Mathematical
Inequalities
& Applications
Volume 18, Number 1 (2015), 51–60 doi:10.7153/mia-18-03

IMPROVED REVERSE ARITHMETIC–GEOMETRIC MEANS INEQUALITIES FOR POSITIVE OPERATORS ON HILBERT SPACE

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(*Communicated by J. Pečarić*)

Abstract. In this paper, employing induction on the given reverse Young inequalities, we obtain more reverse arithmetic-geometric means inequalities for two positive operators. Concretely, following the main result from [13] we obtain reverse ratio type inequalities and reverse difference type inequalities of the refined arithmetic-geometric means inequality for two positive operators on a Hilbert space.

1. Introduction

Throughout this paper, A, B are both positive operators on a Hilbert space H, and $\mathscr{B}_h(H)$ is the semi-space of all bounded linear self-adjoint operators on H. In additive notation $\mathscr{B}^+(H)$ is written as the set of all positive operators in $\mathscr{B}_h(H)$. Besides, we may assume that A and B are invertible without loss of generality,

$$A\nabla_\mu B = (1-\mu)A + \mu B \quad \text{and} \quad A\sharp_\mu B = A^{1/2}(A^{-1/2}BA^{-1/2})^\mu A^{1/2}, \quad \text{where} \quad 0 \leqslant \mu \leqslant 1.$$

When $\mu = 1/2$ we write $A\nabla B$ and $A\sharp B$ for brevity, respectively, see Kubo and Ando [9]. The Specht ratio [11] is defined by

$$S(t) = \frac{t^{\frac{1}{t-1}}}{e\log t^{\frac{1}{t-1}}} \text{ for } t > 0, t \neq 1; \text{ and } S(1) = \lim_{t\to 1} S(t) = 1$$

and has the following properties.

(i) $S(h) = S(1/h) \geqslant 1$ for $h > 0$.

(ii) $S(h)$ is a monotone increasing function on $(1, +\infty)$.

(iii) $S(h)$ is a monotone decreasing function on $(0, 1)$.

Mathematics subject classification (2010): 47A30, 47A63.
Keywords and phrases: Jensen operator, reverse arithmetic-geometric means, Jensen's functional.
This research was partially supported by the National Natural Science Foundation of China (10471085) and Major Project of Henan Education Department (2011A110010).

We start from the reverse arithmetic-geometric mean inequality with the Specht ratio for two positive operators:

THEOREM T. [12] *For invertible operators A and B with $0 < aI_H \leqslant A, B \leqslant bI_H$, we have*

$$\begin{aligned} &\text{(i)} \qquad A\nabla_\mu B \leqslant S(h)A\sharp_\mu B, \\ &\text{(ii)} \quad A\nabla_\mu B - A\sharp_\mu B \leqslant L(1,h)\log S(h)A, \end{aligned}$$

where $L(1,h)$ is defined by $L(a,b) = \frac{a-b}{\log a - \log b}(a \neq b); L(a,a) = a,\ h = b/a$.

These inequalities have recently been improved by Furuichi as follows:

THEOREM F. [2] *If $0 < aI_H \leqslant A, B \leqslant bI_H$, then*

$$\begin{aligned} &\text{(i)} \qquad A\nabla_\mu B - 2r(A\nabla B - A\sharp B) \leqslant S(\sqrt{h})A\sharp_\mu B, \\ &\text{(ii)} \quad A\nabla_\mu B - A\sharp_\mu B - 2r(A\nabla B - A\sharp B) \leqslant L(\sqrt{h},1)\log S(\sqrt{h})bI_H, \end{aligned}$$

where $r = \min\{\mu, 1-\mu\}$, $L(a,b) = \frac{a-b}{\log a - \log b}$, $h = \frac{b}{a}$.

Afterwards, Krnić et al. [8] introduced Jensen's operator and established some bounds for spectra of Jensen's operator. The obtained results were then applied to operator means. In such a way, they get refinements and converses of numerous mean inequalities for Hilbert space operators. See [3, 6–12] for more related developments.

See also [13] for another improvement of the reverse weighted arithmetic-geometric operator mean inequalities. Their proof is independent of [2] but uses [7]:

THEOREM ZF. *If $0 < aA \leqslant B \leqslant bA$ with $a < 1 < b$, then*

$$\begin{aligned} &\text{(i)} \quad A\nabla_\mu B - 2r(A\nabla B - A\sharp B) \leqslant \max\{S(\sqrt{a}), S(\sqrt{b})\}A\sharp_\mu B, \\ &\text{(ii)} \quad A\nabla_\mu B - A\sharp_\mu B - 2r(A\nabla B - A\sharp B) \\ &\qquad\qquad \leqslant \max\{L(\sqrt{1/a},1)\log S(\sqrt{a}), L(\sqrt{1/b},1)\log S(\sqrt{b})\}bA, \end{aligned}$$

where $r = \min\{\mu, 1-\mu\}$.

The aim of this paper is to provide a method to obtain more reverse arithmetic-geometric means inequalities for positive operators on Hilbert space. In the Section 2, we introduce the main lemmas by means of which as well as of the induction employed on Theorem ZF, we obtain reverse ratio type inequalities and reverse difference type inequalities of the refined arithmetic-geometric means inequality for positive operators in the Section 3 and the Section 4, respectively.

2. Main lemmas

LEMMA 2.1. *If A and B are positive operators on Hilbert space, $0 \leqslant \mu, \nu \leqslant 1$, then*

$$A\nabla_{\mu}(A\sharp_{\nu}B) = A\nabla_{\mu\nu}B - \mu(A\nabla_{\nu}B - A\sharp_{\nu}B).$$

Proof.

$$\begin{aligned} A\nabla_{\mu}(A\sharp_{\nu}B) &= (1-\mu)A + \mu A\sharp_{\nu}B \\ &= A - \mu A + \mu\nu A - \mu\nu A + \mu\nu B - \mu\nu B + \mu A\sharp_{\nu}B \\ &= \mu\nu B + (1-\mu\nu)A - \mu[(1-\nu)A + \nu B - A\sharp_{\nu}B] \\ &= A\nabla_{\mu\nu}B - \mu(A\nabla_{\nu}B - A\sharp_{\nu}B). \quad \square \end{aligned}$$

LEMMA 2.2. [7] *If $A, B \in \mathscr{B}^{+}(H)$, $\mathrm{p} = (p_1, p_2) \in \mathbb{R}_{+}^{2}$, then*

$$2\min\{p_1,p_2\}(A\nabla B - A\sharp B) \leqslant \mathfrak{J}(A,B,\mathrm{p}) \leqslant 2\max\{p_1,p_2\}(A\nabla B - A\sharp B),$$

where the operator $\mathfrak{J}\colon \mathscr{B}^{+}(H) \times \mathscr{B}^{+}(H) \times \mathbb{R}_{+}^{2} \longrightarrow \mathscr{B}^{+}(H)$ is defined by

$$\mathfrak{J}(A,B,\mathrm{p}) = (p_1+p_2)\Big[A\nabla_{\frac{p_1}{p_1+p_2}}B - A\sharp_{\frac{p_1}{p_1+p_2}}B\Big].$$

3. Reverse ratio type arithmetic-geometric mean inequalities

First of all, we show a refinement of reverse arithmetic-geometric mean inequality, applying Theorem ZF.

LEMMA 3.1. *If $0 < aA \leqslant B \leqslant bA$ with $a < 1 < b$, and $0 \leqslant \mu \leqslant 1$, then*

$$A\nabla_{\mu}B - 2(r_1+r_2)(A\nabla B - A\sharp B) \leqslant \max\{S(\sqrt[4]{a}), S(\sqrt[4]{b})\}A\sharp_{\mu}B, \tag{3.1}$$

where $r_1 = \min\{\mu, 1-\mu\}$, $r_2 = \min\{|1-2\mu|, 1-|1-2\mu|\}$.

Proof. If $0 \leqslant \mu \leqslant \frac{1}{2}$, then $0 \leqslant 2\mu \leqslant 1$. Since $0 < aA \leqslant B \leqslant bA$ ensures that $\sqrt{a}A \leqslant A\sharp B \leqslant \sqrt{b}A$, by substituting B by $A\sharp B$ and μ by 2μ in (i) of Theorem ZF, it follows that

$$A\nabla_{2\mu}(A\sharp B) - 2\min\{2\mu, 1-2\mu\}[A\nabla(A\sharp B) - A\sharp(A\sharp B)] \leqslant \max\{S(\sqrt[4]{a}), S(\sqrt[4]{b})\}A\sharp_{2\mu}(A\sharp B).$$

By Lemma 2.1 and Lemma 2.2 it follows that

$$A\nabla_{\frac{1}{4}}B - A\sharp_{\frac{1}{4}}B \leqslant 2\max\left\{\frac{1}{4}, \frac{3}{4}\right\}(A\nabla B - A\sharp B) = \frac{3}{2}(A\nabla B - A\sharp B),$$

$$A\nabla(A\sharp B) - A\sharp(A\sharp B) = A\nabla_{\frac{1}{4}}B - \frac{1}{2}(A\nabla B - A\sharp B) - A\sharp_{\frac{1}{4}}B \leqslant (A\nabla B - A\sharp B).$$

Then

$$A\nabla_\mu B-2(\mu+\min\{2\mu,1-2\mu\})(A\nabla B-A\sharp B)\leqslant \max\{S(\sqrt[4]{a}),S(\sqrt[4]{b})\}A\sharp_\mu B. \quad (3.2)$$

If $\frac{1}{2}\leqslant\mu\leqslant 1$, then $0\leqslant 1-\mu\leqslant\frac{1}{2}$. The hypothesis $0<aA\leqslant B\leqslant bA$ admits $0<\frac{1}{b}B\leqslant A\leqslant\frac{1}{a}B$. Then by the inequality (3.2) we have

$$\begin{aligned}&B\nabla_{1-\mu}A-2[(1-\mu)+\min\{2(1-\mu),1-2(1-\mu)\}](B\nabla A-B\sharp A)\\&\leqslant \max\left\{S\left(\frac{1}{\sqrt[4]{b}}\right),S\left(\frac{1}{\sqrt[4]{a}}\right)\right\}B\sharp_{1-\mu}A.\end{aligned}$$

If we notice that $S(\frac{1}{\sqrt[4]{a}})=S(\sqrt[4]{a})$ and $S(\frac{1}{\sqrt[4]{b}})=S(\sqrt[4]{b})$, then we have

$$\begin{aligned}&A\nabla_\mu B-2[(1-\mu)+\min\{2(1-\mu),1-2(1-\mu)\}](A\nabla B-A\sharp B)\\&\leqslant \max\{S(\sqrt[4]{a}),S(\sqrt[4]{b})\}A\sharp_\mu B.\end{aligned}$$

Therefore, for $0\leqslant\mu\leqslant 1$, we have

$$A\nabla_\mu B-2(r_1+r_2)(A\nabla B-A\sharp B)\leqslant \max\{S(\sqrt[4]{a}),S(\sqrt[4]{b})\}A\sharp_\mu B,$$

where $r_1=\min\{\mu,1-\mu\}$, $r_2=\min\{|1-2\mu|,1-|1-2\mu|\}$. □

Replacing the hypothesis $0<aA\leqslant B\leqslant bA$, where $a<1<b$ with $0<aI_H\leqslant A,B\leqslant bI_H$, where $a<b$, we obtain the counterpart of Lemma 3.1.

LEMMA 3.2. *If $0<aI_H\leqslant A,B\leqslant bI_H$ with $a<b$, and $0\leqslant\mu\leqslant 1$, then*

$$A\nabla_\mu B-2(r_1+r_2)(A\nabla B-A\sharp B)\leqslant S(\sqrt[4]{h})A\sharp_\mu B, \quad (3.3)$$

where $r_1=\min\{\mu,1-\mu\}$, $r_2=\min\{|1-2\mu|,1-|1-2\mu|\}$, $h=\frac{b}{a}$.

Proof. Since $0<aI_H\leqslant A,B\leqslant bI_H$ admits that $\sqrt{1/h}A\leqslant A\sharp B\leqslant\sqrt{h}A$, we also substitute B by $A\sharp B$, μ by 2μ when $0\leqslant\mu\leqslant\frac{1}{2}$ and $1-\mu$ by $2(1-\mu)$ when $\frac{1}{2}\leqslant\mu\leqslant 1$ in (i) of Theorem ZF, respectively, then by similar work as in the proof of Lemma 3.1, we can get the required inequality (3.3). □

REMARK 3.3. We easily find that both sides in the inequality (3.3) are less than or equal to those in (i) of Theorem F, so that neither the inequality (3.3) nor (i) of Theorem F is uniformly better than the other.

Besides, if we substitute B by $A\sharp B$, μ by 2μ when $0\leqslant\mu\leqslant\frac{1}{2}$ and $1-\mu$ by $2(1-\mu)$ when $\frac{1}{2}\leqslant\mu\leqslant 1$ in (i) of Theorem F, respectively, then by similar work we get $A\nabla_\mu B-2(r_1+r_2)(A\nabla B-A\sharp B)\leqslant S(\sqrt{h})A\sharp_\mu B$, which could be deduced from Theorem F directly, so we may claim that the above inequality is trivial.

Repeating the above procedure as in Lemma 3.1, that is, employing the induction on Theorem ZF, we can obtain more inequalities as is shown in the sequel.

THEOREM 3.4. *If* $0 < aA \leqslant B \leqslant bA$ *with* $a < 1 < b$*, and* $0 \leqslant \mu \leqslant 1$*, then*

$$A\nabla_\mu B - 2\Sigma_{i=1}^{n} r_i(A\nabla B - A\sharp B) \leqslant \max\{S(\sqrt[2^n]{a}), S(\sqrt[2^n]{b})\}A\sharp_\mu B, \tag{3.4}$$

where $r_1 = \min\{\mu, 1-\mu\} \doteq \min\{\alpha_1, 1-\alpha_1\}$*,* $r_2 = \min\{|1-2\mu|, 1-|1-2\mu|\} \doteq \min\{\alpha_2, 1-\alpha_2\}$*,* $r_{i+1} = \min\{|1-2\alpha_i|, 1-|1-2\alpha_i|\}$*,* $i = 1, 2, \cdots, n-1$.

Proof. When $n = 1$, inequality (3.4) holds by Theorem ZF.

Now, we assume that the inequality (3.4) is valid when $n = m$, that is,

$$A\nabla_\mu B - 2\Sigma_{i=1}^{m} r_i(A\nabla B - A\sharp B) \leqslant \max\{S(\sqrt[2^m]{a}), S(\sqrt[2^m]{b})\}A\sharp_\mu B. \tag{3.5}$$

Our task is to prove (3.4) similarly as in Lemma 3.1. We distinguish the following two cases:

(I) If $0 \leqslant \mu \leqslant \frac{1}{2}$, then $0 \leqslant 2\mu \leqslant 1$. Since $0 < aA \leqslant B \leqslant bA$ ensures that $\sqrt{a}A \leqslant A\sharp B \leqslant \sqrt{b}A$, we replace B by $A\sharp B$ and μ by 2μ in (3.5) and get

$$A\nabla_\mu B - 2(\mu + \Sigma_{i=1}^{m} r_i')(A\nabla B - A\sharp B) \leqslant \max\{S(\sqrt[2^{m+1}]{a}), S(\sqrt[2^{m+1}]{b})\}A\sharp_\mu B, \tag{3.6}$$

where $r_1' = \min\{2\mu, 1-2\mu\} \doteq \min\{\alpha_1', 1-\alpha_1'\}$, $r_2' = \min\{|1-4\mu|, 1-|1-4\mu|\} \doteq \min\{\alpha_2', 1-\alpha_2'\}$, $r_{i+1}' = \min\{|1-2\alpha_i'|, 1-|1-2\alpha_i'|\}$, $i = 1, 2, \cdots, m-1$.

(II) If $\frac{1}{2} \leqslant \mu \leqslant 1$, then $0 \leqslant 1-\mu \leqslant \frac{1}{2}$. The hypothesis $0 < aA \leqslant B \leqslant bA$ admits $0 < \frac{1}{b}B \leqslant A \leqslant \frac{1}{a}B$, then by the inequality (3.6) we have

$$A\nabla_\mu B - 2[(1-\mu) + \Sigma_{i=1}^{m} r_i''](A\nabla B - A\sharp B) \leqslant \max\{S(\sqrt[2^{m+1}]{a}), S(\sqrt[2^{m+1}]{b})\}A\sharp_\mu B.$$

where $r_1'' = \min\{2(1-\mu), 1-2(1-\mu)\} \doteq \min\{\alpha_1'', 1-\alpha_1''\}$, $r_2'' = \min\{|4\mu-3|, 1-|4\mu-3|\} \doteq \min\{\alpha_2'', 1-\alpha_2''\}$, $r_{i+1}'' = \min\{|1-2\alpha_i''|, 1-|1-2\alpha_i''|\}$, $i = 1, 2, \cdots, m-1$.

Combining (I) with (II) we have

$$A\nabla_\mu B - 2(r_1 + \Sigma_{i=1}^{m} \min\{r_i', r_i''\})(A\nabla B - A\sharp B) \leqslant \max\{\sqrt[2^{m+1}]{a}), S(\sqrt[2^{m+1}]{b})\}A\sharp_\mu B,$$

that is,

$$A\nabla_\mu B - 2\Sigma_{i=1}^{m+1} r_i(A\nabla B - A\sharp B) \leqslant \max\{S(\sqrt[2^{m+1}]{a}), S(\sqrt[2^{m+1}]{b})\}A\sharp_\mu B,$$

where $r_1 = \min\{\mu, 1-\mu\} \doteq \min\{\alpha_1, 1-\alpha_1\}$, $r_2 = \min\{|1-2\mu|, 1-|1-2\mu|\} \doteq \min\{\alpha_2, 1-\alpha_2\}$, $r_{i+1} = \min\{|1-2\alpha_i|, 1-|1-2\alpha_i|\}$, $i = 1, 2, \cdots, m$.

This completes the proof. □

In order to show the analogous result holds under the condition $0 < aI_H \leqslant A, B \leqslant bI_H$ with $a < b$, we establish the following result.

THEOREM 3.5. *If* $0 < aI_H \leqslant A, B \leqslant bI_H$ *with* $a < b$*, and* $0 \leqslant \mu \leqslant 1$*, then*

$$A\nabla_\mu B - 2\Sigma_{i=1}^{n} r_i(A\nabla B - A\sharp B) \leqslant S(\sqrt[2^n]{h})A\sharp_\mu B. \tag{3.7}$$

where $r_1 = \min\{\mu, 1-\mu\} \doteq \min\{\alpha_1, 1-\alpha_1\}$*,* $r_2 = \min\{|1-2\mu|, 1-|1-2\mu|\} \doteq \min\{\alpha_2, 1-\alpha_2\}$*,* $r_{i+1} = \min\{|1-2\alpha_i|, 1-|1-2\alpha_i|\}$*,* $i = 1, 2, \cdots, n-1$.

Proof. From Theorem ZF, we get the inequality (3.7) when $n = 1$.

Now, we assume that the inequality (3.7) is valid when $n = m$, that is,

$$A\nabla_\mu B - 2\Sigma_{i=1}^{m} r_i(A\nabla B - A\sharp B) \leqslant S(\sqrt[2^m]{h})A\sharp_\mu B. \tag{3.8}$$

Then, for $n = m+1$, by the similar method we have:

(I) If $0 \leqslant \mu \leqslant \frac{1}{2}$, then $0 \leqslant 2\mu \leqslant 1$. Since $0 < aI_H \leqslant A, B \leqslant bI_H$ ensures that $0 \leqslant \frac{1}{\sqrt{h}}A \leqslant A\sharp B \leqslant \sqrt{h}A$, we substitute B by $A\sharp B$ and μ by 2μ in (3.8) and get

$$A\nabla_\mu B - 2[\mu + \Sigma_{i=1}^{m} r_i'](A\nabla B - A\sharp B) \leqslant S(\sqrt[2^{m+1}]{h})A\sharp_\mu B,$$

where $r_1' = \min\{2\mu, 1-2\mu\} \doteq \min\{\alpha_1', 1-\alpha_1'\}$, $r_2' = \min\{|1-4\mu|, 1-|1-4\mu|\} \doteq \min\{\alpha_2', 1-\alpha_2'\}$, $r_{i+1}' = \min\{|1-2\alpha_i'|, 1-|1-2\alpha_i'|\}$, $i = 1, 2, \cdots, m-1$.

(II) If $\frac{1}{2} \leqslant \mu \leqslant 1$, then $0 \leqslant 1-\mu \leqslant \frac{1}{2}$ and

$$A\nabla_\mu B - 2[(1-\mu) + \Sigma_{i=1}^{m} r_i''](A\nabla B - A\sharp B) \leqslant S(\sqrt[2^{m+1}]{h})A\sharp_\mu B,$$

where $r_1'' = \min\{2(1-\mu), 1-2(1-\mu)\} \doteq \min\{\alpha_1'', 1-\alpha_1''\}$, $r_2'' = \min\{|4\mu-3|, 1-|4\mu-3|\} \doteq \min\{\alpha_2'', 1-\alpha_2''\}$, $r_{i+1}'' = \min\{|1-2\alpha_i''|, 1-|1-2\alpha_i''|\}$, $i = 1, 2, \cdots, m-1$.

Therefore, for $0 \leqslant \mu \leqslant 1$, we have

$$A\nabla_\mu B - 2(r_1 + \Sigma_{i=1}^{m} \min\{r_i', r_i''\})(A\nabla B - A\sharp B) \leqslant S(\sqrt[2^{m+1}]{h})A\sharp_\mu B,$$

that is

$$A\nabla_\mu B - 2\Sigma_{i=1}^{m+1} r_i(A\nabla B - A\sharp B) \leqslant S(\sqrt[2^{m+1}]{h})A\sharp_\mu B,$$

where $r_1 = \min\{\mu, 1-\mu\} \doteq \min\{\alpha_1, 1-\alpha_1\}$, $r_2 = \min\{|1-2\mu|, 1-|1-2\mu|\} \doteq \min\{\alpha_2, 1-\alpha_2\}$, $r_{i+1} = \min\{|1-2\alpha_i|, 1-|1-2\alpha_i|\}$, $i = 1, 2, \cdots, m$.

This completes the proof. □

4. Reverse difference type arithmetic-geometric mean inequalities

In the following lemma we show the corresponding difference type analog of Lemma 3.1.

LEMMA 4.1. *If* $0 < aA \leqslant B \leqslant bA$ *with* $a < 1 < b$, *and* $0 \leqslant \mu \leqslant 1$, *then*

$$\begin{aligned} &A\nabla_\mu B - A\sharp_\mu B - 2(r_1 + r_2)(A\nabla B - A\sharp B) \\ &\leqslant \max\{L(\sqrt[4]{a}, 1)\log S(\sqrt[4]{a}), L(\sqrt[4]{b}, 1)\log S(\sqrt[4]{b})\}\frac{b}{\sqrt{a}}A, \end{aligned}$$

where $r_1 = \min\{\mu, 1-\mu\}$, $r_2 = \min\{|1-2\mu|, 1-|1-2\mu|\}$.

Proof. (I) If $0 \leqslant \mu \leqslant \frac{1}{2}$, then $0 \leqslant 2\mu \leqslant 1$. Since $0 < aA \leqslant B \leqslant bA$ admits that $\sqrt{a}A \leqslant A\sharp B \leqslant \sqrt{b}A$, we substitute B by $A\sharp B$ and μ by 2μ in (ii) of Theorem ZF. Then

$$\begin{aligned}&A\nabla_{2\mu}(A\sharp B) - A\sharp_{2\mu}(A\sharp B) - 2\min\{2\mu, 1-2\mu\}[A\nabla(A\sharp B) - A\sharp(A\sharp B)]\\ &\leqslant \max\{L(\sqrt[4]{1/a},1)\log S(\sqrt[4]{a}), L(\sqrt[4]{1/b},1)\log S(\sqrt[4]{b})\}\sqrt{b}A.\end{aligned}$$

As we showed in the proof of Lemma 3.1, the following inequality holds:

$$A\nabla(A\sharp B) - A\sharp(A\sharp B) = A\nabla_{\frac{1}{4}}B - \frac{1}{2}(A\nabla B - A\sharp B) - A\sharp_{\frac{1}{4}}B \leqslant (A\nabla B - A\sharp B).$$

Hence

$$\begin{aligned}&A\nabla_\mu B - A\sharp_\mu B - 2(\mu + \min\{2\mu, 1-2\mu\})(A\nabla B - A\sharp B)\\ &\leqslant \max\{L(\sqrt[4]{1/a},1)\log S(\sqrt[4]{a}), L(\sqrt[4]{1/b},1)\log S(\sqrt[4]{b})\}\sqrt{b}A.\end{aligned} \tag{4.1}$$

(II) If $\frac{1}{2} \leqslant \mu \leqslant 1$, then $0 \leqslant 1-\mu \leqslant \frac{1}{2}$. The hypothesis $0 < aA \leqslant B \leqslant bA$ ensures $0 < \frac{1}{b}B \leqslant A \leqslant \frac{1}{a}B$. Then by the inequality (4.1) and $S(\frac{1}{\sqrt{a}}) = S(\sqrt{a})$, $L(\frac{1}{\sqrt{a}},1) = \frac{1}{\sqrt{a}}L(\sqrt{a},1)$, we have

$$\begin{aligned}&A\nabla_\mu B - A\sharp_\mu B - 2[(1-\mu) + \min\{2(1-\mu), 1-2(1-\mu)\}](A\nabla B - A\sharp B)\\ &= B\nabla_{1-\mu}A - B\sharp_{1-\mu}A - 2[(1-\mu) + \min\{2(1-\mu), 1-2(1-\mu)\}](B\nabla A - B\sharp A)\\ &\leqslant \max\{L(\sqrt[4]{b},1)\log S(\sqrt[4]{1/b}), L(\sqrt[4]{a},1)\log S(\sqrt[4]{1/a})\}\sqrt{1/a}B\\ &\leqslant \max\{L(\sqrt[4]{b},1)\log S(\sqrt[4]{b}), L(\sqrt[4]{a},1)\log S(\sqrt[4]{a})\}b/\sqrt{a}A.\end{aligned}$$

Combining (I) with (II), then for $0 \leqslant \mu \leqslant 1$, we have

$$\begin{aligned}&A\nabla_\mu B - A\sharp_\mu B - 2(r_1 + r_2)(A\nabla B - A\sharp B)\\ &\leqslant \max\{L(\sqrt[4]{a},1)\log S(\sqrt[4]{a}), L(\sqrt[4]{b},1)\log S(\sqrt[4]{b})\}\frac{b}{\sqrt{a}}A,\end{aligned}$$

where $r_1 = \min\{\mu, 1-\mu\}$, $r_2 = \min\{|1-2\mu|, 1-|1-2\mu|\}$. □

If we put $a = \sqrt{1/h}$, $b = \sqrt{h}$ in Lemma 4.1 and repeat the above procedure, then we obtain more refined difference type arithmetic-geometric mean inequalities.

THEOREM 4.2. *If* $0 < \sqrt{1/h}A \leqslant B \leqslant \sqrt{h}A$ *with* $h > 1$, *and* $0 \leqslant \mu \leqslant 1$, *then*

$$A\nabla_\mu B - A\sharp_\mu B - 2\Sigma_{i=1}^n r_i(A\nabla B - A\sharp B) \leqslant h^{1-\frac{1}{2^n}} L(\sqrt[2^{n+1}]{h},1)\log S(\sqrt[2^{n+1}]{h})A, \tag{4.2}$$

where $h = \frac{b}{a}$, $r_1 = \min\{\mu, 1-\mu\} \doteq \min\{\alpha_1, 1-\alpha_1\}$, $r_2 = \min\{|1-2\mu|, 1-|1-2\mu|\} \doteq \min\{\alpha_2, 1-\alpha_2\}$, $r_{i+1} = \min\{|1-2\alpha_i|, 1-|1-2\alpha_i|\}$, $i = 1, 2, \cdots, n-1$.

Proof. When $n=1$, inequality (4.2) holds by (ii) of Theorem ZF.

Now, we assume that the inequality (4.2) is valid when $n=m$, that is,

$$A\nabla_\mu B - A\sharp_\mu B - 2\Sigma_{i=1}^m r_i(A\nabla B - A\sharp B) \leqslant h^{1-\frac{1}{2^m}} L(\sqrt[2^{m+1}]{h},1)\log S(\sqrt[2^{m+1}]{h})A. \quad (4.3)$$

Our task is to prove (4.2) for $n=m+1$, using the similar way to (4.3) as in Lemma 4.1. We distinguish the following two cases:

(I) If $0\leqslant\mu\leqslant\frac{1}{2}$, then $0\leqslant 2\mu\leqslant 1$. Since $0<\sqrt{1/h}A\leqslant B\leqslant\sqrt{h}A$ admits that $0\leqslant\frac{1}{\sqrt[4]{h}}A\leqslant A\sharp B\leqslant\sqrt[4]{h}A$. Replace B by $A\sharp B$ and μ by 2μ in (3.11), respectively, it follows that

$$A\nabla_\mu B - A\sharp_\mu B - 2(\mu+\Sigma_{i=1}^m r_i')(A\nabla B - A\sharp B) \leqslant h^{\frac{1}{2}(1-\frac{1}{2^m})} L(\sqrt[2^{m+2}]{h},1)\log S(\sqrt[2^{m+2}]{h})A, \quad (4.4)$$

where $r_1'=\min\{2\mu,1-2\mu\}\doteq\min\{\alpha_1',1-\alpha_1'\}$, $r_2'=\min\{|1-4\mu|,1-|1-4\mu|\}\doteq\min\{\alpha_2',1-\alpha_2'\}$, $r_{i+1}'=\min\{|1-2\alpha_i'|,1-|1-2\alpha_i'|\}$, $i=1,2,\cdots,m-1$.

(II) If $\frac{1}{2}\leqslant\mu\leqslant 1$, then $0\leqslant 1-\mu\leqslant\frac{1}{2}$. Since $0<\sqrt{1/h}A\leqslant B\leqslant\sqrt{h}A$ ensures that $0<\sqrt{1/h}B\leqslant A\leqslant\sqrt{h}B$, then by the inequality (4.3) we have

$$A\nabla_\mu B - A\sharp_\mu B - 2[(1-\mu)+\Sigma_{i=1}^m r_i''](A\nabla B - A\sharp B) \leqslant L(\sqrt[2^{m+2}]{h},1)\log S(\sqrt[2^{m+2}]{h})h^{\frac{1}{2}}A.$$

where $r_1''=\min\{2(1-\mu),1-2(1-\mu)\}\doteq\min\{\alpha_1'',1-\alpha_1''\}$, $r_2''=\min\{|4\mu-3|,1-|4\mu-3|\}\doteq\min\{\alpha_2'',1-\alpha_2''\}$, $r_{i+1}''=\min\{|1-2\alpha_i''|,1-|1-2\alpha_i''|\}$, $i=1,2,\cdots,m-1$.

Combining (I) with (II) for $0\leqslant\mu\leqslant 1$ we have

$$A\nabla_\mu B - 2(r_1+\Sigma_{i=1}^m \min\{r_i',r_i''\})(A\nabla B - A\sharp B) \leqslant h^{1-\frac{1}{2^{m+1}}} L(\sqrt[2^{m+2}]{h},1)\log S(\sqrt[2^{m+2}]{h})A,$$

that is,

$$A\nabla_\mu B - 2\Sigma_{i=1}^{m+1} r_i(A\nabla B - A\sharp B) \leqslant h^{1-\frac{1}{2^{m+1}}} L(\sqrt[2^{m+2}]{h},1)\log S(\sqrt[2^{m+2}]{h})A,$$

where $r_1=\min\{\mu,1-\mu\}\doteq\min\{\alpha_1,1-\alpha_1\}$, $r_2=\min\{|1-2\mu|,1-|1-2\mu|\}\doteq\min\{\alpha_2,1-\alpha_2\}$, $r_{i+1}=\min\{|1-2\alpha_i|,1-|1-2\alpha_i|\}$, $i=1,2,\cdots,m$.

This completes the proof. □

REMARK 4.3. We have tried to show the analogous result under the condition of $0<aA\leqslant B\leqslant bA$ with $a<1<b$, but the final result is so complicated that two different inequalities were obtained according to parity of n. As a consequence, we obtain the above simplified elegant form (4.2) under the hypothesis of $0<\sqrt{1/h}A\leqslant B\leqslant\sqrt{h}A$ with $h>1$, which is weaker than the condition of Lemma 4.1.

On the other hand, under the condition of $0<aI_H\leqslant A,B\leqslant bI_H$ with $a<b$ we can easily get the reverse difference type arithmetic-geometric mean inequalities as follows.

THEOREM 4.4. *If $0<aI_H\leqslant A,B\leqslant bI_H$ with $a<b$, and $0\leqslant\mu\leqslant 1$, then*

$$A\nabla_\mu B - A\sharp_\mu B - 2\Sigma_{i=1}^n r_i(A\nabla B - A\sharp B) \leqslant bh^{1-\frac{1}{2^{n-1}}} L(\sqrt[2^n]{h},1)\log S(\sqrt[2^n]{h})I_H, \quad (4.5)$$

where $h=\frac{b}{a}$, $r_1=\min\{\mu,1-\mu\}\doteq\min\{\alpha_1,1-\alpha_1\}$, $r_2=\min\{|1-2\mu|,1-|1-2\mu|\}\doteq\min\{\alpha_2,1-\alpha_2\}$, $r_{i+1}=\min\{|1-2\alpha_i|,1-|1-2\alpha_i|\}$, $i=1,2,\cdots,n-1$.

Proof. When $n=1$, inequality (4.5) holds by (ii) of Theorem F.

Now, we assume that the inequality (4.5) is valid when $n=m$, that is,

$$A\nabla_{\mu}B - A\sharp_{\mu}B - 2\Sigma_{i=1}^{m} r_i(A\nabla B - A\sharp B) \leqslant bh^{1-\frac{1}{2^{m-1}}} L(\sqrt[2^m]{h},1)\log S(\sqrt[2^m]{h}) I_H. \quad (4.6)$$

We only have to prove (4.5) for $n=m+1$, using the similar procedure as in Theorem 4.3 applied to (4.6). We distinguish the following two cases:

(I) If $0\leqslant \mu \leqslant \frac{1}{2}$, then $0\leqslant 2\mu \leqslant 1$. Since $0<aI_H\leqslant A,B\leqslant bI_H$ admits that $0\leqslant \frac{1}{\sqrt{h}}A \leqslant A\sharp B \leqslant \sqrt{h}A$, we replace B by $A\sharp B$ and μ by 2μ in (3.10) when $n=m$, respectively, then

$$A\nabla_{\mu}B - A\sharp_{\mu}B - 2(\mu + \Sigma_{i=1}^{m} r_i')(A\nabla B - A\sharp B) \leqslant bh^{1-\frac{1}{2^m}} L(\sqrt[2^{m+1}]{h},1)\log S(\sqrt[2^{m+1}]{h}) I_H.$$

where $r_1' = \min\{2\mu, 1-2\mu\} \doteq \min\{\alpha_1', 1-\alpha_1'\}$, $r_2' = \min\{|1-4\mu|, 1-|1-4\mu|\} \doteq \min\{\alpha_2', 1-\alpha_2'\}$, $r_{i+1}' = \min\{|1-2\alpha_i'|, 1-|1-2\alpha_i'|\}$, $i=1,2,\cdots,m-1$.

(II) If $\frac{1}{2}\leqslant \mu \leqslant 1$, then $0\leqslant 1-\mu \leqslant \frac{1}{2}$, we have

$$A\nabla_{\mu}B - A\sharp_{\mu}B - 2[(1-\mu) + \Sigma_{i=1}^{m} r_i'')](A\nabla B - A\sharp B) \leqslant bh^{1-\frac{1}{2^m}} L(\sqrt[2^{m+1}]{h},1)\log S(\sqrt[2^{m+1}]{h}) I_H.$$

where $r_1'' = \min\{2(1-\mu), 1-2(1-\mu)\} \doteq \min\{\alpha_1'', 1-\alpha_1''\}$, $r_2'' = \min\{|4\mu-3|, 1-|4\mu-3|\} \doteq \min\{\alpha_2'', 1-\alpha_2''\}$, $r_{i+1}'' = \min\{|1-2\alpha_i''|, 1-|1-2\alpha_i''|\}$, $i=1,2,\cdots,m-1$.

Combining (I) with (II), then, for $0\leqslant \mu \leqslant 1$, we have

$$A\nabla_{\mu}B - 2(r_1 + \Sigma_{i=1}^{m} \min\{r_i', r_i''\})(A\nabla B - A\sharp B) \leqslant bh^{1-\frac{1}{2^m}} L(\sqrt[2^{m+1}]{h},1)\log S(\sqrt[2^{m+1}]{h}) I_H,$$

that is,

$$A\nabla_{\mu}B - 2\Sigma_{i=1}^{m+1} r_i(A\nabla B - A\sharp B) \leqslant bh^{1-\frac{1}{2^m}} L(\sqrt[2^{m+1}]{h},1)\log S(\sqrt[2^{m+1}]{h}) I_H,$$

where $r_1 = \min\{\mu, 1-\mu\} \doteq \min\{\alpha_1, 1-\alpha_1\}$, $r_2 = \min\{|1-2\mu|, 1-|1-2\mu|\} \doteq \min\{\alpha_2, 1-\alpha_2\}$, $r_{i+1} = \min\{|1-2\alpha_i|, 1-|1-2\alpha_i|\}$, $i=1,2,\cdots,m$.

This completes the proof. □

COROLLARY 4.5. *If $0<aI_H\leqslant A,B\leqslant bI_H$ with $a<b$, and $0\leqslant \mu\leqslant 1$, then*

$$A\nabla_{\mu}B - A\sharp_{\mu}B - 2(r_1+r_2)(A\nabla B - A\sharp B) \leqslant b\sqrt{h}L(\sqrt[4]{h},1)\log S(\sqrt[4]{h}) I_H,$$

where $r_1 = \min\{\mu, 1-\mu\}$, $r_2 = \min\{|\,1-2\mu\,|, 1-|\,1-2\mu\,|\}$, $h=\frac{b}{a}$.

REFERENCES

[1] J. BARIĆ, M. MATIĆ AND J. PEČARIĆ, *On the bounds for the normalized Jensen functional and Jensen-Steffensen inequality*, Math. Inequal. Appl., **12**, 2(2009), 413–432.

[2] S. FURUICHI, *On refined Young inequalities and reverse inequalities*, J. Math. Inequal., **5**, 1 (2011), 21–31.

[3] O. HIRZALLAH, F. KITTANEH, M. KRIĆ, N. LOVRIČEVIĆ, J. PEČARIĆ, *Refinements and reverses of means inequalities for Hilert space operators*, Banach J. Math. Anal., **7**, 2 (2013), 15–19.
[4] F. KITTANEH, *Reverse Young inequalities for matrices*, Linear Multilinear Algebra, **59**, 9 (2011), 1031–1037.
[5] F. KITTANEH AND Y. MANASRAH, *Improved Young and Heinz inequalities for matrices*, J. Math. Anal. Appl., **36**, 1 (2010), 262–269.
[6] M. KLARIĆ BAKULA, J. PEČARIĆ AND J. PERIĆ, *On the converse Jensen inequality*, Appl. Math. Comput., **218**, (2012), 6566–6575.
[7] M. KRNIĆ, N. LOVRIČEVIĆ, AND J. PEČARIĆ, *Improved arithmetic-geometric and Heinz means inequalities for Hilbert space operators*, Publ. Math. Debrecen, **80**, (2012), 3–4.
[8] M. KRNIĆ, N. LOVRIČEVIĆ AND J. PEČARIĆ, *Jensen's operator and applications to mean inequalities for operators in Hilbert space*, Bull. Malays. Math. Sci. Soc., **35**, 1 (2012), 1–14.
[9] F. KUBO AND T. ANDO, *Means of positive operators*, Math. Ann., **246**, 3 (1980), 205–224.
[10] J. PEČARIĆ AND P. R. BEESACK, *On Jensen's inequality for convex functions* II, J. Math. Anal. Appl., **118**, (1986), 125–144.
[11] W. SPECHT, *Zer Theorie der elementaren Mittel*, Math. Z., **74**, (1960), 91–98.
[12] M. TOMINAGA, *Specht's ratio in the Young inequality*, Sci. Math. Japon., **55**, 3 (2002), 583–588.
[13] H. L. ZUO, M. FUJII, J. I. FUJII AND Y. SEO, *Upper bound for spectra of Jensen operator and its application to reverse arithmetic-geometric means*, Math. Inequal. Appl., **17**, 2 (2014), 641–648.
[14] H. L. ZUO, G. H. SHI AND M. FUJII, *Refined Young inequality with Kantorovich constant*, J. Math. Inequal., **5**, 4 (2011), 551–556.

(Received February 4, 2013)

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