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ON ONE EXTENSION THEOREM DEALING WITH WEIGHTED ORLICZ–SLOBODETSKII SPACE. ANALYSIS ON CUBE

RAJ NARAYAN DHARA AND AGNIESZKA KAŁAMAJSKA

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Abstract. Having given weight $\tilde{\rho} = \rho\,(\mathrm{dist}(x,\partial Q))$ defined on cube Q and Orlicz function R, we construct the weight $\omega_\rho(\cdot,\cdot)$ defined on $\partial Q\times\partial Q$ and extension operator $\mathrm{Ext}^L : Lip_d(\partial Q) \mapsto Lip(Q)$ from Lipschitz functions defined on ∂Q with certain restricted support to Lipschitz functions defined on Q, independent of ρ and R, in such a way that Ext^L extends to the bounded operator from certain subspace of weighted Orlicz-Slobodetskii space $Y^{R,R}_{\omega_\rho}(\partial Q)$ subordinated to the weight ω_ρ to Orlicz Sobolev space $W^{1,R}_\rho(Q)$. Result is new in the unweighted Orlicz setting for general function R as well as in the weighted L^p setting.

1. Introduction

The purpose of this work is to investigate properties of extension operator from weighted Orlicz-Slobodetskii spaces to the weighted Sobolev spaces of first order. Having the given weight $\tilde{\rho}$ defined on cube Q and Orlicz function R, we contribute to show how to extend every Lipschitz function u defined on the boundary of Q to a Lipschitz function $\tilde{u}$ defined on cube Q in such a way that this extension defines bounded operator from certain space Y subordinated to the weight $\tilde{\rho}$ to Orlicz-Sobolev space $W^{1,R}_{\tilde{\rho}}(Q)$.

The admissible space Y is Orlicz-Slobodetskii space $Y^{R,R}_{\omega_\rho}(\partial Q)$ constructed in the following way. When a domain $\Omega \subseteq \mathbf{R}^n$ (sufficiently regular), two Orlicz functions R_1, R and the weight ω defined on $\partial\Omega\times\partial\Omega$ are fixed, by $Y^{R,R_1}_{\omega}(\partial\Omega)$ we denote the space of all $u \in L^R(\partial\Omega)$, for which the modular quantity

$$I^{R_1}_{\omega}(u,\partial\Omega) := \int_{\partial\Omega}\int_{\partial\Omega} R_1\left(\frac{|u(x)-u(y)|}{|x-y|}\right)\frac{\omega(x,y)}{|x-y|^{n-2}}\,d\sigma(x)d\sigma(y)$$

(where σ is the $n-1$ dimensional Hausdorff measure on $\partial\Omega$) is finite. We equip it with the norm

$$\|u\|_{Y^{R,R_1}_{\omega}(\partial\Omega)} := \|u\|_{L^R(\partial\Omega)} + J^{R_1}_{\omega}(u,\partial\Omega),$$

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involving Luxemburg-type seminorm

$$J_{\omega}^{R_1}(u,\partial\Omega) := \inf\left\{\lambda > 0 : I_{\omega}^{R_1}\left(\frac{u}{\lambda},\partial\Omega\right) \leqslant 1\right\}.$$

For our next discussion if the weights are omitted in the notation, they are mend to be equal to 1 identically.

However in the unweighted L^p-setting the problem of extension and trace operator has been completely solved in the late 50's of the last century (see papers by Aronszajn [2], Slobodetskii [43], Gagliardo [13], see also: Nikolski [39], Lizorkin [31] for rudiments of weighted setting), many important problems are left open when one slightly generalizes setting. Perhaps the crucial ones are the following:

(a) *The problem of trace operator between Orlicz spaces under general growth restriction in the unweighted setting.*
This problem has been undertaken by Nečas ([35], Chapter II, Section 4.3), Fougéres [11, 12] and Lacroix [28] in the 60's and 70's of the last century. It was shown that operator $u \mapsto u|_{\partial\Omega}$ defined on $C^1(\bar{\Omega})$ extends to bounded operator "Tr$''$ from the space $W^{1,R}(\Omega)$ to $Y^{R,R}(\partial\Omega)$, provided that R^* - the Legendre conjugate function to R satisfies the Δ_2-condition. Moreover, in that case the operator $\mathrm{Tr} : W^{1,R}(\Omega) \to Y^{R,R}(\partial\Omega)$ is a surjection. Then the question is what happens if one relaxes the assumption that R^* satisfies the Δ_2-condition. In the recent paper [17] by Miroslav Krbec and second author it is shown that in general case we have embedding $\mathrm{Tr} : W^{1,R}(\Omega) \to Y^{R,R_1}(\partial\Omega)$ where the pair of Orlicz functions (R_1,R) is a Kita pair (see Section 2.3). It is known that we always have $R_1 \prec R$ and moreover, $R_1 \sim R$ if and only if the Legendre conjugate to R satisfies the Δ_2 condition (see Remark 2.6). It is not known if trace operator obtained in the paper [17] (which relates to the general situation) is a surjection.

(b) *The problem of extension operator in the unweighted Orlicz setting.*
It is clear from description of Problem (a) that the following problem arises in connection with the question about surjectivity of operator "Tr". Having pair of Orlicz functions (R_1,R) and function $u \in Y^{R,R_1}(\partial\Omega)$, can we extend it to a function $\tilde{u} \in W^{1,R}(\Omega)$ defined in the whole of Ω in such a way that $\tilde{u}|_{\partial\Omega} = u$ (in some sense which we do not explain in details here)? This is possible when $R_1 = R$ and R^* satisfies the Δ_2-condition, ([35], Chapter II, Section 4.3,[11, 12, 28]), but in general case answer is unknown. Partial contribution, when we look for an extension within the same Orlicz space R (i.e. $R_1 = R$), under some special assumptions (Assumption B) which make R to behave like logarithmic $LLogL$-space, can be found in [18], Theorem 5.1. This approach is based on the study of regularity for solutions to the heat equation

$$\begin{cases} \tilde{u}_t(x,t) = \Delta_x \tilde{u}(x,t) & \text{in } \Omega \times (0,T), \\ \tilde{u}(x,0) = u \in Y^{R,R}(\partial\Omega) & \text{for } x \in \Omega, \end{cases}$$

when Ω is a bounded domain with the sufficiently regular boundary.

(c) *Trace and extension operator in the weighted L^p-setting.*
We found very few sources for trace embedding into weighted Orlicz Slobodetskii-type spaces. One of them is paper by Lacroix [29]. The target space there is defined in a very abstract way and it is not possible to recognize it in practice. Another source is interesting paper by Kokilashvili [24], where the author gives necessary and sufficient conditions for a function given on the boundary of a domain to be the trace of some function with first order partial derivatives in weighted Orlicz-Sobolev space. Result given there seems to be quite general. However, analysis there is restricted to the class of measures satisfying certain conditions, being in general rather hard to be verified in practice and the statements do not involve the definition of Orlicz-Slobodetskii space, directly.
Some authors investigate weighted Sobolev spaces $W^{1,p}_\rho(\Omega)$ and ask for trace and embedding theorems for that spaces [27, 35, 45]. Perhaps rudiments of trace embedding and extension theorems in the weighted L^p-setting can be found in the paper by Nikolskii [39] (written in 1953, before fundamental paper by Slobodetskii [43] obtained in 1958), which dealt with power measure $\mathrm{dist}(x,\partial\Omega)^\alpha$ and $p=2$, in the form not involving Slobodetskii type spaces directly. Extensions within that class of measures can be found in works by Lizorkin [31], Vasarin [44], Portnov [41] Kudryavcev [25] (Section 9), Nečas [36], Nekvinda and Pick [37], Nekvinda [38] and Kim [21]. See also our Example 7.1, part (b).
Generalization of embedding and extension theorems to weighted Sobolev type spaces defined on interval by means of semigroups of operators (weighted B-spaces) can be found in paper by Lions [30].

We also mention few interesting related sources dealing with trace embedding theorems: [7] for embeddings from Orlicz-Sobolev spaces into Orlicz spaces (unweighted, see also earlier related result [8]), Theorem 9.14 in [27] for embedding theorems from weighted Sobolev spaces into weighted L^p-spaces and Theorem 2.2 from page 291 in [35] for embeddings from weighted Sobolev space into unweighted L^p-space defined on the boundary. Those sources do not undertake the problem of embedding into Slobodetskii- type spaces.

Weighted Sobolev Spaces are basic tool to study degenerated PDEs. As one of the several possible examples, let us consider the following boundary value problem of elliptic type.

$$\begin{cases} -\mathrm{div}\,(\tilde\rho(x)\nabla u(x)) = f & \text{in } \ \Omega \\ \qquad\qquad u = g & \text{in } \partial\Omega. \end{cases} \tag{1.1}$$

Suppose that g belongs to some function space Y and there exists bounded extension operator: $Y \to W^{1,2}_{\tilde\rho}(\Omega)$. In particular there exists $\Psi_g \in W^{1,2}_{\tilde\rho}(\Omega)$ such that $\Psi_g|_{\partial\Omega} = g$ in some sense which we will explain later. Let us substitute $v := u - \Psi_g$ and denote $W^{1,2}_{\tilde\rho,0}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ in $W^{1,2}_{\tilde\rho}(\Omega)$. Then the problem is equivalent to the following:

$$\begin{cases} P((u-\Psi_g)+\Psi_g) = f & \text{in } \ \Omega \\ \qquad u-\Psi_g = 0 & \text{on } \partial\Omega \end{cases} \iff \begin{cases} Pv = f - P\Psi_g & \text{in } \ \Omega \\ \qquad v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $Pw = -\mathrm{div}(\tilde{\rho}\nabla w)$. Let us consider Hilbert space $H = W^{1,2}_{\tilde{\rho},0}(\Omega)$ and assume that $f \in H^*$. Simple observation gives $P\Psi_g \in H^*$, so that last equation translates as

$$\begin{cases} Pv = F \text{ in } \Omega \\ \quad v \quad \in H, \end{cases} \tag{1.3}$$

where $F = f - P\Psi_g \in H^*$. With suitable assumptions on the admitted weight $\tilde{\rho}$ one can easily prove existence of last equation by Lax Milgram theorem. In particular we also have the solution of (1.1) and boundary data interprets as $u - \Psi_g \in W^{1,2}_{\tilde{\rho},0}(\Omega)$.

This way tools used to prove existence for homogeneous boundary data combined with the right extension theorems can be used to prove existence results for boundary value problems with the nonhomogeneous boundary data.

In some other cases we could also immediately deduce nonexistence for (1.1), having the trace embedding theorem $\mathrm{Tr}: W^{1,2}_{\tilde{\rho}}(\Omega) \to Y$ and knowing that $g \notin Y$.

For some other example motivations to consider weighted Sobolev spaces we refer to books: [5, 27, 35], papers [22, 23, 32, 34, 40] and to their references. For motivations to consider Orlicz-Sobolev spaces we refer e.g. to [1, 3, 6, 10, 14, 15].

We are interested in the following problem. Having the given weight $\tilde{\rho}$ defined on Ω, construct another weight $\omega_{\tilde{\rho}}(\cdot,\cdot)$ such that every Lipschitz function determined on $\partial\Omega$ can be extended to a Lipschitz function determined on Ω in such a way that this extension defines bounded operator from weighted Orlicz-Slobodetskii type space $Y = Y^{R,R}_{\omega_{\tilde{\rho}}}(\partial\Omega)$ subordinated to the weight $\omega_{\tilde{\rho}}$ to the space $W^{1,R}_{\tilde{\rho}}(\Omega)$. Here we approach this question under certain additional assumptions. For this, we analyze the special case when Ω is a cube $(-\frac{1}{2},\frac{1}{2})^n$ and function u is supported in its bottom wall $(-\frac{1}{2},\frac{1}{2})^{n-1} \times \{-\frac{1}{2}\}$. Moreover, we consider $\tilde{\rho}(x) = \rho(\mathrm{dist}(x,\partial\Omega))$. The weight $\omega = \omega_\rho(x,y)$ is ceratin transform of weight ρ and does not depend on the choice of Orlicz space R (see formulae (4.1)). When ρ is nonincreasing, integrable and satisfies the $\Delta_{\frac{1}{2}}$-condition: $\rho(\frac{1}{2}t) \leqslant C\rho(t)$, we have $\omega_\rho(x,y) \sim \rho(|x-y|)$ (see Theorem 7.2). Main tools we use are convolution techniques. The technique to use convolution for extension was used earlier by other authors also (see e.g. [26]), but our approach requires careful analysis. In particular it is important to recognize that the convolution operator satisfies certain first order PDE (see (3.3)) and also certain pointwise estimates for convolution obtained in Section 5.2 inspired by similar techniques from [18].

Let us mention that the result seems to be new for the weighted approach in the homogeneous case $R(\lambda) = \lambda^p$. It is also new when R is a general Orlicz function (without any additional assumptions) in the unweighted setting. For last issue it partially contributes to the open question related to trace operator described in discussion of problem (a) and we gave partial answer on problems (b) and (c).

Generalization dealing with general Lipschitz boundary domain will be provided in [9].

2. Notation and preliminaries

2.1. Basic notation

Let $\Omega \subset \mathbb{R}^n$ be an open set. By $C^\infty(\bar{\Omega})$ we mean set of functions which have smooth extension to certain open neighborhood of $\bar{\Omega}$. If f is defined on a set $A \subseteq \mathbb{R}^n$, by $f\chi_A$ we mean the function f extended by 0 outside A. Having two functions Φ, Ψ defined on $[0,\infty)$ we will say that Ψ dominates Φ ($\Phi \prec \Psi$) if there exist constants $C_1, C_2 > 0$ such that $\Phi(x) \leqslant C_1\Psi(C_2 x)$ for every $x > 0$. Functions Φ, Ψ are called equivalent if $\Psi \prec \Phi$ and $\Phi \prec \Psi$. The notation "$\lesssim$" will be used in usual manner, namely, if $\Phi, \Psi : \mathscr{A} \to \mathbb{R}$ are given functions, where $\mathscr{A}$ is some abstract domain (it can be either a subset of Euclidean space, as well as a set of functions), we will write that $\Phi \lesssim \Psi$ if there is a constant $C > 0$ such that $\Phi(a) \leqslant C\Psi(a)$, for every $a \in \mathscr{A}$. When $n \in \mathbf{N}$, we denote: $Q' = (-\frac{1}{2}, \frac{1}{2})^{n-1}$, $Q = Q' \times (0,1)$, and $tA := \{tx : x \in A\}$ whenever A is an arbitrary subset of an Euclidean space. In particular $tQ' = (-\frac{t}{2}, \frac{t}{2})^{n-1}$, $tQ = tQ' \times (0,t)$. If X is a subset in an Euclidean space, by $Lip(X)$ we denote Lipschitz functions defined on X, while the notation $Lip_0(X)$ stands for Lipschitz functions with compact support in X. In our notation the symbol $d\sigma(x)$ stands for the $n-1$-dimensional Hausdorff measure. If X is measurable space, by $L^1_+(X)$ we mean all nonnegative, measurable and integrable functions defined on X. Symbol $[a]$ stands for an integer part of real number a.

2.2. Orlicz, Orlicz-Sobolev and Orlicz-Slobodetskii spaces equipped with weights

In the sequel we assume that all weight functions in our considerations on domains of their definition X belong to $L^1_+(X)$. Moreover, all domains considered here are cubes.

2.2.1. Orlicz space

We start with the definition of Orlicz space.

DEFINITION 2.1. The function $\Psi : [0,\infty) \to [0,\infty)$ is called Orlicz function if it is nondecreasing, convex and satisfies conditions: $\Psi(0) = 0$ and $\lim_{t\to\infty} \Psi(t) = +\infty$.

We will write that $\Psi \in \Delta_2$ if it satisfies the Δ_2-condition: $\Psi(2\lambda) \leqslant C\Psi(\lambda)$, for every $\lambda > 0$, with a constant C independent of λ. Symbol $\Psi \in \Delta_2^c$ will mean that the *Legendre conjugate* of Ψ, $\Psi^*(s) := \sup_{t>0}\{st - \Psi(t)\}$, satisfies the Δ_2-condition.

We are now ready to define Orlicz space. Of our interest will be that one defined on domain and on its boundary.

A. Orlicz space on domain.

Let Ψ be an Orlicz function and $\rho : \Omega \to (0,\infty)$ be a given weight function. The space

$$L^\Psi_\rho(\Omega) := \{f \in L^1_{loc}(\Omega) : \int_\Omega \Psi(s|f(x)|)\,\rho(x)\,dx < \infty \text{ for some } s > 0\}$$

is called *weighted Orlicz space* with weight ρ. It is a Banach space with the *Luxemburg norm*:

$$\|f\|_{L^\Psi_\rho(\Omega)} := \inf\left\{\lambda > 0 : \int_\Omega \Psi\left(\frac{|f(x)|}{\lambda}\right)\rho(x)\,dx \leqslant 1\right\}.$$

As is well known, when $\Psi(\lambda) = \lambda^p$ and $p \geqslant 1$, then $L^\Psi_\rho(\Omega) = L^p_\rho(\Omega)$. See e.g. [42].

B. Orlicz space on the boundary of domain.

Similarly, we define the weighted Orlicz space on the boundary of the domain:

$$L^\Psi_\tau(\partial\Omega) := \{f \in L^1_{loc}(\partial\Omega) : \int_{\partial\Omega} \Psi(s|f(x)|)\,\tau(x)\,d\sigma(x) < \infty \text{ for some } s > 0\},$$

with the norm:

$$\|f\|_{L^\Psi_\tau(\partial\Omega)} := \inf\left\{\lambda > 0 : \int_{\partial\Omega} \Psi\left(\frac{|f(x)|}{\lambda}\right)\tau(x)\,d\sigma(x) \leqslant 1\right\},$$

where $\tau : \partial\Omega \to (0,\infty)$ is a given weight function defined on the boundary of Ω.

The same notation will be used for vector functions, $u : \Omega \to \mathbb{R}^m$, with the formal difference that instead of $|u(x)|$ we shall work with the Euclidean norm of the vector $u(x)$.

We will be using the following statement (see e.g. [4], Proposition 2).

PROPOSITION 2.2. *Let M be a Young function and (X,μ) be the measurable space equipped with the measure μ. Then the expression*

$$\|f\|_{L^\Psi(X,\mu),\alpha} := \inf\left\{\lambda > 0 : \int_X \Psi\left(\frac{|f(x)|}{\lambda}\right)\mu(dx) \leqslant \alpha\right\}.$$

defines a complete norm on

$$L^\Psi(X,\mu) := \{f \in L^1_{loc}(X) : \int_\Omega \Psi(s|f(x)|)\,\mu(dx) < \infty \text{ for some } s > 0\}$$

for each $\alpha \in (0,\infty)$. Moreover, all norms $\|\cdot\|_{L^\Psi(X,\mu),\alpha}$, $\alpha \in (0,\infty)$ are equivalent.

2.2.2. Orlicz-Sobolev space

Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded domain, $k \in \mathbb{N}$, and $\Psi : [0,\infty) \to [0,\infty)$ be a given Orlicz function. The weighted *Orlicz-Sobolev space* with weight ρ, $W^{k,\Psi}_\rho(\Omega)$ is the linear set

$$\{u \in L^1_{loc}(\Omega) : D^\alpha u \in L^\Psi_\rho(\Omega) \text{ for every } \alpha : |\alpha| \leqslant k\} \tag{2.1}$$

equipped with the norm

$$\|u\|_{W^{k,\Psi}_\rho(\Omega)} := \sum_{\alpha:|\alpha|\leqslant k} \|D^\alpha u\|_{L^\Psi_\rho(\Omega)}.$$

Here $D^\alpha u$ means the distributional derivative of u. We will be dealing with $k = 1$. For more information we refer e.g. [6].

Symbol $W^{1,\Phi}_{\rho,L}(\Omega)$ will denote the completion of Lipschitz functions in the norm of the space $W^{1,\Phi}_\rho(\Omega)$.

2.2.3. Orlicz-Slobodetskii space $Y^{\Psi,\Phi}$

A. Orlicz-Slobodetskii space in domain.

Let $\omega \in L^1(\Omega \times \Omega)$ be the given weight (in particular ω is non-negative a.e.). Moreover, let Ψ and Φ be the given two Orlicz functions. By $Y_\omega^{\Psi,\Phi}(\Omega)$ we denote the space of all $u \in L^\Psi(\Omega)$, for which the quantity

$$I_\omega^\Phi(u,\Omega) := \int_\Omega \int_\Omega \Phi\left(\frac{|u(x)-u(y)|}{|x-y|}\right) \frac{\omega(x,y)}{|x-y|^{n-1}}\, dxdy \tag{2.2}$$

is finite. We equip it with the norm

$$\|u\|_{Y_\omega^{\Psi,\Phi}(\Omega)} := \|u\|_{L^\Psi(\Omega)} + J_\omega^\Phi(u,\Omega),$$

involving Luxemburg-type seminorm

$$J_\omega^\Phi(u,\Omega) := \inf\left\{\lambda > 0 : I_\omega^\Phi\left(\frac{u}{\lambda},\Omega\right) \leqslant 1\right\}.$$

B. Orlicz-Slobodetskii space on the boundary of domain.

The same type of space can be defined on the boundary of Ω, with a given weight $\omega(x,y) \in L^1(\partial\Omega \times \partial\Omega)$. Namely, when

$$I_\omega^\Phi(u,\partial\Omega) := \int_{\partial\Omega} \int_{\partial\Omega} \Phi\left(\frac{|u(x)-u(y)|}{|x-y|}\right) \frac{\omega(x,y)}{|x-y|^{n-2}}\, d\sigma(x) d\sigma(y),$$

we define the space

$$Y_\omega^{\Psi,\Phi}(\partial\Omega) := \left\{u \in L^\Psi(\partial\Omega) : \text{there exists } s > 0; I_\omega^\Phi(su,\partial\Omega) < \infty\right\}$$

equipped with the norm

$$\|u\|_{Y_\omega^{\Psi,\Psi}(\partial\Omega)} := \|u\|_{L^\Psi(\partial\Omega)} + J_\omega^\Phi(u,\partial\Omega),$$

where

$$J_\omega^\Phi(u,\partial\Omega) := \inf\left\{\lambda > 0 : I_\omega^\Phi\left(\frac{u}{\lambda},\partial\Omega\right) \leqslant 1\right\}.$$

In the similar way as before we define spaces: $Y_{\omega,L}^{\Psi,\Phi}(\Omega)$, $Y_{\rho,L}^{\Psi,\Phi}(\partial\Omega)$ as the completion of Lipschitz functions in the space $Y_\omega^{\Psi,\Phi}(\Omega)$ and $Y_\rho^{\Psi,\Phi}(\partial\Omega)$, respectively.

REMARK 2.3. If $\omega \equiv 1$ and $\Psi(\lambda) = \Phi(\lambda) = |\lambda|^p$, $1 < p < \infty$, then we have

$$\|u\|_{Y^{\Psi,\Phi}(\partial\Omega)} \sim \|u\|_{L^p(\partial\Omega)} + \left(\int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{p+n-2}}\, d\sigma(x)\, d\sigma(y)\right)^{1/p},$$

which is the norm of u in the Slobodetskii space $W^{1-\frac{1}{p},p}(\partial\Omega)$, see e.g. [26].

2.3. Embedding theorem (unweighted case)

We will use the following assumptions.

ASSUMPTION A. (Kita pair, [20]) We assume that $a,b:[0,\infty)\to[0,\infty)$ are strictly positive continuous functions such that

(a) $\int_0^1 a(s)/s\,ds<\infty$, $\int_1^\infty \frac{a(s)}{s}\,ds=+\infty$;

(b) $b(\cdot)$ is non-decreasing, $\lim_{s\to\infty} b(s)=+\infty$.

(c) there exist constants $c_1>0, s_0\geqslant 0$ such that

$$\int_0^s \frac{a(t)}{t}\,dt\leqslant c_1 b(c_1 s)\ \text{ for all } s>s_0, \tag{2.3}$$

and in the case $s_0>0$ mapping $s\mapsto \frac{a(s)}{s}$ is bounded when $s\neq 0$ is close to 0.

We define

$$\Phi(t):=\int_0^t a(s)\,ds \quad\text{and}\quad \Psi(t):=\int_0^t b(s)\,ds,\ \text{where } t\geqslant 0. \tag{2.4}$$

Operator of trace.
Let us recall the concept of the trace of a function.

Suppose that for given Orlicz-functions Φ and Ψ there is an inequality:

$$\|u\|_{Y^{\Psi,\Phi}(\partial\Omega)}\leqslant D\|u\|_{W^{1,\Psi}(\Omega)},$$

satisfied for every Lipschitz function u defined on $\bar{\Omega}$. Let $u\in W_L^{1,\Psi}(\Omega)$ and consider any sequence of Lipschitz functions u_m converging to u in the norm of $W^{1,\Psi}(\Omega)$. Then $\{u_m\}$ is a Cauchy sequence in $Y^{\Psi,\Phi}(\partial\Omega)$ (norm convergence) so that it converges to some element $\widehat{u}\in Y_L^{\Psi,\Phi}(\partial\Omega)$. It is easy to check that $\widehat{u}$ is independent of the choice of Lipschitz sequence $\{u_m\}$, converging to u. It allows to extend the standard definition of the trace operator:

$$\mathrm{Tr}\,u:=\lim_{m\to\infty} u_m=\widehat{u}\in Y_L^{\Psi,\Phi}(\partial\Omega). \tag{2.5}$$

REMARK 2.4. In the same way we can define the trace operator in weighted case

$$\mathrm{Tr}: W_{\rho,L}^{1,\Psi}(\Omega)\longmapsto Y_{\omega,L}^{\Phi,\Psi}(\partial\Omega),$$

if we only have the inequality

$$\|u\|_{Y_\omega^{\Psi,\Phi}(\partial\Omega)}\leqslant D\|u\|_{W_\rho^{1,\Psi}(\Omega)},$$

holding within Lipschitz functions. In that case, when sequence of Lipschitz functions $\{u_m\}$ converges to u in $W_\rho^{1,\Psi}(\Omega)$, then sequence of restrictions $\{u_m|_{\partial\Omega}\}$ converges to some $\widehat{u}$ in $Y_\omega^{\Psi,\Phi}(\partial\Omega)$, and we have

$$\mathrm{Tr}\,u:=\lim_{m\to\infty} u_m=\widehat{u}\in Y_{\omega,L}^{\Psi,\Phi}(\partial\Omega). \tag{2.6}$$

The following theorem was obtained in [17].

THEOREM 2.5. (embedding theorem) *Let the N-functions Φ and Ψ satisfy the Assumption A and Ω be a bounded domain of class $\mathscr{C}^{0,1}$. Then we have:*

(i) There is an inequality:

$$\|u\|_{Y^{\Psi,\Phi}(\partial\Omega)} \leqslant D\|u\|_{W^{1,\Psi}(\Omega)}, \tag{2.7}$$

with D independent of u-an arbitrary Lipschitz function defined on $\bar{\Omega}$;

(ii) The trace operator $\mathrm{Tr}: W_L^{1,\Psi}(\Omega) \mapsto Y_L^{\Psi,\Phi}(\partial\Omega)$ is well defined by (2.5) and for every $u \in W_L^{1,\Psi}(\Omega)$ we have

$$\|\mathrm{Tr}\, u\|_{Y^{\Psi,\Phi}(\partial\Omega)} \leqslant D\|u\|_{W^{1,\Psi}(\Omega)}, \tag{2.8}$$

where D is the same as in (2.7).

We refer e.g. [26] to definition of $\mathscr{C}^{0,1}$ class.

REMARK 2.6. It is known that Ψ always dominates Φ whenever (Φ,Ψ) is the Kita pair. Moreover, we have $\Psi \sim \Phi$ if and only if either Ψ^* or Φ^* satisfies the Δ_2-condition. Moreover, the conditions: (Ψ^* *satisfies the Δ_2-condition*) and (Φ^* *satisfies the Δ_2-condition*) are equivalent (see e.g. Proposition 5.1 in [19]).

3. Construction of extension operator

Let $\phi \in Lip_0(\mathbb{R}^{n-1})$, $0 \leqslant \phi \leqslant 1$ be the Lipschitz compactly supported function such that $\phi \equiv 1$ in some neighborhood of zero and $\int_{\mathbb{R}^{n-1}} \phi(x)dx = 1$. To have better control on constants we will choose function having the special form:

$$\phi(x_1,\ldots,x_{n-1}) = \psi(x_1)\cdot\ldots\cdot\psi(x_{n-1}), \tag{3.1}$$

where ψ is the Lipschitz one variable even function defined by

$$\psi(t) = \begin{cases} 1 & \text{when } 0 \leqslant t \leqslant \frac{1}{4}, \\ -2t+\frac{3}{2} & \text{when } \frac{1}{4} \leqslant t \leqslant \frac{3}{4}, \\ 0 & \text{when } \quad t > \frac{3}{4}. \end{cases}, \quad \text{for } t \geqslant 0. \tag{3.2}$$

In particular $\operatorname{supp}\phi \subseteq [-\frac{3}{2},\frac{3}{2}]^{n-1} = 3Q'$, $\phi \equiv 1$ on the set $[-\frac{1}{4},\frac{1}{4}]^{n-1} = \frac{1}{2}Q'$. Moreover, let

$$\phi_t(x) = t^{-(n-1)}\phi(\frac{x}{t}).$$

We have the following remark. Its easy proof is left to the reader.

REMARK 3.1. Function ϕ_t satisfies the following properties

1.

$$\int_{\mathbb{R}^{n-1}} \phi_t(x)\,dx = 1 \text{ for every } t > 0,$$

2.

$$\lim_{t\to 0}\int_{\mathbb{R}^{n-1}} R(x)\phi_t(x)\,dx = R(0) \text{ for every } R \in C_0^1(\mathbb{R}^{n-1}),$$

3.

$$\begin{aligned} \frac{\partial}{\partial t}\phi_t(x) &= -\mathrm{div}(g_t(x)), \text{ where} \\ g_t(x) &= \frac{1}{t^{n-1}} g(\frac{x}{t}), \text{ and } g(x) = (\phi(x)x_1,\dots,\phi(x)x_{n-1}), \end{aligned} \tag{3.3}$$

and divergence is taken with respect to x.

Let $u \in Lip_0(Q' \times \{0\})$. At first we will be dealing with the extension of u defined by formulae

$$\widetilde{u}(x,t) = u(\cdot,0) * \phi_t(x) := \begin{cases} \int_{\mathbb{R}^{n-1}} u(y,0)\phi_t(x-y)\,dy & \text{when } t > 0, \\ u(x,0) & \text{when } t = 0. \end{cases} \tag{3.4}$$

It is easy to check that $\widetilde{u} \in Lip(\bar{Q})$.

The key role in our paper will be the estimates for $\widetilde{u}$.

4. Formulation of main results

Let $\rho : [0,1] \to [0,\infty)$ be a given weight function, $\int_0^1 \rho(t)dt < \infty$ and let us define the following transforms (global and local) of the weight ρ, defined on $\mathbb{R}^{n-1}$:

$$\begin{aligned} \omega_\rho(z) &:= |z|^{n-1}\int_0^1 \frac{1}{t^n}\chi_{\{\frac{z}{t}\in(-\frac{3}{4},\frac{3}{4})^{n-1}\}}\rho(t)\,dt, \quad z \in \mathbf{R}^{n-1}, \\ \omega_{\rho,\kappa}(z) &:= |z|^{n-1}\int_0^\kappa \frac{1}{t^n}\chi_{\{\frac{z}{t}\in(-\frac{3}{4},\frac{3}{4})^{n-1}\}}\rho(t)\,dt,\ \kappa \in (0,1), \quad z \in \mathbf{R}^{n-1}. \end{aligned} \tag{4.1}$$

We will deal with weighted Sobolev space $W^{1,R}_{\tilde{\rho}}(Q)$ where $\tilde{\rho}(x',t) = \rho(t)$.

Our first result describes properties of a convolution operator.

THEOREM 4.1. *Let R be the given convex function, $n \geqslant 2$, $Q = Q' \times (0,1)$, $Q' = (-\frac{1}{2},\frac{1}{2})^{n-1}$, $v : Q' \to \mathbf{R}$ be Lipschitz and compactly supported in $(1-d)Q'$, where $d \in (0,1)$ and*

$$\widetilde{v}(x,t) = v * \phi_t(x) := \begin{cases} \int_{\mathbb{R}^{n-1}} v(y)\phi_t(x-y)\,dy & \text{when } t > 0, \\ v(x) & \text{when } t = 0. \end{cases} \tag{4.2}$$

Moreover, let $\rho : [0,1] \to [0,\infty)$ be a given weight function, $\int_0^1 \rho(t)dt = C(\rho) < \infty$ and $\tilde{\rho}(x',t) = \rho(t)$. Then we have:

(i)

$$\int_{Q'}\int_0^1 R\left(|\widetilde{v}(x,t)|\right)\rho(t)\,dxdt \leqslant C(\rho)\int_{Q'} R\left(|v(x)|\right)\,dx. \tag{4.3}$$

In particular, when R is an Orlicz function, then there exists a constant $\tilde{B}_1$ independent of u such that

$$\|\widetilde{v}\|_{L^R_{\tilde{\rho}}(Q)} \leqslant \tilde{B}_1\|v\|_{L^R(Q')}.$$

(ii)

$$\int_{Q'}\int_0^1 R(|\nabla\widetilde{v}|)\,\rho(t)\,dtdx \leqslant L\int_{x\in Q'}\int_{y\in Q'} R\left(\frac{I|v(y)-v(x)|}{|x-y|}\right)\frac{\omega_\rho(x-y)}{|x-y|^{n-2}}\,dydx \\ + \frac{C(\rho)}{2}\int_{Q'} R(J|v(x)|)\,dx, \tag{4.4}$$

where $I=\frac{5n}{2}\left(\frac{3}{2}\right)^n\sqrt{\frac{n-1}{3}}$, $J=\left(\frac{3}{2d}\right)^{n-1}\frac{n+7}{2}$, $L=\frac{1}{2}\left(\frac{4}{3}\right)^n\frac{e}{\sqrt{n-1}}$, ω_ρ is defined by (4.1) and $\nabla\widetilde{v}$ denotes full gradient of $\widetilde{v}$.

In particular, when R is an Orlicz function, then there exists constant $\tilde{B}_2$ independent of u such that

$$\|\nabla\widetilde{v}\|_{L^R_{\tilde{\rho}}(Q)} \leqslant \tilde{B}_2\|v\|_{Y^{R,R}_{\omega_\rho}(Q')}.$$

(iii) When R is an Orlicz function, then there exists constant $\tilde{B}_3$ independent of u such that

$$\|\widetilde{v}\|_{W^{1,R}_{\tilde{\rho}}(Q)} \leqslant \tilde{B}_2\|v\|_{Y^{R,R}_{\omega_\rho}(Q')}.$$

As a consequence we obtain the following result which applies to the extension operator.

THEOREM 4.2. *Let R be the given convex function, $Q=Q'\times(0,1)$, $Q'=(-\frac{1}{2},\frac{1}{2})^{n-1}$, $d\in(0,1)$ and let us consider*

(a) the transformation of weights

$$\mathscr{T}_d : L^1_+(0,1)\mapsto L^1_+\left(\{Q'\times\{0\}\}\times\{Q'\times\{0\}\}\right),$$

$$\rho\mapsto\omega_{\rho,\frac{d}{3}}(x'-y') =: \tilde{\omega}^d_\rho(x,y) \overset{(4.1)}{=} |x-y|^{n-1}\int_0^{\frac{d}{3}}\frac{1}{t^n}\chi_{\{\frac{x'-y'}{t}\in(-\frac{3}{4},\frac{3}{4})^{n-1}\}}\rho(t)\,dt,$$

where we use the notation $z=(z',0)\in Q'\times\{0\}$;

(b) the subspace of $Y^{R,R}_{\omega,L}(\partial Q)$ depending on $d\in(0,1)$

$$Y^{R,R}_{\omega,L,d}(\partial Q):=\{v\in Y^{R,R}_{\omega,L}(\partial Q): \mathrm{supp} v\subseteq(1-d)Q'\times\{0\}\}.$$

Then there exists a linear extension operator:

$$\mathrm{Ext}: Y^{R,R}_{\tilde{\omega}^d_\rho,L,d}(Q'\times\{0\})\mapsto W^{1,R}_{\rho(\mathrm{dist}(\cdot,\partial Q)),L}(Q) \tag{4.5}$$

such that for $\tilde{u}:=\mathrm{Ext}(u)$, where u is Lipschitz and we have

$$\begin{aligned}\int_Q R(|\widetilde{u}|)\,\rho(\mathrm{dist}(x,\partial Q))\,dx &\leqslant D(\rho)\int_{\partial Q}R(|u(x)|)\,d\sigma(x),\\ \int_Q R(|\nabla\widetilde{u}|)\,\rho(\mathrm{dist}(x,\partial Q))\,dx &\leqslant \tilde{L}\int_{\partial Q}\int_{\partial Q}R\left(\frac{\tilde{I}|u(x)-u(y)|}{|x-y|}\right)\frac{\tilde{\omega}^d_\rho(x,y)}{|x-y|^{n-2}}\,d\sigma(y)\,d\sigma(x)\\ &\quad+\frac{3D(\rho)}{4}\int_{\partial Q}R(\tilde{J}|u(x)|)\,d\sigma(x),\end{aligned} \tag{4.6}$$

where $\tilde{I}=5n\left(\frac{3}{2}\right)^n\cdot\sqrt{\frac{n-1}{3}}$, $\tilde{J}=\left(\frac{3}{2d}\right)^{n-1}(n+7)$, $D(\rho)=\int_0^{\frac{d}{3}}\rho(t)dt$, $\tilde{L}=\frac{1}{4}\left(\frac{4}{3}\right)^n\frac{e}{\sqrt{n-1}}$.

In particular, when R is an Orlicz function, then there exists a constant $\tilde{B}_3=\tilde{B}_3(n,\rho,d)$ such that

$$\|\mathrm{Ext}(u)\|_{W^{1,R}_{\rho(\mathrm{dist}(\cdot,\partial Q))}(Q)}\leqslant\tilde{B}_3\|u\|_{Y^{R,R}_{\tilde{\omega}^d_\rho}(Q')},$$

for every $u\in Y^{R,R}_{\tilde{\omega}^d_\rho,L}(Q'\times\{0\})$ supported in $(1-d)Q'\times\{0\}$,

REMARK 4.3. We always have $\omega_{\rho,\frac{d}{3}}\leqslant\omega_\rho$. Localization of the weights and more precise statement allows to admit weights ρ which might not be integrable on the whole interval $(0,1)$ but they are integrable on $(0,\frac{d}{3})$.

The remaining part of the paper is devoted to the proof of the above results and discussion.

5. Properties of convolution operator

Analysis in this section is restricted to the case when function ϕ used to construct convolution operator has the special form (3.1). It is clear that the choice of other Lipschitz compactly supported function does not change formulations of our statements qualitatively but only the constants can change. Our choice is dictated by the goal to obtain inequalities with certain control of constants.

Through this section we suppose that assumptions of Theorem 4.1 are satisfied, in particular R is the given convex function, $Q=Q'\times(0,1)$, $Q'=(-\frac{1}{2},\frac{1}{2})^{n-1}$, $v:Q'$ is Lipschitz and compactly supported in $(1-d)Q'$, where $d\in(0,1)$ and

$$\widetilde{v}(x,t)=v*\phi_t(x):=\begin{cases}\int_{\mathbb{R}^{n-1}}v(y)\phi_t(x-y)\,dy & \text{when } t>0,\\ v(x) & \text{when } t=0.\end{cases}$$

Moreover, we have given a weight function $\rho : [0,1] \to [0,\infty)$, $\int_0^1 \rho(t)dt < \infty$ and $\tilde{\rho}(x',t) = \rho(t)$.

5.1. Integral estimates for function $\widetilde{v}$

In this subsection we give the proof of Theorem 4.1, part (i).

Proof of Theorem 4.1, part (i). We have

$$\mathscr{I} := \int_{Q'} \int_0^1 R\left(|\widetilde{v}(x',t)|\right) \rho(t)\,dt = \int_{Q'} \int_0^1 R\left((\phi_t * v)(x')\right) \rho(t)\,dt.$$

Applying Jensen's inequality (as $\phi_t(x'-y')dy' = p(dy')$ is a probability measure, therefore $R\left(\int_{\mathbb{R}^{n-1}} \phi_t(x'-y')v(y')dy'\right) \leqslant \int_{\mathbb{R}^{n-1}} R(v(y'))\,p(dy') = (R(v)*\phi_t)(x')$), we get:

$$\begin{aligned}
\mathscr{I} &\leqslant \int_{Q'} \int_0^1 (\phi_t * R(v))(x')\ \rho(t)\,dt\,dx' \\
&= \int_{x'\in Q'} \int_{t\in(0,1)} \int_{y'\in\mathbb{R}^{n-1}} \frac{1}{t^{n-1}} \phi\left(\frac{x'-y'}{t}\right) R(v(y'))\ \rho(t)\ dy'dtdx' \\
&\leqslant \int_{y'\in Q'} \left(\int_0^1 \left\{\int_{x'\in\mathbb{R}^{n-1}} \frac{1}{t^{n-1}} \phi\left(\frac{x'-y'}{t}\right) dx'\right\} \rho(t)dt\right) R(v(y'))\ dy',
\end{aligned}$$

because v is supported in Q'. This implies the thesis as integral in brackets $\{\cdot\}$ equals 1 and $\int_0^1 \rho(t)dt = C(\rho)$.

Last part of the statement follows in the rather standard way, but for reader's convenience we submit the proof. We take $\lambda = \|v\|_{L^R(Q')}$, $\varepsilon > 0$ and substitute the function $v_1 := \frac{v}{\lambda+\varepsilon}$ to the just derived inequality (4.3) getting:

$$\int_{Q'} \int_0^1 R\left(\frac{|\widetilde{v}(x,t)|}{\lambda+\varepsilon}\right) \rho(t)\,dxdt \leqslant C(\rho) \int_{Q'} R\left(\frac{|v(x)|}{\lambda+\varepsilon}\right) dx \leqslant C(\rho).$$

Consequently, using the notation in Proposition 2.2, when $\tilde{\rho}(x',t) = \rho(t)$ and $\mu(dx) = \tilde{\rho}(x)dx$, $x = (x',t)$, and $\alpha = C(\rho)$ we get

$$\|\frac{\tilde{v}}{\lambda+\varepsilon}\|_{L^R(Q,\mu),\alpha} \leqslant 1.$$

Therefore by Proposition 2.2 there exists constant $\tilde{B}_1 > 0$ such that $\|\frac{\tilde{v}}{\lambda+\varepsilon}\|_{L^R_{\tilde{\rho}}(Q)} \leqslant \tilde{B}_1$. Last condition is equivalent to the fact that $\|\tilde{v}\|_{L^R_{\tilde{\rho}}(Q)} \leqslant \tilde{B}_1(\lambda+\varepsilon)$, which after letting $\varepsilon \to 0$ gives the result. $\square$

EXAMPLE 5.1. When $\rho(t) = t^\alpha, \alpha > -1$, we obtain

$$\int_{Q'} \int_0^1 R(|\widetilde{v}(x,t)|)\,t^\alpha\,dxdt \leqslant \frac{1}{\alpha+1} \int_{Q'} R(|v(x)|)\,dx.$$

As an immediate corollary we obtain the following statement which applies to the unweighted case.

COROLLARY 5.2. *Let R be any given convex function, v be Lipschitz function supported in Q' and $\widetilde{u}$ be defined by expression (3.4). Then we have*

$$\int_{Q'}\int_{(0,1)} R(|\widetilde{v}(x)|)\,dxdt \leqslant \int_{Q'} R(|v(x)|)\,dx.$$

In particular, when R is an Orlicz function, we have

$$\|\widetilde{v}\|_{L^R(Q)} \leqslant \|v\|_{L^R(Q')}.$$

5.2. Pointwise estimates for derivatives of $\widetilde{v}$

In this section we obtain pointwise estimates for all derivatives of $\tilde{v}$. They were inspired by similar type of estimates from [17] and seem to be of separate interest. We start with the following simple lemma which will be used in the sequel.

LEMMA 5.3. *Let $n \geqslant 2$, $d \in (0,1)$, $\alpha > 0$, $w(z,t) = \frac{1}{t^\alpha}\widetilde{w}(\frac{z}{t})$; $z \in Q' = (-\frac{1}{2},\frac{1}{2})^{n-1}$ and $\widetilde{w}$ be bounded and supported in $\frac{3}{2}Q'$. Let $x = (x_1,x') \in \mathbb{R}\times\mathbb{R}^{n-2}$ and consider the following expressions*

$$I_+(x,t) = \int_{y'\in[-\frac{1}{2},\frac{1}{2}]^{n-2}} w\left(x_1+\frac{1}{2},x'-y',t\right)dy',$$
$$I_-(x,t) = \int_{y'\in[-\frac{1}{2},\frac{1}{2}]^{n-2}} w\left(x_1-\frac{1}{2},x'-y',t\right)dy',$$

for $n > 2$ and

$$I_+(x,t) = w\left(x+\frac{1}{2},t\right),\ I_-(x,t) = w\left(x-\frac{1}{2},t\right),$$

for $n = 2$. Then for any $x \in (1-d)Q'$, and any $t > 0$, we have

$$|I_+(x,t)|,\ |I_-(x,t)| \leqslant \left(\frac{3}{2d}\right)^\alpha \|\tilde{w}\|_\infty. \tag{5.1}$$

Proof. We prove estimation for I_+, because for I_- the arguments are almost the same. When

$$\left|\frac{x_1+\frac{1}{2}}{t}\right| > \frac{3}{4}, \tag{5.2}$$

we have $\frac{(x_1+\frac{1}{2},\cdot)}{t} \notin \frac{3}{2}Q'$. Thus, as $\widetilde{w}(\cdot)$ is supported in $\frac{3}{2}Q'$, we have $w((x_1+\frac{1}{2},\cdot),t) = 0$ and so $I_+(x,t) = 0$. Moreover, when $x \in (1-d)Q'$, then $x_1 \in \left(-\frac{1}{2}(1-d),\frac{1}{2}(1-d)\right)$

and $x_1+\frac{1}{2}\in\left(\frac{d}{2},1-\frac{d}{2}\right)$. Consequently, $\left|x_1+\frac{1}{2}\right|>\frac{d}{2}$. When $d>\frac{3}{2}t$ condition (5.2) is satisfied. Therefore in case $n>2$,

$$
\begin{aligned}
|I_+(x,t)| &\leqslant |I_+(x,t)|\chi_{\{t\geqslant\frac{2}{3}d\}}\leqslant\int_{y'\in[-\frac{1}{2},\frac{1}{2}]^{n-2}}\frac{1}{t^\alpha}\widetilde{w}\left(\frac{x_1+\frac{1}{2}}{t},\frac{x'-y'}{t}\right)dy'\cdot\chi_{\{t\geqslant\frac{2}{3}d\}}\\
&\leqslant\frac{1}{\left(\frac{2}{3}d\right)^\alpha}\|\widetilde{w}\|_\infty,
\end{aligned}
$$

for $\alpha>0$. In case of $n=2$ we also have $|I_+(x,t)|\leqslant\frac{1}{\left(\frac{2}{3}d\right)^\alpha}\|\widetilde{w}\|_\infty$, by simplification of the above arguments. □

Our next result describes the pointwise estimation of the spatial gradient of $\tilde{v}$.

LEMMA 5.4. *Let $n\geqslant 2$,*

$$
P_k(z,t)=\frac{1}{t^{n-1}}\cdot\frac{|z|}{t}\cdot\chi_{\{\frac{z}{t}\in\frac{3}{2}Q'\}}\cdot\chi_{\{|\frac{z_k}{t}|\geqslant\frac{1}{4}\}} \tag{5.3}
$$

Then for any $k\in\{1,\dots,n-1\}$, v being Lipschitz and compactly supported in $(1-d)Q'$, where $d\in(0,1)$, we have

$$
\left|\frac{\partial\widetilde{v}}{\partial x_k}(x,t)\right|\leqslant 2\int_{Q'}P_k(x-y,t)\left(\frac{|v(x)-v(y)|}{|x-y|}\right)dy+C|v(x)|,
$$

where $C=\left(\frac{3}{2d}\right)^{n-1}$.

Proof. Let

$$
\begin{aligned}
\phi_{k,t}(z) &:= \frac{\partial}{\partial z_k}\left(\frac{1}{t^{n-1}}\phi\left(\frac{z}{t}\right)\right)=\frac{1}{t}\left(\frac{1}{t^{n-1}}\frac{\partial\phi}{\partial z_k}\left(\frac{z}{t}\right)\right)\\
&= \frac{1}{t^n}\psi\left(\frac{z_1}{t}\right)\cdot\ldots\cdot\left(2\cdot\chi_{\{|\frac{z_k}{t}|\in(\frac{1}{4},\frac{3}{4})\}}\operatorname{sgn}(z_k)\right)\cdot\ldots\cdot\psi\left(\frac{z_{n-1}}{t}\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{\partial\widetilde{v}}{\partial x_k}(x,t) &= \left(\phi_{k,t}(\cdot)*v\right)(x)=\int_{Q'}\phi_{k,t}(x-y)v(y)\,dy\\
&= \int_{Q'}\phi_{k,t}(x-y)\left(v(y)-v(x)\right)dy+v(x)\int_{Q'}\phi_{k,t}(x-y)\,dy\\
&= \int_{Q'}\phi_{k,t}(x-y)|x-y|\left(\frac{v(y)-v(x)}{|x-y|}\right)dy+v(x)\int_{Q'}\phi_{k,t}(x-y)\,dy\\
&=: A(x,t)+v(x)I_k(x,t).
\end{aligned}
$$

It is clear that $|\phi_{k,t}(z)z|\leqslant 2P_k(z,t)$. Therefore the only nontriviality is to estimate second term above. For simplicity let us assume that $k=1$ as the remaining computations are the same. Moreover, as $v(\cdot)$ is supported in $(1-d)Q'$, it suffices to estimate $I_1(x,t)$

when $x \in (1-d)Q'$. We will show that in such case I_1 is estimated by constant independent on x and t. To verify this, we use Gauss-Ostrogradsky Theorem, to get

$$I_1 = \begin{cases} \int_{\partial Q'} \phi_t(x-y) n_1(y) d\sigma(y), & \text{when } n > 2, \\ -\left[\phi_t\left(x+\frac{1}{2}\right) - \phi_t\left(x-\frac{1}{2}\right)\right], & \text{when } n = 2, \end{cases}$$

where $n_1(y)$ is first coordinate of an outer normal vector to $\partial Q'$ at y. Clearly, n_1 is nonzero only on subsets of $\partial Q'$: $\{-\frac{1}{2}\} \times [-\frac{1}{2}, \frac{1}{2}]^{n-2}$ (here $n_1 = -1$) and $\{\frac{1}{2}\} \times [-\frac{1}{2}, \frac{1}{2}]^{n-2}$ (here $n_1 = 1$). Let us consider first $n > 2$. After decomposing $x = (x_1, x') \in [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]^{n-2}$ and $y = (y_1, y') \in [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]^{n-2}$, we observe that

$$\begin{aligned} I_1 &= \frac{-1}{t^{n-1}} \left\{ \int_{y' \in [-\frac{1}{2}, \frac{1}{2}]^{n-2}} \phi\left(\frac{(x_1 + \frac{1}{2}, x' - y')}{t}\right) dy' \right. \\ &\quad \left. - \int_{y' \in [-\frac{1}{2}, \frac{1}{2}]^{n-2}} \phi\left(\frac{(x_1 - \frac{1}{2}, x' - y')}{t}\right) dy' \right\} \\ &= -\left\{ \int_{y' \in [-\frac{1}{2}, \frac{1}{2}]^{n-2}} \frac{1}{t^{n-1}} \phi\left(\frac{(x_1 + \frac{1}{2}, x' - y')}{t}\right) dy' \right. \\ &\quad \left. - \int_{y' \in [-\frac{1}{2}, \frac{1}{2}]^{n-2}} \frac{1}{t^{n-1}} \phi\left(\frac{(x_1 - \frac{1}{2}, x' - y')}{t}\right) dy' \right\} =: -\{a - b\}. \end{aligned}$$

Both terms a and b in the expression of I_1 are of the same sign (both are positive) and according to Lemma 5.3 they obey the same estimation:

$$a, b \leqslant \left(\frac{3}{2d}\right)^{n-1} \|\phi\|_\infty = \left(\frac{3}{2d}\right)^{n-1}.$$

Consequently, $|I_1| \leqslant \left(\frac{3}{2d}\right)^{n-1} = C$. The same estimation holds when $n = 2$ by simplification of the above arguments. This finishes the proof. □

We are now to estimate the derivative of $\widetilde{v}$ with respect to the last variable.

LEMMA 5.5. *Let $n \geqslant 2$ and*

$$\mathcal{Q}(z,t) = \frac{1}{t^{n-1}} \frac{|z|}{t} \chi_{\{\frac{z}{t} \in \frac{3}{2} Q'\}}.$$

Then for any $k \in \{1, \ldots, n-1\}$, v being Lipschitz and compactly supported in $(1-d)Q'$, $d \in (0,1)$, we have

$$\left| \frac{\partial \widetilde{v}}{\partial t}(x,t) \right| \leqslant D \int_{Q'} \mathcal{Q}(x-y,t) \left(\frac{|v(x) - v(y)|}{|x-y|} \right) dy + E|v(x)|, \text{ where}$$
$$D = (n-1)\frac{5}{2}, \;\; E = (n-1)\left(\frac{3}{2d}\right)^{n-1} \frac{1}{4}.$$

Proof. According to property (3.3) and the fact that v is supported in Q', we have:

$$\begin{aligned}
\frac{\partial \widetilde{v}}{\partial t}(x,t) &= \frac{\partial}{\partial t}(\phi_t * v)(x) = \left((\frac{\partial}{\partial t}\phi_t) * v \right)(x) = -\{(\mathrm{div}\,(g_{1,t},\dots,g_{n-1,t})) * v\} \\
&= -\sum_{k=1}^{n-1} \left\{ \left(D_k g_{k,t}\right) * v) \right\}(x) = -\sum_{k=1}^{n-1} \int_{\mathbb{R}^{n-1}} \left(\left(D_k g_{k,t}\right)(x-y) \right) v(y)dy \\
&= -\sum_{k=1}^{n-1} \int_{Q'} \left(\left(D_k g_{k,t}\right)(x-y) \right) v(y)dy =: -\sum_{k=1}^{n-1} T_k(x,t), \qquad (5.4)
\end{aligned}$$

where $g_{k,t}(x) = \frac{1}{t^{n-1}} \frac{x_k}{t} \psi(\frac{x_1}{t}) \cdot \dots \cdot \psi(\frac{x_{n-1}}{t})$ with ψ defined in (3.2). Moreover, for every $k \in \{1,\dots,n-1\}$:

$$\begin{aligned}
T_k(x,t) &= \int_{Q'} \left(\left(D_k g_{k,t}\right)(x-y) \right) (v(y)-v(x))\, dy + v(x) \int_{Q'} \left(\left(D_k g_{k,t}\right)(x-y) \right) dy \\
&= \int_{Q'} \left\{ \left(\left(D_k g_{k,t}\right)(x-y) \right) |x-y| \right\} \left(\frac{v(y)-v(x)}{|x-y|} \right) dy \\
&\quad + v(x) \int_{Q'} \left(\left(D_k g_{k,t}\right)(x-y) \right) dy \\
&=: I_k(x) + v(x) J_k(x). \qquad (5.5)
\end{aligned}$$

We will estimate every term: I_k and J_k separately.

Estimations for I_k.

Let $A_k(z,t) = \left(\left(D_k g_{k,t}\right)(z) \right) |z|$. We will show that

$$|A_k(z,t)| \leqslant \frac{5}{2} \mathscr{Q}(z,t). \qquad (5.6)$$

Indeed, we have

$$|D_k \phi(x_1,\dots,x_{n-1})| \leqslant 2\chi_{\{\frac{1}{4} < |\frac{x_k}{t}| < \frac{3}{4}\}} \cdot \chi_{\{\frac{z}{t} \in \frac{3}{2}Q'\}}$$

and (5.6) follows by the following estimates:

$$\begin{aligned}
|\left(D_k g_{k,t}\right)(z)|\,|z| &\leqslant \frac{1}{t^{n-1}} \left\{ 2\chi_{\{\frac{1}{4} < |\frac{z_k}{t}| < \frac{3}{4}\}} \cdot \frac{1}{t} \cdot \frac{|z_k|}{t} + \frac{1}{t} \right\} \chi_{\{\frac{z}{t} \in \frac{3}{2}Q'\}} |z| \\
&\leqslant \frac{5}{2} \frac{1}{t^{n-1}} \frac{|z|}{t} \chi_{\{\frac{z}{t} \in \frac{3}{2}Q'\}} = \frac{5}{2} \mathscr{Q}(z,t).
\end{aligned}$$

Estimations for J_k.

When $x \in Q' \setminus (1-d)Q'$ we have $v(x) = 0$ and hence $v(x)J_k(x) = 0$. Therefore it suffices to provide the estimates for J_k only when $x \in (1-d)Q'$. We will do it for $k=1$ and $n>2$ only. The case $n=2$ is more simpler.

Using the Gauss-Ostrogradsky Theorem, we get

$$J_1(x) = \int_{\partial Q'} g_{1,t}(x-y) n_1(y) d\sigma(y),$$

where n_1 is first coordinate of an outer normal vector to $\partial Q'$ at y. Obviously n_1 is nonzero only on subsets of $\partial Q'$: $\{-\frac{1}{2}\} \times [-\frac{1}{2}, \frac{1}{2}]^{n-2}$ ($n_1 = -1$) and $\{\frac{1}{2}\} \times [-\frac{1}{2}, \frac{1}{2}]^{n-2}$ ($n_1 = 1$). After decomposing $x = (x_1, x') \in [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]^{n-2}$ and $y = (y_1, y') \in [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]^{n-2}$, we get that

$$\begin{aligned} J_1(x,t) = &- \left\{ \int_{y' \in [-\frac{1}{2}, \frac{1}{2}]^{n-2}} g_{1,t} \left(x_1 + \frac{1}{2}, x' - y' \right) dy' \right. \\ &\left. - \int_{y' \in [-\frac{1}{2}, \frac{1}{2}]^{n-2}} g_{1,t} \left(x_1 - \frac{1}{2}, x' - y' \right) dy' \right\} \\ =: &- \{a - b\}. \end{aligned}$$

and a and b are of the same sign, so that $|J_1| \leqslant \max\{|a|, |b|\}$. We note that $\widetilde{g}_{1,t}(\cdot)$ is supported in $\frac{3}{2}Q'$ and $g_{1,t}(\cdot) = \frac{1}{t^{n-1}} \left\{ \phi\left(\frac{x}{t}\right) \frac{x_1}{t} \right\}$. Therefore by Lemma 5.1 applied to a and b, where $\widetilde{w}(z) = \phi(z) z_1$ and $\alpha = n-1$, we obtain

$$|J_1(x,t)| \leqslant \left(\frac{3}{2d} \right)^{n-1} \|\widetilde{w}\|_\infty = \left(\frac{3}{2d} \right)^{n-1} \frac{1}{4}. \tag{5.7}$$

Final assertion is obtained after summing up the estimations: (5.4),(5.5),(5.6),(5.7) with respect to $k \in \{1, \cdots, n-1\}$. □

We arrive at the following result which seems to be of separate interest, it provides pointwise estimates for the full gradient of $\widetilde{v}$.

THEOREM 5.6. *Let* $n \geqslant 2$, $Q = Q' \times (0,1)$, $Q' = (-\frac{1}{2}, \frac{1}{2})^{n-1}$, $v : Q' \to \mathbf{R}$ *be Lipschitz and compactly supported in* $(1-d)Q'$, *where* $d \in (0,1)$, *moreover,*

$$\widetilde{v}(x,t) = v * \phi_t(x) := \begin{cases} \int_{\mathbb{R}^{n-1}} v(y) \phi_t(x-y)\, dy & \text{when } t > 0, \\ v(x) & \text{when } t = 0. \end{cases}$$

and

$$\mathscr{Q}(z,t) = \frac{1}{t^{n-1}} \frac{|z|}{t} \chi_{\{\frac{z}{t} \in \frac{3}{2}Q'\}}. \tag{5.8}$$

Then for any $k \in \{1, \ldots, n-1\}$, *we have for* $(x,t) \in Q$

$$|\nabla \widetilde{v}(x,t)| \leqslant F \int_{Q'} \mathscr{Q}(x-y,t) \left(\frac{|v(x) - v(y)|}{|x-y|} \right) dy + G|v(x)|, \text{ where}$$

$F = \frac{5n}{2}$, $G = \left(\frac{3}{2d}\right)^{n-1} \frac{n+7}{4}$ *and* $\nabla \widetilde{v}$ *denotes full gradient of* $\widetilde{v}$.

Proof. We observe that $P_k(z,t) \leqslant \mathscr{Q}(z,t)$, where P_k is as in Lemma 5.4. According to Lemmas 5.4 and 5.5, after changing the notation $x_n := t$ we observe that all partial derivatives of $\tilde{u}$ obey the same estimations:

$$\left| \frac{\partial \tilde{v}}{\partial x_k}(x) \right| \leqslant D_{1,k} \int_{Q'} \mathscr{Q}(x'-y', x_n) \left(\frac{|v(x') - v(y')|}{|x'-y'|} \right) dy' + D_{2,k} |v(x')|,$$

where $x=(x',x_n)\in Q$, $(D_{1,k},D_{2,k})=(2,(\frac{3}{2d})^{n-1})$ when $k\in\{1,\dots,n-1\}$ and $(D_{1,n},D_{2,n})=((n-1)\frac{5}{2},(n-1)\left(\frac{3}{2d}\right)^{n-1}\frac{1}{4})$. To compute constants efficiently we denote:

$$z_k := \frac{\partial\tilde{v}}{\partial x_k}(x),\; z:=(z_1,\dots,z_n),\; a:=\int_{Q'}\mathscr{Q}(x'-y',x_n)\left(\frac{|v(x')-v(y')|}{|x'-y'|}\right)dy',$$
$$b:=|v(x')|,\; D_1:=(D_{1,1},\dots,D_{1,n}),\; D_2:=(D_{2,1},\dots,D_{2,n}).$$

As we have: $|z_k|\leqslant D_{1,k}a+D_{2,k}b$ and $D_{i,k},a,b$ are nonnegative, this implies:

$$|z|\leqslant |D_1|a+|D_2|b.$$

Now result follows from simple estimation:

$$|D_1|\leqslant n\frac{5}{2},\qquad |D_2|\leqslant\left(\frac{3}{2d}\right)^{n-1}\left|\left(1,\dots,1,\frac{n-1}{4}\right)\right|\leqslant\left(\frac{3}{2d}\right)^{n-1}\frac{n+7}{4},$$

after we switch to our previous notation $(t=x_n)$. □

5.3. Integral estimates for function $\nabla\widetilde{v}$

We are now to prove Theorem 4.1, part (ii).

Proof. [Proof of Theorem 4.1, part (ii)]

(a) The case $\rho\equiv 1$.
By the convexity argument $\left(R(a+b)\leqslant\frac{1}{2}R(2a)+\frac{1}{2}R(2b)\right)$ and Theorem 5.6:

$$\begin{aligned}R(|\nabla\widetilde{v}(x,t)|)&\leqslant\frac{1}{2}R\left(\int_{Q'}\left(\frac{\mathscr{Q}(x-y,t)}{C(x,t)}\right)\left(\frac{2FC(x,t)|v(y)-v(x)|}{|x-y|}\right)dy\right)\\&\quad+\frac{1}{2}R(2G|v(x)|)\\&\leqslant\frac{1}{2}\int_{Q'}\left(\frac{\mathscr{Q}(x-y,t)}{C(x,t)}\right)R\left(\frac{2FH|v(y)-v(x)|}{|x-y|}\right)dy\\&\quad+\frac{1}{2}R(2G|v(x)|)=:\frac{1}{2}A(x,t)+\frac{1}{2}R(2G|v(x)|),\end{aligned}\tag{5.9}$$

where $F=\frac{5n}{2}$, $G=\left(\frac{3}{2d}\right)^{n-1}\frac{n+7}{4}$ (so that $2G=J$), $\mathscr{Q}$ is defined by (5.8), $C(x,t)\leqslant H$ (H will be established later) and

$$C(x,t):=\int_{Q'}\mathscr{Q}(x-y,t)\,dy=\int_{\{y\in Q':\frac{x-y}{t}\in\frac{3}{2}Q'\}}\frac{1}{t^{n-1}}\frac{|x-y|}{t}\,dy.$$

We will show that

$$\frac{\sqrt{n-1}}{e}\left(\frac{3}{4}\right)^n\leqslant C(x,t)\leqslant\frac{1}{2}\left(\frac{3}{2}\right)^n\sqrt{\frac{n-1}{3}},\tag{5.10}$$

whenever $t \in (0,1)$. We observe that

$$C(x,t) = \int_{\{y \in Q' : \frac{x-y}{t} \in \frac{3}{2}Q'\}} \frac{1}{t^{n-1}} \frac{|x-y|}{t} \, dy \overset{z:=x-y}{=} \frac{1}{t^n} \int_{\{z \in x+Q', z \in \frac{3}{2}tQ'\}} |z| \, dz.$$

Moreover,

$$C((x_1,\dots,x_{k-1},-x_k,x_{k+1},\dots,x_{n-1}),t) = C((x_1,\dots,x_{k-1},x_k,x_{k+1},\dots,x_{n-1}),t). \tag{5.11}$$

Therefore for our estimations we can assume that $x_1,\dots,x_{n-1} \geqslant 0$. Note that for $x \in \{(x_1,\dots,x_{n-1}) : x_i \geqslant 0,\ i = 1,\dots,n-1\}$ sets

$$\mathscr{B}(x,t) := \{x + Q'\} \cap \{\frac{3}{2}tQ'\}$$

obey the inclusion property:

$$\mathscr{B}((x_1,\dots,x_k,\dots,x_{n-1}),t) \supseteq \mathscr{B}((x_1,\dots,\bar{x}_k,\dots,x_{n-1}),t)$$

whenever $\bar{x}_k \geqslant x_k \geqslant 0$. Therefore when $x_1,\dots,x_{n-1}$ are nonnegative, biggest value of $C(x,t)$ is achieved for x having possible small coordinates, i.e. at $x = (0,\dots,0)$, while smallest value is achieved when x has the possibly big coordinates in Q'. We estimate (roughly, more precise estimations would be if we used the assumption $x \in (1-d)Q'$):

$$C\Big(\Big(\frac{1}{2},\dots,\frac{1}{2}\Big),t\Big) \leqslant C(x,t) \leqslant C((0,\dots,0),t). \tag{5.12}$$

To give the precise value of $C((\frac{1}{2},\dots,\frac{1}{2}),t)$ we observe that

$$\Big\{\Big(\frac{1}{2},\dots,\frac{1}{2}\Big) + Q'\Big\} \cap \Big\{\frac{3}{2}tQ'\Big\} = [0,1]^{n-1} \cap \Big\{\frac{3}{2}tQ'\Big\} = \Big[0, \min\Big\{1, \frac{3}{4}t\Big\}\Big]^{n-1} = \Big[0, \frac{3}{4}t\Big]^{n-1}.$$

Therefore

$$C\Big(\Big(\frac{1}{2},\dots,\frac{1}{2}\Big),t\Big) = \frac{1}{t^n} \int_{[0,\frac{3}{4}t]^{n-1}} |z| \, dz. \tag{5.13}$$

By the property (5.11) for $\varepsilon_1,\dots\varepsilon_{n-1} \in \{+1,-1\}$ we have

$$C\Big(\Big(\varepsilon_1\frac{1}{2},\dots,\varepsilon_{n-1}\frac{1}{2}\Big),t\Big) = C\Big(\Big(\frac{1}{2},\dots,\frac{1}{2}\Big),t\Big).$$

and as

$$\mathscr{B}(0,t) = \{Q'\} \cap \Big\{\frac{3}{2}tQ'\Big\} = \Big[-\min\Big\{\frac{1}{2},\frac{3}{4}t\Big\}, \min\Big\{\frac{1}{2},\frac{3}{4}t\Big\}\Big]^{n-1} \subseteq \Big[-\frac{3}{4}t, \frac{3}{4}t\Big]^{n-1},$$

we obtain

$$C((0,\dots,0),t) \leqslant \frac{1}{t^n} \int_{[-\frac{3}{4}t,\frac{3}{4}t]^{n-1}} |z| \, dz =: \mathscr{H}. \tag{5.14}$$

Moreover, let $\bar{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_{n-1}) \in \{-1,1\}^{n-1}$ and let $I_{\bar{\varepsilon}}(t)$ be the range of cube $I(t) = [0, \frac{3}{4}t]^{n-1}$ under the mapping $(x_1, \dots, x_{n-1}) \mapsto (\varepsilon_1 x_1, \dots, \varepsilon_{n-1} x_{n-1})$. It is clear that $a_{\bar{\varepsilon}} := \frac{1}{t^n} \int_{I_{\bar{\varepsilon}}(t)} |z|\, dz$ is the same for every $\bar{\varepsilon} \in \{-1,1\}^{n-1}$. Therefore

$$\mathscr{H} = \sum_{\bar{\varepsilon} \in \{-1,1\}^{n-1}} a_{\bar{\varepsilon}} = 2^{n-1} \frac{1}{t^n} \int_{I(t)} |z|\, dz = 2^{n-1} C\Big(\Big(\frac{1}{2}, \dots, \frac{1}{2}\Big), t\Big). \tag{5.15}$$

To estimate $C(\frac{1}{2}, \dots, \frac{1}{2}), t)$ given by (5.13) we note that

$$I_{n-1}(y) := \int_{[0,y]^{n-1}} |z| dz = |y|^n I_{n-1}(1).$$

Moreover, Schwartz inequality yields

$$I_{n-1}(1) = \int_{z \in [0,1]^{n-1}} \left(\sqrt{\sum_{i=1}^{n-1} z_i^2} \right) dz \leqslant \left(\int_{z \in [0,1]^{n-1}} \sum_{i=1}^{n-1} z_i^2 dz \right)^{\frac{1}{2}} \leqslant \sqrt{\frac{n-1}{3}}.$$

On the other hand, using the inequality between arithmatic and geometric means, we have

$$\begin{aligned} I_{n-1}(1) &= \int_{z \in [0,1]^{n-1}} \left(\sqrt{\sum_{i=1}^{n-1} z_i^2} \right) dz \geqslant \sqrt{n-1} \int_{z \in [0,1]^{n-1}} \prod_{i=1}^{n-1} z_i^{\frac{1}{n-1}} dz \\ &= \sqrt{n-1} \prod_{i=1}^{n-1} \int_0^1 z_i^{\frac{1}{n-1}} dz_i = \sqrt{n-1}\, \frac{1}{(1+1/(n-1))^{n-1}} \geqslant \frac{\sqrt{n-1}}{e}. \end{aligned}$$

Therefore

$$\frac{\sqrt{n-1}}{e} \left(\frac{3}{4}\right)^n \leqslant C((\frac{1}{2}, \dots, \frac{1}{2}), t) \leqslant \left(\frac{3}{4}\right)^n \sqrt{\frac{n-1}{3}}. \tag{5.16}$$

Now inequalities (5.10) follow from (5.16), (5.12), (5.14), (5.15).

Integrating $A(x,t)$ with respect to $t \in (0,1)$, we get from (5.9) and (5.10):

$$\int_{(0,1)} A(x,t)\, dt \leqslant \int_{Q'} \left\{ \int_{(0,1)} \frac{\mathscr{Q}(x-y,t)}{C(x,t)} dt \right\} R\left(\frac{2FH|v(y) - v(x)|}{|x-y|} \right) dy, \tag{5.17}$$

where $H = \frac{1}{2} \left(\frac{3}{2}\right)^n \sqrt{\frac{n-1}{3}}$, so that $2FH = \frac{5n}{2} \left(\frac{3}{2}\right)^n \sqrt{\frac{n-1}{3}} = I$.

Now we will estimate the term in bracket $\{\}$ on right hand side in (5.17). By formulae (5.8) we have for any $z \in \mathbb{R}^{n-1}$

$$\begin{aligned} \int_{(0,1)} \frac{\mathscr{Q}(z,t)}{C(x,t)} dt &= |z| \int_{(0,1)} \frac{1}{t^n} \chi_{\{\frac{z}{t} \in \frac{3}{2} Q'\}} \cdot \frac{1}{C(x,t)} dt \\ &\overset{(5.10)}{\leqslant} \frac{1}{|z|^{n-2}} \left(|z|^{n-1} \int_0^1 \frac{1}{t^n} \chi_{\{\frac{z}{t} \in \frac{3}{2} Q'\}} 1 dt \right) \frac{e}{\sqrt{n-1}} \left(\frac{4}{3}\right)^n \overset{(4.1)}{=} a_0 \cdot \frac{\omega_{\rho \equiv 1}(z)}{|z|^{n-2}}, \end{aligned} \tag{5.18}$$

where $a_0 = \frac{e}{\sqrt{n-1}}\left(\frac{4}{3}\right)^n$.

Therefore

$$\int_{x\in Q'}\int_{(0,1)}\frac{1}{2}A(x,t)\,dt\,dx \leqslant \frac{a_0}{2}\int_{x\in Q'}\int_{y\in Q'} R\left(\frac{I|v(y)-v(x)|}{|x-y|}\right)\frac{\omega_{\rho\equiv 1}(x-y)}{|x-y|^{n-2}}\,dydx \tag{5.19}$$

and $\frac{a_0}{2} = L$. This gives (4.4) in case $\rho \equiv 1$.

(b) The case of general ρ.

We multiply both sides of the pointwise inequality (5.9) by a weight ρ and integrate the inequality over $(0,1)$ first, then over Q'. We get:

$$\begin{aligned}\int_{Q'}\int_0^1 R(|\nabla\tilde{v}|)\,\rho(t)\,dt\,dx \leqslant &\frac{1}{2}\int_{Q'}\int_0^1 A(x,t)\,\rho(t)\,dt\,dx\\ &+\frac{1}{2}\int_{Q'}\int_0^1 R(J|v(x)|)\,\rho(t)\,dt\,dx,\\ =&\frac{1}{2}\int_{Q'}\int_0^1 A(x,t)\,\rho(t)\,dt\,dx\\ &+\frac{1}{2}\int_{Q'} R(J|v(x)|)\int_0^1 \rho(t)\,dt\,dx =: \mathscr{I}_1+\mathscr{I}_2.\end{aligned}$$

We will estimate the terms: $\mathscr{I}_1$ and $\mathscr{I}_2$ separately. Obviously, as $\int_0^1 \rho(t)\,dt = C(\rho) < \infty$, we get

$$\mathscr{I}_2 \leqslant \frac{C(\rho)}{2}\int_{Q'} R(J|v(x)|)\,dx.$$

To deal with $\mathscr{I}_1$, we deduce that:

$$\int_{Q'}\int_0^1 A(x,t)\,\rho(t)\,dt\,dx \leqslant \int_{Q'}\int_{Q'}\left\{\int_0^1 \frac{\mathscr{Q}(x-y,t)}{C(x,t)}\rho(t)\,dt\right\} R\left(\frac{I|v(y)-v(x)|}{|x-y|}\right)dydx,$$

and by obvious modification of (5.18), we have

$$\int_0^1 \frac{\mathscr{Q}(x-y,t)}{C(x,t)}\rho(t)\,dt \leqslant a_0\cdot\frac{\omega_\rho(x-y)}{|x-y|^{n-2}}.$$

Therefore,

$$\mathscr{I}_1 = \frac{1}{2}\int_{Q'}\int_0^1 A(x,t)\,\rho(t)\,dt\,dx \leqslant L\int_{x\in Q'}\int_{y\in Q'} R\left(\frac{I|v(y)-v(x)|}{|x-y|}\right)\frac{\omega_\rho(x-y)}{|x-y|^{n-2}}\,dy\,dx.$$

This ends the proof of (4.4).

Last part of the statement as well as the proof of part (iii) follows by similar arguments as that used to finish the proof of part (i). □

REMARK 5.7. More exact computations in case $\rho \equiv 1$ can be provided. Namely, we have $\omega_{\rho\equiv 1}(z) = |z|^{n-1}\int_{(0,1)} \frac{1}{t^n}\chi_{\{\frac{z}{t}\in\frac{3}{2}Q'\}}dt$ and

$$\{z : \frac{z}{t} \in \frac{3}{2}Q'\} \subseteq \bigcap_{i=1}^{n-1}\{z : t \geqslant \frac{4}{3}|z_i|\} \subseteq \{z : t \geqslant \frac{\frac{4}{3}|z|}{\sqrt{n-1}}\}.$$

We have for $p(z) = \min\{\frac{\frac{4}{3}|z|}{\sqrt{n-1}}, 1\}$:

$$\begin{aligned}\omega_{\rho\equiv 1}(z) &\leqslant |z|^{n-1}\int_{p(z)}^{1}\frac{1}{t^n}\,dt = |z|^{n-1}\frac{1}{-n+1}t^{-n+1}|_{p(z)}^{1} = |z|^{n-1}\frac{1}{n-1}\left(p(z)^{-(n-1)}-1\right)\\ &\leqslant |z|^{n-1}\frac{1}{n-1}\left(\frac{|z|}{\sqrt{n-1}}\frac{4}{3}\right)^{-(n-1)} = (\sqrt{n-1})^{n-3}\left(\frac{3}{4}\right)^{n-1} \equiv Const.\end{aligned}$$

Therefore when $\rho \equiv 1$ we arrive at the following inequality, which is the special variant of (4.4):

$$\begin{aligned}\int_{Q'}\int_0^1 R(|\nabla\tilde{v}|)\,\rho(t)\,dt\,dx \leqslant \tilde{L}\int_{x\in Q'}\int_{y\in Q'} R\left(\frac{I|v(y)-v(x)|}{|x-y|}\right)\frac{1}{|x-y|^{n-2}}\,dy\,dx \qquad (5.20)\\ +\frac{C(\rho)}{2}\int_{Q'} R(J|v(x)|)\,dx,\end{aligned}$$

where $I = \frac{5n}{2}\left(\frac{3}{2}\right)^n\sqrt{\frac{n-1}{3}}$, $J = \left(\frac{3}{2d}\right)^{n-1}\frac{n+7}{2}$, $\tilde{L} = \frac{1}{2}\left(\frac{4}{3}\right)^n\frac{e}{\sqrt{n-1}}(\sqrt{n-1})^{n-3}\left(\frac{3}{4}\right)^{n-1} = (\sqrt{n-1})^{n-4}\frac{2e}{3}$, and $\nabla\tilde{v}$ denotes full gradient of $\tilde{v}$.

6. Proof of Theorem 4.2

In this section we prove Theorem 4.2.

Proof of Theorem 4.2. We will use the notation $Q_{a,b} := aQ' \times (0,b)$ where $a,b > 0$. The proof follows by several simple steps. Some of their proofs are omitted.

Step 1. We observe that for any integrable function w defined on $Q' \times \{0\}$, we have

$$\int_{Q'\times\{0\}} w(x)d\sigma(x) = \int_{Q'} w(x',0)dx'. \qquad (6.1)$$

This follows from the fact that $I(x') = (x',0)$ defines the map for $Q' \times \{0\}$ and $|DI| = 1$ (where DI is the jacobian matrix of the change of variables).

Step 2. Let $d' \in (0,1)$ be a given number. We observe that when $x = (x',t) \in Q_{1-d',d'} \subseteq Q$ then $t = \mathrm{dist}(x,\partial Q)$. In particular

$$\rho(t) = \rho\left(\mathrm{dist}(x,\partial Q)\right) \text{ when } x = (x',t) \in Q_{1-d',d'} \subseteq Q.$$

Step 3. Let

$$w(x',t) = u(\cdot,0) * \phi_t(x) := \begin{cases} \int_{\mathbb{R}^{n-1}} u(y',0)\phi_t(x'-y')\,dy' & \text{when } t > 0,\\ u(x,0) & \text{when } t = 0.\end{cases} \qquad (6.2)$$

We observe that w restricted to $Q_{1,\frac{d}{3}}$ is supported in $Q_{1-\frac{d}{2},\frac{d}{3}} \subseteq Q_{1,\frac{d}{3}}$.

Indeed, when $x=(x',t)\in Q_{1,\frac{d}{3}}\setminus Q_{1-\frac{d}{2},\frac{d}{3}}$, we have $x'\in Q'\setminus(1-\frac{d}{2})Q'$, consequently $\mathrm{dist}\{x',(1-d)Q'\}\geqslant \frac{d}{2}$. Therefore for every $y'\in(1-d)Q'$ we have $|x'-y'|\geqslant \frac{d}{2}$. In particular for every such y' and $t\leqslant \frac{d}{3}$ we have $\frac{|x'-y'|}{t}\geqslant \frac{3}{2}$, and so $\phi_t(x'-y')=0$. As $v(y')=u(y',0)$ is supported in $(1-d)Q'$, we get for such (x',t)

$$\int_{\mathbb{R}^{n-1}} u(y',0)\phi_t(x'-y')\,dy' = \int_{(1-d)Q'} u(y',0)\phi_t(x'-y')\,dy' = 0.$$

Therefore $w(x',t)=0$.

Step 4. We define extension operator $\mathrm{Ext}(u)$ by expression

$$\tilde{u} = \mathrm{Ext}(u)(x',t) := \tilde{\phi}(t)\cdot w(x',t),$$

where w is the same as in (6.2), $\tilde{\phi}(t) = \begin{cases} -\frac{3}{d}t+1 & \text{when } t\in(0,\frac{d}{3}) \\ 0 & \text{when } \quad t\geqslant \frac{d}{3} \end{cases}$.

We will show that $\tilde{u}$ satisfies the required properties. For that, we introduce the notation $v(z)=u(z,0)$, $z\in Q'$,

$$\begin{aligned} M(\lambda) &:= \int_{\partial Q}\int_{\partial Q} R\left(\lambda\frac{|u(x)-u(y)|}{|x-y|}\right)\frac{\tilde{\omega}_\rho^d(x,y)}{|x-y|^{n-2}}\,d\sigma(y)\,d\sigma(x) \\ &\overset{Step\ 1}{=} \int_{x\in Q'}\int_{y\in Q'} R\left(\lambda\frac{|v(x)-v(y)|}{|x-y|}\right)\frac{\omega_\rho\chi_{(0,\frac{d}{3})}(x-y)}{|x-y|^{n-2}}\,dy\,dx, \\ N(\lambda) &:= \int_{\partial Q} R(\lambda|u(x)|)\,d\sigma(x) = \int_{Q'} R(\lambda|v(x')|)\,dx'. \end{aligned}$$

We start with the computation of

$$I_1 := \int_Q R(|\tilde{u}|)\,\rho(\mathrm{dist}(x,\partial Q))\,dx.$$

As $R(0)=0$, we note that $R(|\tilde{u}|)$ is supported in $(1-\frac{d}{2})Q'\times(0,\frac{d}{3})\subseteq(1-\frac{d}{3})Q'\times(0,\frac{d}{3}) = Q_{1-d',d'}$ where $d'=\frac{d}{3}$ (Step 3). On the other hand, according to Step 2 we have $\rho(\mathrm{dist}(x,\partial Q))=\rho(t)$ when $x=(x',t)\in Q_{1-d',d'}$. This gives

$$\begin{aligned} I_1 &= \int_{x'\in(1-\frac{d}{3})Q'}\int_0^{\frac{d}{3}} R\left(\tilde{\phi}(t)|w(x',t)|\right)\rho(t)\,dx'dt \\ &\leqslant \int_{x'\in Q'}\int_0^1 R\left(|w(x',t)|\right)(\rho\chi_{(0,\frac{d}{3})})(t)\,dx'dt \\ &\overset{Theorem\ 4.1, part(i)}{\leqslant} D(\rho)\int_{Q'} R(|v(x')|)dx' = D(\rho)N(1). \end{aligned}$$

Therefore first inequality in (4.6) follows. To deal with

$$I_2 := \int_Q R(|\nabla\tilde{u}|)\,\rho(\mathrm{dist}(x,\partial Q))\,dx,$$

we note that

$$\nabla\tilde{u}(x',t) = \tilde{\phi}(t)\nabla w(x',t) + \tilde{\phi}(t)(0,\dots,0,-\frac{3}{d}\chi_{(0,\frac{d}{3})}(t)\cdot w(x',t)).$$

Hence

$$|\nabla\tilde{u}(x',t)| \leqslant |\nabla w(x',t)| + \frac{3}{d}|w(x',t)|\chi_{(0,\frac{d}{3})}(t).$$

Consequently,

$$I_2 = \int_{x'\in(1-\frac{d}{3})Q'}\int_0^{\frac{d}{3}} R\left(|\nabla\tilde{u}(x',t)|\right)\rho(t)\,dt \leqslant \frac{1}{2}\int_{x'\in(1-\frac{d}{3})Q'}\int_0^{\frac{d}{3}} R\left(2|\nabla w(x',t)|\right)\rho(t)\,dt$$
$$+\frac{1}{2}\int_{x'\in(1-\frac{d}{3})Q'}\int_0^{\frac{d}{3}} R\left(\frac{6}{d}|w(x',t)|\right)\rho(t)\,dt =: A_1 + A_2.$$

Moreover, using Theorem 4.1, part (ii), one easily obtains

$$A_1 \leqslant \frac{1}{2}\int_{x'\in Q'}\int_0^1 R\left(2|\nabla w(x',t)|\right)(\rho\chi_{(0,\frac{d}{3})})(t)\,dt \leqslant \frac{1}{2}\left\{LM(2I) + \frac{D(\rho)}{2}N(2J)\right\},$$

while by Theorem 4.1, part (i)

$$A_2 \leqslant \frac{1}{2}D(\rho)N(\frac{6}{d}) \leqslant \frac{1}{2}D(\rho)N(2J)$$

as $N(\cdot)$ is nondecreasing. From there second inequality in (4.6) follows. □

7. Admissible weights in Theorems 4.1 and 4.2

The expression (4.1) defines the transform ω_ρ of the given weight ρ defined on interval. Below we compute several examples which illustrate results of Theorems 4.1 and 4.2.

EXAMPLE 7.1. (a) Considering $\rho \equiv 1$ we get $\omega_\rho \leqslant Const$ (see Remark 5.7) and so we retrieve the classical unweighted result.

(b) Let $\rho(t) = t^\alpha$, $-1 < \alpha < n-1$. An easy computation shows that:

$$|z|^{n-1}\int_0^1 \frac{1}{t^n}\chi_{\{\frac{z}{t}\in\frac{3}{2}Q'\}}t^\alpha dt \leqslant |z|^{n-1}\int_{\min\{\frac{4}{3\sqrt{n-1}}|z|,1\}}^1 t^{\alpha-n}dt$$
$$\leqslant |z|^{n-1}\frac{1}{n-1-\alpha}\left(\frac{4}{3\sqrt{n-1}}|z|\right)^{\alpha-(n-1)}$$
$$= \frac{1}{n-1-\alpha}\left(\frac{4}{3\sqrt{n-1}}\right)^{\alpha-(n-1)}|z|^\alpha$$
$$\sim |z|^\alpha.$$

Therefore statements hold with $\omega(z) \sim |z|^{\alpha}$.

This result can be compared with an old result by Lizorkin [31], Theorem 3 and by Vasarin [44]. Namely, it was shown in [31] that in case when $\Omega \subseteq \mathbf{R}^n$ has C^2-boundary, $-1 \leqslant \alpha < p-1$ and $R(\lambda) = \lambda^p$, there exist trace embedding operator $\mathrm{Tr} : W^{1,p}_{(\mathrm{dist}(x,\partial\Omega))^{\alpha}}(\Omega) \to Y^{R,R}_{|x-y|^{\alpha}}(\partial\Omega)$ and this operator is a surjection. The case $0 \leqslant \alpha < 1$ was studied by Vasarin [44], while the special case $p=2$ was obtained earlier by Nikolski [39] (not involving Slobodetskii type space directly).

(c) If $\rho(t) = t^{\alpha}\left(\ln\left(2+\frac{1}{t}\right)\right)^{\beta}$, $-1 < \alpha < n-1, \beta > 0$. By similar computations as before, we get:

$$
\begin{aligned}
&|z|^{n-1}\int_{C_1|z|}^{1} t^{\alpha-n}\left(\ln\left(2+\frac{1}{t}\right)\right)^{\beta} dt \\
&\leqslant \left(\ln\left(2+\frac{1}{C_1|z|}\right)\right)^{\beta} |z|^{n-1}\int_{\min\{\frac{4}{3\sqrt{n-1}}|z|,1\}}^{1} t^{\alpha-n}\,dt \\
&\leqslant \frac{1}{n-1-\alpha}\left(\frac{4}{3\sqrt{n-1}}\right)^{\alpha-(n-1)} |z|^{\alpha}\left(\ln\left(2+\frac{1}{\frac{4}{3\sqrt{n-1}}|z|}\right)\right)^{\beta} \\
&\sim |z|^{\alpha}\left(\ln\left(2+\frac{1}{|z|}\right)\right)^{\beta}.
\end{aligned}
$$

Considering $\beta = 0$ we retrieve previous result.

(d) When $\rho(t)$ is an arbitrary nonincreasing integrable function, we get

$$
\begin{aligned}
\omega_{\rho}(z) &\leqslant |z|^{n-1}\int_{\min\{C_1|z|,1\}}^{1} t^{-n}\rho(t)\,dt \leqslant \rho(C_1|z|)\cdot|z|^{n-1}\int_{\min\{\frac{4}{3\sqrt{n-1}}|z|,1\}}^{1} t^{-n}\,dt \\
&\leqslant E\cdot\rho(C_1|z|),
\end{aligned}
$$

where $C_1 = \frac{4}{3\sqrt{n-1}}$, $E = (\sqrt{n-1})^{n-3}\left(\frac{3}{4}\right)^{n-1}$.

(e) when $\rho(t)$ is an arbitrary integrable nonincreasing function and satisfies the $\Delta_{\frac{1}{2}}$-condition: $\rho(\frac{1}{2}t) \leqslant \mathscr{D}\rho(t)$, with $\mathscr{D}$ independent on t, we get for $C_1 < 1$

$$
\rho(C_1|z|) \leqslant F\cdot\rho(|z|),
$$

where $F = \mathscr{D}^{-[log_2(C_1)]}$. In particular Theorem 4.1 holds with $\omega_{\rho}(z)$ substituted by $G\cdot\rho(|z|)$ where $G = \mathscr{D}^{-[log_2(\frac{4}{3\sqrt{n-1}})]}(\sqrt{n-1})^{n-3}\left(\frac{3}{4}\right)^{n-1}$ when $n > 2$ and $E = \frac{3}{4}$ when $n = 2$.

As a corollary, we obtain the following statement which seems to be of separate interest.

THEOREM 7.2. *Let R be the given convex function, $Q=Q'\times(0,1)$, $Q'=(-\frac{1}{2},\frac{1}{2})^{n-1}$, $d\in(0,1)$, $\rho\in L^1((0,\frac{d}{3}))$ is an arbitrary nonincreasing weight function and satisfies the $\Delta_{1/2}$-condition: $\rho(\frac{1}{2}t)\leqslant \mathscr{D}\rho(t)$ (with $\mathscr{D}$ independent on t), $\omega(x,y):=\rho(|x-y|)$ and let us consider the subspace of $Y^{R,R}_{\omega,L}(\partial Q)$ depending on d*

$$Y^{R,R}_{\omega,L,d}(\partial Q):=\{v\in Y^{R,R}_{\omega,L}(\partial Q):\mathrm{supp} v\subseteq(1-d)Q'\times\{0\}\}.$$

Then there exists a linear extension operator:

$$\mathrm{Ext}: Y^{R,R}_{\rho(|x-y|),L,d}(Q'\times\{0\})\mapsto W^{1,R}_{\rho(\mathrm{dist}(\cdot,\partial Q)),L}(Q)$$

such that for $\tilde{u}:=\mathrm{Ext}(u)$ we have

$$\int_Q R(|\tilde{u}|)\,\rho(\mathrm{dist}(x,\partial Q))\,dx\leqslant D(\rho)\int_{\partial Q}R(|u(x)|)\,d\sigma(x),$$

$$\int_Q R(|\nabla\tilde{u}|)\,\rho(\mathrm{dist}(x,\partial Q))\,dx\leqslant \tilde{K}\int_{\partial Q}\int_{\partial Q}R\left(\frac{\tilde{I}|u(x)-u(y)|}{|x-y|}\right)\frac{\rho(|x-y|)}{|x-y|^{n-2}}\,d\sigma(y)\,d\sigma(x)$$
$$+\frac{3D(\rho)}{4}\int_{\partial Q}R(\tilde{J}|u(x)|)\,d\sigma(x),$$

where $\tilde{I}=5n\left(\frac{3}{2}\right)^n\sqrt{\frac{n-1}{3}}$, $\tilde{J}=\left(\frac{3}{2d}\right)^{n-1}(n+7)$, $D(\rho)=\int_0^{\frac{d}{3}}\rho(t)dt$, $\tilde{K}=\mathscr{D}^{-[\log_2(\frac{4}{3\sqrt{n-1}})]}(\sqrt{n-1})^{n-4}\frac{e}{3}$ when $n>2$ and $\tilde{K}=\frac{e}{4}$ when $n=2$.

In particular, when R is an Orlicz function, then there exists a constant $\tilde{B}_3=\tilde{B}_3(n,\rho,d)$ such that

$$\|\,\mathrm{Ext}(u)\|_{W^{1,R}_{\rho(\mathrm{dist}(\cdot,\partial Q))}(Q)}\leqslant\tilde{B}_3\|u\|_{Y^{R,R}_{\rho(|x-y|)}(Q')},$$

for every $u\in Y^{R,R}_{\rho(|x-y|),d,L}(Q'\times\{0\})$ supported in $(1-d)Q'\times\{0\}$.

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Raj Narayan Dhara
Temps Frequence, Femto-ST, Université de Franche-Comté
26, Chemin de l'Epitaphe, 25030 Besançon cedex, France
and Faculty of Mathematics, Informatics and Mechanics
University of Warsaw
ul. Banacha 2, 02-097 Warszawa, Poland
e-mail: r.dhara@mimuw.edu.pl

Agnieszka Kałamajska
Institute of Mathematics, University of Warsaw
ul. Banacha 2, 02-097 Warszawa, Poland
e-mail: kalamajs@mimuw.edu.pl