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THE MAXIMAL OPERATOR OF MARCINKIEWICZ–FEJÉR MEANS WITH RESPECT TO WALSH–KACZMARZ–FOURIER SERIES

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Abstract. In the paper [4, Theorem 1] Gát, Goginava and the author proved that the maximal operator $\sigma^{\kappa,*}$ of Marcinkiewicz-Fejér means of Walsh-Kaczmarz-Fourier series, is bounded from the dyadic Hardy space H_p into the space L_p for $p > 2/3$. Moreover, Goginava and the author showed that $\sigma^{\kappa,*}$ is not bounded from the Hardy space $H_{2/3}$ to the space $L_{2/3}$ [6, Theorem 1]. The main aim of this paper is to show that the maximal operator $\tilde{\sigma}^{\kappa,*} f := \sup_{n\in\mathbb{P}} \frac{|\sigma_n^\kappa f|}{\log^{3/2}(n+1)}$, is bounded from the Hardy space $H_{2/3}$ into the space $L_{2/3}$. Moreover, we prove that the order of deviant behavior of the nth Walsh-Kacmarz-Marcinkiewicz-Fejér mean is exactly $\log^{3/2}(n+1)$ in the endpoint $p = 2/3$.

1. Definitions and notation

Now, we give a brief introduction to the theory of dyadic analysis [1, 11, Chapter 1]. Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N} := \mathbb{P} \cup \{0\}$. Denote $\mathbb{Z}_2$ the discrete cyclic group of order 2, that is $\mathbb{Z}_2 = \{0,1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on $\mathbb{Z}_2$ is given such that the measure of a singleton is 1/2. Let G be the complete direct product of the countable infinite copies of the compact groups $\mathbb{Z}_2$. The elements of G are sequences of the form $x = (x_0, x_1, \ldots, x_k, \ldots)$ with coordinates $x_k \in \{0,1\}$ $(k \in \mathbb{N})$. The group operation on G is the coordinate-wise addition, the measure (denoted by μ) is the product measure and the topology is the product topology. The compact Abelian group G is called the Walsh group. A base for the neighbourhoods of G can be given in the following way:

$$I_0(x) := G, \quad I_n(x) := I_n(x_0, \ldots, x_{n-1}) := \{y \in G : y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots)\},$$

$(x \in G, n \in \mathbb{N})$. These sets are called dyadic intervals. Let $0 = (0 : i \in \mathbb{N}) \in G$ denote the null element of G, and $I_n := I_n(0)$ $(n \in \mathbb{N})$. Set $e_n := (0, \ldots, 0, 1, 0, \ldots) \in G$, the nth coordinate of which is 1 and the rest are zeros $(n \in \mathbb{N})$.

For $k \in \mathbb{N}$ and $x \in G$ denote

$$r_k(x) := (-1)^{x_k}$$

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the kth Rademacher function. If $n \in \mathbb{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$ can be written, where $n_i \in \{0,1\} \quad (i \in \mathbb{N})$, i. e. n is expressed in the number system of base 2. Denote the order of n by $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$, that is $2^{|n|} \leqslant n < 2^{|n|+1}$.

The Walsh-Paley system is defined as the product system of Rademacher functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (x \in G, n \in \mathbb{P}).$$

The Walsh-Kaczmarz functions are defined by $\kappa_0 = 1$ and for $n \geqslant 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}.$$

The set of Walsh-Kaczmarz functions and the set of Walsh-Paley functions is the same in dyadic blocks. Namely,

$$\{\kappa_n : 2^k \leqslant n < 2^{k+1}\} = \{w_n : 2^k \leqslant n < 2^{k+1}\}$$

for all $k \in \mathbb{P}$ and $\kappa_0 = w_0$.

V. A. Skvortsov (see [14, page 142]) gave a relation between the Walsh-Kaczmarz functions and the Walsh-Paley functions by the help of the transformation $\tau_A : G \to G$ defined by

$$\tau_A(x) := (x_{A-1}, x_{A-2}, \ldots, x_1, x_0, x_A, x_{A+1}, \ldots)$$

for $A \in \mathbb{N}$. By the definition of τ_A, we have

$$\kappa_n(x) = r_{|n|}(x) w_{n-2^{|n|}}(\tau_{|n|}(x)) \quad (n \in \mathbb{N}, x \in G).$$

The Dirichlet kernels and the Fejér kernels are defined by

$$D_n^{\alpha} := \sum_{k=0}^{n-1} \alpha_k, \quad K_n^{\alpha}(x) := \frac{1}{n} \sum_{k=0}^{n-1} D_k^{\alpha}(x),$$

where $\alpha_n = w_n$ (for all $n \in \mathbb{P}$) or κ_n (for all $n \in \mathbb{P}$), $D_0^{\alpha} := 0$. The 2^nth Dirichlet kernels have a closed form (see e.g. [11, page 7])

$$D_{2^n}^{w}(x) = D_{2^n}^{\kappa}(x) = D_{2^n}(x) = \begin{cases} 0, & \text{if } x \notin I_n, \\ 2^n, & \text{if } x \in I_n. \end{cases} \tag{1}$$

The σ-algebra generated by the 2-dimensional cube of measure 2^{-2k} will be denoted by F_k $(k \in \mathbb{N})$. Denote by $f = \left(f^{(n)}, n \in \mathbb{N}\right)$ the one-parameter martingale with respect to $(F_n, n \in \mathbb{N})$ (for details see, e. g. [17, 18, Chapter 1]). The maximal function of a martingale f is defined by $f^* = \sup_{n \in \mathbb{N}} \left| f^{(n)} \right|$. For $0 < p < \infty$ the Hardy martingale space $H_p(G^2)$ consists of all martingales for which $\|f\|_{H_p} := \|f^*\|_p < \infty$.

The Kronecker product $(\alpha_{n,m} : n,m \in \mathbb{N})$ of two Walsh-(Kaczmarz) system is said to be the two-dimensional Walsh-(Kaczmarz) system. That is,

$$\alpha_{n,m}(x^1,x^2) = \alpha_n(x^1)\alpha_m(x^2).$$

If $f \in L\left(G^2\right)$, then the number $\widehat{f}^{\alpha}(n,m) := \int\limits_{G^2} f\alpha_{n,m}$ $(n,m \in \mathbb{N})$ is said to be the (n,m)th Walsh-(Kaczmarz)-Fourier coefficient of f. We can extend this definition to martingales in the usual way (see Weisz [17, 18, Chapter 1]). Denote by $S^{\alpha}_{n,m}$ the (n,m)th rectangular partial sum of the Walsh-(Kaczmarz)-Fourier series of a martingale f. Namely,

$$S^{\alpha}_{n,m}(f;x^1,x^2) := \sum_{k=0}^{n-1}\sum_{i=0}^{m-1} \widehat{f}^{\alpha}(k,i)\alpha_{k,i}(x^1,x^2).$$

The Marcinkiewicz-Fejér means of a martingale f are defined by

$$\sigma^{\alpha}_{n}\left(f;x^1,x^2\right) := \frac{1}{n}\sum_{k=0}^{n-1} S^{\alpha}_{k,k}(f;x^1,x^2).$$

The two-dimensional Dirichlet kernels and Marcinkiewicz-Fejér kernels are defined by

$$D^{\alpha}_{k,l}(x^1,x^2) := D^{\alpha}_{k}(x^1)D^{\alpha}_{l}(x^2),\ \ K^{\alpha}_{n}(x^1,x^2) := \frac{1}{n}\sum_{k=0}^{n-1} D^{\alpha}_{k,k}(x^1,x^2).$$

The nth Marcinkiewicz-Fejér kernel has got a decomposition

$$\begin{aligned} nK^{\kappa}_{n}\left(x^1,x^2\right) &= 1 + \sum_{j=0}^{|n|-1} 2^j D_{2^j,2^j}\left(x^1,x^2\right) + \sum_{j=0}^{|n|-1} 2^j D_{2^j}\left(x^1\right) r_j\left(x^2\right) K^{w}_{2^j}\left(\tau_j\left(x^2\right)\right) \\ &+ \sum_{j=0}^{|n|-1} 2^j D_{2^j}\left(x^2\right) r_j\left(x^1\right) K^{w}_{2^j}\left(\tau_j\left(x^1\right)\right) \\ &+ \sum_{j=0}^{|n|-1} 2^j r_j\left(x^1+x^2\right) K^{w}_{2^j}\left(\tau_j\left(x^1\right),\tau_j\left(x^2\right)\right) \qquad (2) \\ &+ (n-2^{|n|})(D_{2^{|n|},2^{|n|}}(x^1,x^2) + D_{2^{|n|}}(x^1)r_{|n|}(x^2)K^{w}_{n-2^{|n|}}(\tau_{|n|}(x^2)) \\ &+ D_{2^{|n|}}(x^2)r_{|n|}(x^1)K^{w}_{n-2^{|n|}}(\tau_{|n|}(x^1)) \\ &+ r_{|n|}(x^1+x^2)K^{w}_{n-2^{|n|}}(\tau_{|n|}(x^1),\tau_{|n|}(x^2))) \end{aligned}$$

for $\left(x^1,x^2\right) \in G^2$ (see [9, Lemma 2.1]).

For the martingale f we consider the maximal operators

$$\sigma^{\kappa,*}f(x^1,x^2) = \sup_{n\in\mathbb{P}} |\sigma^{\kappa}_{n}(f;x^1,x^2)|, \qquad \tilde{\sigma}^{\kappa,*}f(x^1,x^2) = \sup_{n\in\mathbb{P}} \frac{|\sigma^{\kappa}_{n}(f;x^1,x^2)|}{\log^{3/2}(n+1)}.$$

In 1948 Šneider [15, page 184] introduced the Walsh-Kaczmarz system and showed that the inequality

$$\limsup_{n\to\infty} \frac{D_n^\kappa(x)}{\log n} \geqslant C > 0$$

holds a.e. In 1974 Schipp [12, Corollary 3] and Young [16, page 354] proved that the Walsh-Kaczmarz system is a convergence system. Skvortsov in 1981 [14, Theorem 2] showed that the Fejér means with respect to the Walsh-Kaczmarz system converge uniformly to f for any continuous functions f. Gát [2, Theorem 1] proved for any integrable functions, that the Fejér means with respect to the Walsh-Kaczmarz system converge almost everywhere to the function. He showed that the maximal operator $\sigma^{\kappa,*}$ of Walsh-Kaczmarz-Fejér means is of weak type $(1,1)$ and of type (p,p) for all $1 < p \leqslant \infty$. Gát's result was generalized by Simon [13, Theorem 1], who showed that the maximal operator $\sigma^{\kappa,*}$ is of type (H_p, L_p) for $p > 1/2$.

In the endpoint case $p = 1/2$ Goginava [5, Theorem 2] proved that the maximal operator $\sigma^{\kappa,*}$ is not of type $(H_{1/2}, L_{1/2})$ and Weisz [19, Theorem 5, page 162] showed that the maximal operator is of weak type $(H_{1/2}, L_{1/2})$.

In the paper [7, Theorem 3.1, Theorem 3.2] Goginava and the author proved that the maximal operator $\tilde{\sigma}^{\kappa,*}$ defined by

$$\tilde{\sigma}^{\kappa,*} := \sup_{n\in\mathbb{P}} \frac{|\sigma_n^\kappa f|}{\log^2(n+1)}$$

is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$. It was also proved that for any non-decreasing function $\varphi : \mathbb{P} \to [1,\infty)$ satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{\log^2(n+1)}{\varphi(n)} = +\infty \tag{3}$$

then the maximal operator $\sup_{n\in\mathbb{P}} \frac{|\sigma_n^\kappa f|}{\varphi(n)}$ is not bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$. In other words the order of deviant behaviour of the nth Walsh-Kaczmarz-Fejér mean is exactly $\log^2(n+1)$ in our special sense.

In 1939 for the two-dimensional trigonometric Fourier partial sums $S_{j,j}(f)$ Marcinkiewicz [8] has proved for $f \in L\log L([0,2\pi]^2)$ that the means

$$\sigma_n f = \frac{1}{n}\sum_{j=1}^{n} S_{j,j}(f)$$

converge a.e. to f as $n \to \infty$. Zhizhiashvili [21, page 1116] improved this result for $f \in L([0,2\pi]^2)$. We mention that the result of Marcinkiewicz and Zhizhiashvili and the boundedness of the maximal operator from H_p to L_p $(2/3 < p < \infty)$ was proved by Weisz for Walsh-Fourier series [20, Theorem 3].

In [4, Theorem 1] it was proved that the maximal operator

$$\sigma^{\kappa,*} f := \sup_{n\in\mathbb{P}} |\sigma_n^\kappa f| = \sup_{n\in\mathbb{P}} \frac{1}{n}\left|\sum_{j=0}^{n-1} S_{j,j}^\kappa f\right|$$

is bounded from the Hardy space H_p to the space L_p for $p > 2/3$.

In the paper [6, Theorem 1] Goginava and the author showed that, $\sigma^{\kappa,*} f$ is not bounded from the Hardy space $H_{2/3}$ to the space $L_{2/3}$.

The main aim of this paper is to investigate that what does happen in the endpoint $p = 2/3$. We show that the maximal operator $\tilde{\sigma}^{\kappa,*} f := \sup_{n\in\mathbb{P}} \frac{|\sigma_n^\kappa f|}{\log^{3/2}(n+1)}$, is bounded from the Hardy space $H_{2/3}$ into the space $L_{2/3}$. Moreover, we prove that for any non-decreasing function $\varphi : \mathbb{P} \to [1,\infty)$ satisfying the analogue of condition (3), that is

$$\overline{\lim_{n\to\infty}} \frac{\log^{3/2}(n+1)}{\varphi(n)} = +\infty \tag{4}$$

then the maximal operator $\sup_{A\in\mathbb{P}} \frac{|\sigma_n^\kappa f|}{\varphi(n)}$ is not bounded from the Hardy space $H_{2/3}$ to the space $L_{2/3}$. That is, the order of deviant behaviour of the nth Walsh-Kacmarz-Marcinkiewicz-Fejér mean is exactly $\log^{3/2}(n+1)$ in the endpoint case $p = 2/3$. Analogue of this result for Walsh-Marcinkiewicz-Fejér mean is given in [10, Theorem 1, Theorem 2].

2. Auxiliary propositions and main results

First, we formulate our main theorems.

THEOREM 1. *The maximal operator $\tilde{\sigma}^{\kappa,*}$ is bounded from the Hardy space $H_{2/3}$ to the space $L_{2/3}$.*

THEOREM 2. *Let $\varphi : \mathbb{P} \to [1,\infty)$ be a non-decreasing function satisfying the condition (4). Then the maximal operator*

$$\sup_{n\in\mathbb{P}} \frac{|\sigma_n^\kappa f|}{\varphi(n)}$$

is not bounded from the Hardy space $H_{2/3}$ to the space $L_{2/3}$.

To prove our Theorem 1 we need the following Lemmas [3, page 480–482], [4, Lemma 8, 9, 10]:

LEMMA 1. (Gát, Goginava, Nagy ([3], page 481)) *Let $x \in I_N(x_0,\ldots,x_{l-1},x_l = 1, 0,\ldots,0)$ and $j > N$. Then*

$$\int_{I_N} K_{2^j}^w(\tau_j(x+t))d\mu(t) \leqslant \frac{c}{2^l} 1_{I_N(0,\ldots,0,x_l=1,0,\ldots,0)}(x).$$

LEMMA 2. (Gát, Goginava, Nagy ([4], Lemma 8)) *Let $n < 2^{A+1}$, $A > N$ and $x \in I_N(x_0,\ldots,x_m = 1,0,\ldots,0,x_l = 1,0,\ldots,0)$, $l = 0,\ldots,N-1$, $m = -1,0,\ldots,l$. Then*

$$\int_{I_N} n\,|K_n^w(\tau_A(x+t))|\,d\mu(t) \leqslant c\frac{2^A}{2^{m+l}},$$

where

$$I_N\left(x_0,\ldots,x_m=1,0,\ldots,0,x_l=1,0,\ldots,0\right)$$
$$:=I_N\left(0,\ldots,0,x_l=1,0,\ldots,0\right),\ \textit{for } m=-1.$$

LEMMA 3. (Gát, Goginava, Nagy ([3], page 480)) *Let* $x^1\in I_N(x_0^1,\ldots,x_{s-1}^1,x_s^1=1,0,\ldots,0)$ *and* $x^2\in I_N(x_0^2,\ldots,x_{l-1}^2,x_l^2=1,0,\ldots,0)$ *for* $0\leqslant s\leqslant l<N$. *Then for* $j>N$ *we have*

$$\int_{I_N\times I_N} K_{2^j}^w\left(\tau_j\left(x^1+t^1\right),\tau_j\left(x^2+t^2\right)\right)d\mu(t^1,t^2)$$
$$\leqslant c\sum_{m=s}^{l}2^{-l-m}1_{I_N\left(x_0^1,\ldots,x_{s-1}^1,x_s^1=1,0,\ldots,0,x_m^2=1,0,\ldots,0,x_l^2=1,0,\ldots,0\right)}(x^2).$$

LEMMA 4. (Gát, Goginava, Nagy ([3], page 482)) *Let* $\left(x^1,x^2\right)\in I_N\times I_N(x_0^2,\ldots,x_{l-1}^2,x_l^2=1,0,\ldots,0)$, $l=0,\ldots,N-1$. *Then for* $j>N$ *we have*

$$\int_{I_N\times I_N} K_{2^j}^w\left(\tau_j\left(x^1+t^1\right),\tau_j\left(x^2+t^2\right)\right)d\mu(t^1,t^2)$$
$$\leqslant c\sum_{s=0}^{l}2^{-l-s}1_{I_N\left(0,\ldots,0,x_s^2=1,0,\ldots,0,x_l^2=1,0\ldots,0\right)}\left(x^2\right).$$

LEMMA 5. (Gát, Goginava, Nagy ([4], Lemma 9)) *Let* $(x^1,x^2)\in I_N(x_0^1,\ldots,x_{m^1}^1=1,0,\ldots,0)\times I_N(x_0^2,\ldots,x_{m^2}^2=1,0,\ldots,0)$, $m^1\leqslant m^2$, $A>N$ *and* $n<2^{A+1}$. *Then*

$$\int_{I_N\times I_N} n\left|K_n^w\left(\tau_A\left(x^1+t^1\right),\tau_A\left(x^2+t^2\right)\right)\right|d\mu(t^1,t^2)$$
$$\leqslant c\left\{\sum_{r=0}^{m^1-1}\sum_{q^2=m^1}^{m^2}\frac{2^A}{2^{m^2+q^2+r}}1_{I_N\left(x_0^2,\ldots,x_r^2,x_{r+1}^1,\ldots,x_{m^1-1}^1,x_{m^1}^2,0,\ldots,0,x_{q^2}^2=1,0,\ldots,0,x_{m^2}^2=1,0,\ldots,0\right)}\left(x^2\right)\right.$$
$$\left.+\sum_{r=m^1}^{m^2-1}\frac{2^A}{2^{m^1+m^2+r}}\sum_{q^2=r}^{m^2}1_{I_N\left(x_0^2,\ldots,x_r^2,0,\ldots,0,x_{q^2}^2=1,0\ldots,0,x_{m^2}^2=1,0,\ldots,0\right)}\left(x^2\right)\right\}.$$

LEMMA 6. (Gát, Goginava, Nagy ([4], Lemma 10)) *Let* $(x^1,x^2)\in I_N(0,\ldots,0)\times I_N(x_0^2,\ldots,x_{m^2}^2=1,0,\ldots,0)$, $A>N$ *and* $n<2^{A+1}$. *Then*

$$\int_{I_N\times I_N} n\left|K_n^w\left(\tau_A\left(x^1+t^1\right),\tau_A\left(x^2+t^2\right)\right)\right|dt^1dt^2$$
$$\leqslant c\left\{\sum_{r=0}^{m^2}\frac{2^A}{2^{m^2+r}}\sum_{q^2=r}^{m^2}1_{I_N\left(x_0^2,\ldots,x_r^2,0,\ldots,0,x_{q^2}^2=1,0\ldots,0,x_{m^2}^2=1,0,\ldots,0\right)}\left(x^2\right)\right\}.$$

A bounded measurable function a is a p-atom, if there exists a dyadic two-dimensional cube I^2, such that

a) $\int_{I^2} a d\mu = 0$,

b) $\|a\|_\infty \leqslant \mu(I^2)^{-1/p}$,

c) $\operatorname{supp} a \subseteq I^2$.

The operator T is said to be p-quasi-local, if there exists a constant c_p such that

$$\int_{\overline{I^2}} |Ta|^p d\mu \leqslant c_p < \infty \tag{5}$$

for every p-atom a, where the dyadic cube I^2 is the support of the p-atom a.

LEMMA 7. (Weisz ([18], Theorem 1.34)) *Suppose that the operator T is sub-linear and p-quasi-local for any $0 < p \leqslant 1$. If T is bounded from L_∞ to L_∞, then*

$$\|Tf\|_p \leqslant c_p \|f\|_{H_p} \quad \textit{for all } f \in H_p.$$

3. Proofs of the theorems

First, we prove Theorem 1.

Proof of Theorem 1. By the help of Lemma 7 we prove that the operator $\tilde{\sigma}^{\kappa,*}$ is of type $(H_{2/3}, L_{2/3})$. The boundedness from the space L_∞ to the space L_∞ follows from the inequality

$$\|K_n^\kappa\|_1 \leqslant c$$

for all $n \in \mathbb{N}$ (see [4, Corollary 3]). The proof will be complete, if we show that the maximal operator $\tilde{\sigma}^{\kappa,*}$ is $2/3$-quasi-local (see inequality (5)).

Let a be an arbitrary $2/3$-atom with $\operatorname{supp} a = I^2$, and $\mu(I^2) = 2^{-2N}$. Without loss of generality, we may assume that $I^2 := I_N \times I_N$.

It is simple to see that $\sigma_n^\kappa a = 0$ if $n \leqslant 2^N$. Therefore, we suppose that $n > 2^N$.

We write that

$$\begin{aligned}\int_{\overline{I_N^2}} |\tilde{\sigma}^{\kappa,*} a|^{2/3} d\mu &= \int_{I_N \times \overline{I_N}} |\tilde{\sigma}^{\kappa,*} a|^{2/3} d\mu + \int_{\overline{I_N} \times I_N} |\tilde{\sigma}^{\kappa,*} a|^{2/3} d\mu + \int_{\overline{I_N} \times \overline{I_N}} |\tilde{\sigma}^{\kappa,*} a|^{2/3} d\mu \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

By equality (2) and the property a) of an $2/3$-atom for any $(x^1, x^2) \in \overline{I_N^2}$ we have that

$$\begin{aligned} n\sigma_n^\kappa a(x^1, x^2) = \int_{I_N^2} a(t^1, t^2) &\left(\sum_{j=N+1}^{|n|-1} 2^j D_{2^j}\left(x^1 + t^1\right) r_j\left(x^2 + t^2\right) K_{2^j}^w\left(\tau_j\left(x^2 + t^2\right)\right) \right. \\ &+ \sum_{j=N+1}^{|n|-1} 2^j D_{2^j}\left(x^2 + t^2\right) r_j\left(x^1 + t^1\right) K_{2^j}^w\left(\tau_j\left(x^1 + t^1\right)\right) \\ &+ \sum_{j=N+1}^{|n|-1} 2^j r_j\left(x^1 + t^1 + x^2 + t^2\right) K_{2^j}^w\left(\tau_j\left(x^1 + t^1\right), \tau_j\left(x^2 + t^2\right)\right) \end{aligned}$$

$$+(n-2^{|n|})D_{2^{|n|}}(x^1+t^1)r_{|n|}(x^2+t^2)K^w_{n-2^{|n|}}(\tau_{|n|}(x^2+t^2))$$
$$+(n-2^{|n|})D_{2^{|n|}}(x^2+t^2)r_{|n|}(x^1+t^1)K^w_{n-2^{|n|}}(\tau_{|n|}(x^1+t^1))$$
$$\left.+(n-2^{|n|})r_{|n|}(x^1+t^1+x^2+t^2)K^w_{n-2^{|n|}}(\tau_{|n|}(x^1+t^1),\tau_{|n|}(x^2+t^2))\right)d\mu(t^1,t^2)$$
$$=: L_n^1a(x^1,x^2)+L_n^2a(x^1,x^2)+L_n^3a(x^1,x^2)+L_n^4a(x^1,x^2)+L_n^5a(x^1,x^2)+L_n^6a(x^1,x^2).$$

First, we discuss the integral I_3. On the set $\overline{I_N}\times\overline{I_N}$ we have that $L_n^1a=L_n^2a=L_n^4a=L_n^5a=0$. Thus, we write that

$$\sigma_n^{\kappa}a(x^1,x^2)=\frac{L_n^3a(x^1,x^2)}{n}+\frac{L_n^6a(x^1,x^2)}{n}.$$

For a $2/3$-atom a we have that

$$\|a\|_\infty\leqslant 2^{3N}. \tag{6}$$

This yields

$$I_3\leqslant\frac{c2^{2N}}{N}\int_{\overline{I_N}\times\overline{I_N}}\left(\sup_{n>2^N}\frac{1}{n}\sum_{j=N+1}^{|n|-1}2^j\int_{I_N^2}\left|K^w_{2^j}\left(\tau_j\left(x^1+t^1\right),\tau_j\left(x^2+t^2\right)\right)\right|d\mu(t^1,t^2)\right.$$
$$\left.+\sup_{n>2^N}\frac{n-2^{|n|}}{n}\int_{I_N^2}\left|K^w_{n-2^{|n|}}\left(\tau_{|n|}\left(x^1+t^1\right),\tau_{|n|}\left(x^2+t^2\right)\right)\right|d\mu(t^1,t^2)\right)^{2/3}d\mu(x^1,x^2)$$

Now, we decompose the set $\overline{I_N}$ in the following way

$$\overline{I_N}=\bigcup_{s=0}^{N-1}J_N^s, \tag{7}$$

where $J_N^s:=I_N(x_0,x_1,\ldots,x_{s-1},x_s=1,0,\ldots,0)$. By this, we get that

$$I_3\leqslant\frac{c2^{2N}}{N}\sum_{l=0}^{N-1}\sum_{s=0}^{l}\int_{J_N^s\times J_N^l}\left(\sup_{n>2^N}\frac{1}{n}\sum_{j=N+1}^{|n|-1}2^j\int_{I_N^2}\left|K^w_{2^j}\left(\tau_j\left(x^1+t^1\right),\tau_j\left(x^2+t^2\right)\right)\right|d\mu(t^1,t^2)\right.$$
$$\left.+\sup_{n>2^N}\frac{n-2^{|n|}}{n}\int_{I_N^2}\left|K^w_{n-2^{|n|}}\left(\tau_{|n|}\left(x^1+t^1\right),\tau_{|n|}\left(x^2+t^2\right)\right)\right|d\mu(t^1,t^2)\right)^{2/3}d\mu(x^1,x^2)$$
$$+\frac{c2^{2N}}{N}\sum_{l=0}^{N-1}\sum_{s=l+1}^{N-1}\int_{J_N^s\times J_N^l}\left(\sup_{n>2^N}\frac{1}{n}\sum_{j=N+1}^{|n|-1}2^j\int_{I_N^2}\left|K^w_{2^j}\left(\tau_j\left(x^1+t^1\right),\tau_j\left(x^2+t^2\right)\right)\right|d\mu(t^1,t^2)\right.$$
$$\left.+\sup_{n>2^N}\frac{n-2^{|n|}}{n}\int_{I_N^2}\left|K^w_{n-2^{|n|}}\left(\tau_{|n|}\left(x^1+t^1\right),\tau_{|n|}\left(x^2+t^2\right)\right)\right|d\mu(t^1,t^2)\right)^{2/3}d\mu(x^1,x^2)$$
$$=: I_3^1+I_3^2.$$

We investigate I_3^1 (I_3^2 can be discussed analogously). Using Lemma 3, Lemma 5 and the inequality

$$\left(\sum_{k=0}^{\infty} a_k\right)^p \leqslant \sum_{k=0}^{\infty} a_k^p \quad (a_k \geqslant 0,\ 0 < p \leqslant 1) \tag{8}$$

we immediately get that

$$\begin{aligned}
I_3^1 \leqslant & \frac{c2^{2N}}{N} \sum_{l=0}^{N-1} \sum_{s=0}^{l} \int_{J_N^s \times J_N^l} \left(\sum_{m=s}^{l} 2^{-l-m} 1_{I_N\left(x_0^1,\ldots,x_{s-1}^1,x_s^1=1,0,\ldots,0,x_m^2=1,0,\ldots,0,x_l^2=1,0,\ldots,0\right)}(x^2) \right. \\
& + \sum_{r=0}^{s-1} \sum_{q^2=s}^{l} 2^{-l-q^2-r} 1_{I_N(x_0^2,\ldots,x_r^2,x_{r+1}^1,\ldots,x_{s-1}^1,x_s^2,0,\ldots,0,x_{q^2}^2=1,0,\ldots,0,x_l^2=1,0,\ldots,0)}(x^2) \\
& \left. + \sum_{r=s}^{l-1} 2^{-s-l-r} \sum_{q^2=r}^{l} 1_{I_N(x_0^2,\ldots,x_r^2,0,\ldots,0,x_{q^2}^2=1,0,\ldots,0,x_l^2=1,0,\ldots,0)}(x^2) \right)^{2/3} d\mu(x^1,x^2) \\
\leqslant & \frac{c2^{2N}}{N} \sum_{l=0}^{N-1} \sum_{s=0}^{l} \int_{J_N^s \times J_N^l} \left(\sum_{m=s}^{l} 2^{(-l-m)2/3} 1_{I_N\left(x_0^1,\ldots,x_{s-1}^1,x_s^1=1,0,\ldots,0,x_m^2=1,0,\ldots,0,x_l^2=1,0,\ldots,0\right)}(x^2) \right. \\
& + \sum_{r=0}^{s-1} \sum_{q^2=s}^{l} 2^{(-l-q^2-r)2/3} 1_{I_N(x_0^2,\ldots,x_r^2,x_{r+1}^1,\ldots,x_{s-1}^1,x_s^2,0,\ldots,0,x_{q^2}^2=1,0,\ldots,0,x_l^2=1,0,\ldots,0)}(x^2) \\
& \left. + \sum_{r=s}^{l-1} 2^{(-s-l-r)2/3} \sum_{q^2=r}^{l} 1_{I_N(x_0^2,\ldots,x_r^2,0,\ldots,0,x_{q^2}^2=1,0,\ldots,0,x_l^2=1,0,\ldots,0)}(x^2) \right) d\mu(x^1,x^2) \\
=: & \sum_1 + \sum_2 + \sum_3 .
\end{aligned}$$

We discuss $\sum_1$.

$$\begin{aligned}
\sum_1 \leqslant & \frac{c2^{2N}}{N} \sum_{l=0}^{N-1} \sum_{s=0}^{l} \sum_{x_0^1=0}^{1} \cdots \sum_{x_{s-1}^1=0}^{1} \sum_{m=s}^{l} \times \\
& \times \int_{I_N(x_0^1,\ldots,x_{s-1}^1,x_s^1=1,0,\ldots,0) \times I_N(x_0^1,\ldots,x_{s-1}^1,x_s^1=1,0,\ldots,0,x_m^2=1,0,\ldots,0,x_l^2=1,0,\ldots,0)} 2^{\frac{-2l-2m}{3}} d\mu(x^1,x^2) \\
\leqslant & \frac{c2^{2N}}{N} \sum_{l=0}^{N-1} 2^{-2l/3} \sum_{s=0}^{l} 2^s 2^{-2s/3} 2^{-2N} \leqslant \frac{c}{N} \sum_{l=0}^{N} 2^{-2l/3} 2^{l/3} \leqslant c.
\end{aligned}$$

Analogously,

$$\begin{aligned}
\sum_2 \leqslant & \frac{c2^{2N}}{N} \sum_{l=0}^{N-1} 2^{-2l/3} \sum_{s=0}^{l} \sum_{r=0}^{s-1} 2^{-2r/3} \sum_{q^2=s}^{l} 2^{-2q^2/3} 2^{-(N-s)} 2^{-(N-r)} \\
\leqslant & \frac{c}{N} \sum_{l=0}^{N-1} 2^{-2l/3} 2^{2l/3} \leqslant \frac{c}{N} N \leqslant c.
\end{aligned}$$

Now, $\sum_3$ follows.

$$\begin{aligned}\sum_3 &\leqslant \frac{c2^{2N}}{N}\sum_{l=0}^{N-1}2^{-2l/3}\sum_{s=0}^{l}2^{-2s/3}\sum_{r=s}^{l-1}2^{-2r/3}\sum_{q^2=r}^{l}2^{-(N-s)}2^{-(N-r)}\\ &\leqslant \frac{c}{N}\sum_{l=0}^{N-1}2^{-2l/3}2^{2l/3}\leqslant \frac{c}{N}N\leqslant c.\end{aligned}$$

Second, we discuss the integral I_1 (the discussion of I_2 goes similarly). On the set $I_N \times \overline{I_N}$ we have that $L_n^2 a = L_n^5 a = 0$. That is,

$$\sigma_n^{\kappa} a(x^1,x^2) = \frac{L_n^1 a(x^1,x^2)}{n} + \frac{L_n^3 a(x^1,x^2)}{n} + \frac{L_n^4 a(x^1,x^2)}{n} + \frac{L_n^6 a(x^1,x^2)}{n}.$$

Moreover, by inequality (6), decomposition (7), Lemma 1, 2, 4 and 6 we have that

$$\begin{aligned} I_1 \leqslant \frac{c2^{2N}}{N}\sum_{l=0}^{N-1}\int_{I_N\times J_N^l}\Bigg(&\sup_{n>2^N}\frac{1}{n}\sum_{j=N+1}^{|n|-1}2^j\int_{I_N}K_{2^j}^w\left(\tau_j\left(x^2+t^2\right)\right)d\mu(t^2)\\ &+\sup_{n>2^N}\frac{1}{n}\sum_{j=N+1}^{|n|-1}2^j\int_{I_N^2}K_{2^j}^w\left(\tau_j\left(x^1+t^1\right),\tau_j\left(x^2+t^2\right)\right)d\mu(t^1,t^2)\\ &+\sup_{n>2^N}\frac{n-2^{|n|}}{n}\int_{I_N}|K_{n-2^{|n|}}^w\left(\tau_{|n|}\left(x^2+t^2\right)\right)|d\mu(t^2)\\ &+\sup_{n>2^N}\frac{n-2^{|n|}}{n}\int_{I_N^2}|K_{n-2^{|n|}}^w\left(\tau_{|n|}\left(x^1+t^1\right),\tau_{|n|}\left(x^2+t^2\right)\right)|d\mu(t^1,t^2)\Bigg)^{2/3}d\mu(x^1,x^2)\\ \leqslant \frac{c2^{2N}}{N}\sum_{l=0}^{N-1}\int_{I_N\times J_N^l}\Bigg(&\sup_{n>2^N}\frac{1}{n}\sum_{j=N+1}^{|n|-1}2^{j-l}1_{I_N(0,\ldots,0,x_l^2=1,0,\ldots,0)}(x^2)\\ &+\sup_{n>2^N}\frac{1}{n}\sum_{j=N+1}^{|n|-1}2^j\sum_{s=0}^{l}2^{-l-s}1_{I_N(0,\ldots,0,x_s^2=1,0,\ldots,0,x_l^2=1,0,\ldots,0)}(x^2)\\ &+\sup_{n>2^N}\frac{1}{n}\sum_{m=-1}^{l}\frac{2^{|n|}}{2^{l+m}}1_{I_N(x_0^2,\ldots,x_{m-1}^2,x_m^2=1,0,\ldots,0,x_l^2=1,0,\ldots,0)}(x^2)\\ &+\sup_{n>2^N}\frac{1}{n}\sum_{r=0}^{l}\frac{2^{|n|}}{2^{l+r}}\sum_{q^2=r}^{l}1_{I_N(x_0^2,\ldots,x_r^2,0,\ldots,0,x_{q^2}^2=1,0,\ldots,0,x_l^2=1,0,\ldots,0)}(x^2)\Bigg)^{2/3}d\mu(x^1,x^2)\end{aligned}$$

A simple consideration and inequality (8) yield that

$$
\begin{aligned}
I_1 \leqslant & \frac{c2^N}{N}\sum_{l=0}^{N-1} 2^{-2l/3}\int_{J_N^l} 1_{I_N(0,\dots,0,x_l^2=1,0,\dots,0)}(x^2)d\mu(x^2)\\
&+\frac{c2^N}{N}\sum_{l=0}^{N-1}\sum_{s=0}^{l} 2^{(-l-s)2/3}\int_{J_N^l} 1_{I_N(0,\dots,0,x_s^2=1,0,\dots,0,x_l^2=1,0,\dots,0)}(x^2)d\mu(x^2)\\
&+\frac{c2^N}{N}\sum_{l=0}^{N-1}\sum_{m=-1}^{l} 2^{(-l-m)2/3}\int_{J_N^l} 1_{I_N(x_0^2,\dots,x_{m-1}^2,x_m^2=1,0,\dots,0,x_l^2=1,0,\dots,0)}(x^2)d\mu(x^2)\\
&+\frac{c2^N}{N}\sum_{l=0}^{N-1}\sum_{r=0}^{l} 2^{(-l-r)2/3}\sum_{q^2=r}^{l}\int_{J_N^l} 1_{I_N(x_0^2,\dots,x_r^2,0,\dots,0,x_{q^2}^2=1,0,\dots,0,x_l^2=1,0,\dots,0)}(x^2)d\mu(x^2)\\
=: & \sum^{1}+\sum^{2}+\sum^{3}+\sum^{4}.
\end{aligned}
$$

Thus,

$$\sum^{1}\leqslant \frac{c2^N}{N}\sum_{l=0}^{N-1}2^{-2l/3}2^{-N}\leqslant \frac{c}{N}\leqslant c,$$

$$\sum^{2}\leqslant \frac{c2^N}{N}\sum_{l=0}^{N-1}\sum_{s=0}^{l}2^{(-l-s)2/3}2^{-N}\leqslant \frac{c}{N}\leqslant c,$$

$$\sum^{3}\leqslant \frac{c2^N}{N}\sum_{l=0}^{N-1}\sum_{m=-1}^{l}2^{(-l-m)2/3}2^{-(N-m)}\leqslant \frac{c}{N}\sum_{l=0}^{N-1}2^{-2l/3}2^{l/3}\leqslant \frac{c}{N}\leqslant c$$

and

$$\sum^{4}\leqslant \frac{c2^N}{N}\sum_{l=0}^{N-1}\sum_{r=0}^{l}2^{(-l-r)2/3}\sum_{q^2=r}^{l}2^{-(N-r)}\leqslant \frac{c}{N}\sum_{l=0}^{N-1}2^{-2l/3}\sum_{r=0}^{l}(l-r+1)2^{r/3}\leqslant \frac{c}{N}\leqslant c.$$

This completes the proof of Theorem 1. □

Next, we prove Theorem 2. We use the counterexample function and the idea given in [10, page 642], but we have to make the necessary changes. Let

$$f_A(x^1,x^2) := (D_{2^{A+1}}(x^1)-D_{2^A}(x^1))(D_{2^{A+1}}(x^2)-D_{2^A}(x^2)).$$

A simple calculation yields

$$\widehat{f}_A^{\kappa}(i,k)=\begin{cases}1, & \text{if } i,k=2^A,\dots,2^{A+1}-1,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$S_{i,j}^{\kappa}(f;x^1,x^2)=$$

$$\begin{cases}(D_i^{\kappa}(x^1)-D_{2^A}(x^1))(D_j^{\kappa}(x^2)-D_{2^A}(x^2)), & \text{if } i,j=2^A+1,\dots,2^{A+1}-1,\\ f_A(x^1,x^2), & \text{if } i,j\geqslant 2^{A+1},\\ 0, & \text{otherwise.}\end{cases}$$

We have
$$f_A^*(x^1,x^2) = \sup_{n\in\mathbb{N}} |S_{2^n,2^n}(f_A;x^1,x^2)| = |f_A(x^1,x^2)|$$
and
$$\|f_A\|_{H_{2/3}} = \|f_A^*\|_{2/3} = c2^{-A}. \tag{9}$$

Proof of Theorem 2. We can write the nth Walsh-Kaczmarz-Dirichlet kernel in the following form:
$$D_n^\kappa(x) = D_{2^{|n|}}(x) + r_{|n|}(x)D^w_{n-2^{|n|}}(\tau_{|n|}(x)).$$

Thus, we have for a non-decreasing function φ that
$$\begin{aligned}
\tilde{\sigma}^{\kappa,*} f_A(x^1,x^2) &= \sup_{n\in\mathbb{P}} \frac{|\sigma_n^\kappa(f_A;x^1,x^2)|}{\varphi(n)} \geqslant \max_{t:1\leqslant 2^t\leqslant 2^A} \frac{|\sigma^\kappa_{2^A+2^t}(f_A;x^1,x^2)|}{\varphi(2^A+2^t)}\\
&\geqslant \max_{t:1\leqslant 2^t\leqslant 2^A} \frac{1}{(2^A+2^t)\varphi(2^A+2^t)} \left|\sum_{k=0}^{2^A+2^t-1} S^\kappa_{k,k}(f_A;x^1,x^2)\right|\\
&= \max_{t:1\leqslant 2^t\leqslant 2^A} \frac{1}{2^{A+1}\varphi(2^{A+1})} \left|\sum_{k=2^A+1}^{2^A+2^t-1} r_A(x^1)D^w_{k-2^A}(\tau_A(x^1))r_A(x^2)D^w_{k-2^A}(\tau_A(x^2))\right|\\
&\geqslant \frac{1}{2^{A+1}\varphi(2^{A+1})} \max_{t:1\leqslant 2^t\leqslant 2^A} 2^t|K^w_{2^t}(\tau_A(x^1),\tau_A(x^2))|.
\end{aligned}$$

By this and inequality (9) we obtain
$$\frac{\|\tilde{\sigma}^{\kappa,*} f_A\|_{2/3}}{\|f_A\|_{H_{2/3}}} \geqslant \frac{c}{2^A\varphi(2^{A+1})2^{-A}} \left(\int_{G^2} \max_{t:1\leqslant 2^t\leqslant 2^A} (2^t|K^w_{2^t}(\tau_A(x^1),\tau_A(x^2))|)^{2/3} d\mu(x^1,x^2)\right)^{3/2}.$$

Now, we decompose the set G as the following disjoint union
$$G = I_A \cup \bigcup_{t=0}^{A-1} J_t^A,$$
where $J_t^A := \{x\in G : x_{A-1},\dots,x_{A-t} = 0, x_{A-t-1} = 1\}$ for $A > t \geqslant 1$ and $J_0^A := \{x \in G : x_{A-1} = 1\}$. Notice that $\tau_A(J_t^A) = I_t\backslash I_{t+1}$. It is easy to show that, for $(x^1,x^2)\in I_s\times I_s$
$$K^w_{2^s}(x^1,x^2) = \frac{(2^s-1)(2^{s+1}-1)}{6}.$$

Therefore,
$$\begin{aligned}
&\int_{G\times G} \max_{t:1\leqslant 2^t\leqslant 2^A} (2^t|K^w_{2^t}(\tau_A(x^1),\tau_A(x^2))|)^{2/3} d\mu(x^1,x^2)\\
&\qquad \geqslant \sum_{s=1}^{A-1} \int_{J_s^A\times J_s^A} \max_{t:1\leqslant 2^t\leqslant 2^A} (2^t K^w_{2^t}(\tau_A(x^1),\tau_A(x^2)))^{2/3} d\mu(x^1,x^2)
\end{aligned}$$

$$\begin{aligned}
&\geqslant \sum_{s=1}^{A-1} \int_{(I_s\setminus I_{s+1})\times(I_s\setminus I_{s+1})} (2^s K_{2^s}^w(x^1,x^2))^{2/3} d\mu(x^1,x^2) \\
&\geqslant c \sum_{s=1}^{A-1} \int_{(I_s\setminus I_{s+1})\times(I_s\setminus I_{s+1})} \left(2^{3s}\right)^{2/3} d\mu(x^1,x^2) \\
&\geqslant c(A-1).
\end{aligned}$$

That is,

$$\frac{\|\tilde{\sigma}^{\kappa,*} f_A\|_{2/3}}{\|f_A\|_{H_{2/3}}} \geqslant \frac{c(A+1)^{3/2}}{\varphi(2^{A+1})}$$

for A big enough.

From now, the proof goes analogously as in the paper [10, page 644] we did. This completes the proof of this theorem. □

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