_{S^p} \ll \omega f\left(\frac{1}{\sigma}\right)_{S^p}$$

and basic properties of the modulus of continuity $\omega f(\cdot)_{S^p}$. □

3.5. Proof of Theorem 5

If $(a_{n,k})_{k=0}^{\infty} \in MS$, then $(a_{n,k})_{k=0}^{\infty} \in GM({}_2\beta)$ and using Theorem 4 we obtain

$$\left\| T^q_{n,A,\gamma} f \right\| \ll \left\{ \sum_{k=0}^{\infty} a_{n,k} E^q_{\frac{\alpha k}{2^{[c]+1}}}(f)_{S^p} \right\}^{1/q} = \left\{ \sum_{k=0}^{\infty} \sum_{m=k2^{[c]}}^{(k+1)2^{[c]}-1} a_{n,m} E^q_{\frac{\alpha k}{2^{[c]+1}}}(f)_{S^p} \right\}^{1/q}$$

$$\leqslant \left\{ \sum_{k=0}^{\infty} E^q_{\frac{\alpha k}{2}}(f)_{S^p} \sum_{m=k2^{[c]}}^{(k+1)2^{[c]}-1} a_{n,m} \right\}^{1/q} \leqslant \left\{ \sum_{k=0}^{\infty} 2^{[c]} E^q_{\frac{\alpha k}{2}}(f)_{S^p} a_{n,k2^{[c]}} \right\}^{1/q}$$

$$\leqslant \left\{ 2^{[c]} \sum_{k=0}^{\infty} E^q_{\frac{\alpha k}{2}}(f)_{S^p} a_{n,k} \right\}^{1/q} \ll \left\{ \sum_{k=0}^{\infty} E^q_{\frac{\alpha k}{2}}(f)_{S^p} a_{n,k} \right\}^{1/q}.$$

This ends our proof. □

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Włodzimierz Łenski
University of Zielona Góra
Faculty of Mathematics, Computer Science and Econometrics
65-516 Zielona Góra, ul. Szafrana 4a
Poland
e-mail: W.Lenski@wmie.uz.zgora.pl

Bogdan Szal
University of Zielona Góra
Faculty of Mathematics, Computer Science and Econometrics
65-516 Zielona Góra, ul. Szafrana 4a
Poland
e-mail: B.Szal@wmie.uz.zgora.pl