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HILBERT–TYPE INEQUALITIES INVOLVING DIFFERENTIAL OPERATORS, THE BEST CONSTANTS, AND APPLICATIONS

VANDANJAV ADIYASUREN, TSERENDORJ BATBOLD AND MARIO KRNIĆ

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Abstract. Motivated by some recent results, in this article we derive several Hilbert-type inequalities with a differential operator, regarding a general homogeneous kernel. Moreover, we show that the constants appearing on the right-hand sides of these inequalities are the best possible. The general results are then applied to some particular examples of homogeneous kernels and compared with previously known from the literature.

1. Introduction

Although classical, the famous Hilbert inequality (see [6]) is still of interest to numerous authors. Recently, Azar [2], obtained a new form of this inequality including a differential operator. In order to state that result and summarize our further discussion, we start by giving some notation. We denote by $\mathscr{D}_+^n$, $n \geqslant 0$, a differential operator defined by $\mathscr{D}_+^n f(x) = f^{(n)}(x)$, where $f^{(n)}$ stands for the n-th derivative of a function $f : \mathbb{R}_+ \to \mathbb{R}$. In addition, throughout this article, Λ_+^n denotes the set of non-negative measurable functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that $f^{(n)}$ exists a.e. on $\mathbb{R}_+$, $f^{(n)}(x) > 0$, a.e. on $\mathbb{R}_+$, and $f^{(k)}(0) = 0$, $k = 0,1,2,\dots,n-1$.

Now, the above mentioned form of the Hilbert inequality obtained in [2] reads as follows: Let p and q be non-negative mutually conjugate parameters, i.e. $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, let $\lambda > n\max\{p,q\}$, and let $A = \frac{\Gamma(\frac{\lambda}{p}-n)\Gamma(\frac{\lambda}{q}-n)}{\Gamma(\lambda)}$, where Γ is a usual Gamma function. Then the inequality

$$\begin{aligned}&\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\frac{f(x)g(y)}{(x+y)^\lambda}dxdy\\ &< A\left[\int_{\mathbb{R}_+} x^{p(n+1)-\lambda-1}\left(\mathscr{D}_+^n f(x)\right)^p dx\right]^{\frac{1}{p}}\left[\int_{\mathbb{R}_+} y^{q(n+1)-\lambda-1}\left(\mathscr{D}_+^n g(y)\right)^q dy\right]^{\frac{1}{q}}\end{aligned} \tag{1}$$

holds for all $f,g \in \Lambda_+^n$, provided that the integrals on its right-hand side converge. In addition, the constant A is the best possible in the sense that it can not be replaced

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with a smaller constant so that (1) still holds for all $f,g \in \Lambda_+^n$. The above inequality may be regarded as a generalization of a classical Hilbert inequality since for $n=0$, $p=q=2$, and $\lambda=1$, we obtain the non-weighted inequality with the previously known sharp constant $A=\pi$ (for more details, see [6]).

The main objective of this paper is to extend inequality (1) to hold for a general homogeneous function. Recall that a function $K_\lambda : \mathbb{R}_+^2 \to \mathbb{R}$ is said to be homogeneous of degree $-\lambda$, $\lambda>0$, if $K_\lambda(tx,ty)=t^{-\lambda}K_\lambda(x,y)$ for every $x,y,t \in \mathbb{R}_+$. In addition, for such homogeneous function we define the constant $k(\alpha)$ as

$$k(\alpha)=\int_{\mathbb{R}_+} K_\lambda(1,t)t^\alpha dt,$$

provided that the above integral converges for $-1<\alpha<\lambda-1$.

In this regard, Perić and Vuković [10], obtained the following pair of equivalent Hilbert-type inequalities: Assume that $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, $K_\lambda : \mathbb{R}_+^2 \to \mathbb{R}$ is a homogeneous function of degree $-\lambda$, $\lambda>0$, and let $\alpha_1,\alpha_2 \in (-1,\lambda-1)$ be real parameters such that $\alpha_1+\alpha_2=\lambda-2$. Then the inequalities

$$\begin{aligned}&\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K_\lambda(x,y)f(x)g(y)dxdy\\ <&k(\alpha_2)\left[\int_{\mathbb{R}_+} x^{-p\alpha_1-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{\mathbb{R}_+} y^{-q\alpha_2-1}g^q(y)dy\right]^{\frac{1}{q}}\end{aligned} \tag{2}$$

and

$$\begin{aligned}&\left[\int_{\mathbb{R}_+} y^{(p-1)(1+q\alpha_2)}\left(\int_{\mathbb{R}_+} K_\lambda(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}}\\ <&k(\alpha_2)\left[\int_{\mathbb{R}_+} x^{-p\alpha_1-1}f^p(x)dx\right]^{\frac{1}{p}}\end{aligned} \tag{3}$$

hold for all measurable, a.e. positive functions f and g, provided that the integrals on the right-hand sides of these relations converge. In addition, the constant $k(\alpha_2)$ is the best possible in both inequalities (see also [8]). The equivalence means that one inequality implies the other and vice versa (for more details, see [7] and [10]). Inequalities (2) and (3) will be an important tool in our extension of relation (1) to a homogeneous case.

Moreover, in contrast to the technique of proving in [2], our generalization will be established by virtue of another famous inequality, that is, the Hardy inequality. In 1928, Hardy [5], proved an estimate for the integration operator (or the Hardy operator)

$$\mathscr{H}f(x)=\int_0^x f(t)dt,$$

from which the first weighted modification of the Hardy inequality followed, namely the inequality

$$\int_{\mathbb{R}_+} x^{-r}(\mathscr{H}f(x))^p dx < \left(\frac{p}{r-1}\right)^p \int_{\mathbb{R}_+} x^{p-r}f^p(x)dx, \tag{4}$$

valid with $p>1$, $r>1$, $0<\int_{\mathbb{R}_+} x^{p-r} f^p(x)dx<\infty$, where the constant $\left(\frac{p}{r-1}\right)^p$ is the best possible (for more details, see [6], Theorem 330, and [9]).

The paper is divided into three sections as follows: After this Introduction, in Section 2 we derive extensions of inequalities (2) and (3), in view of relation (1). Moreover, we also derive dual forms of these inequalities, based on the application of a dual Hardy inequality. It is interesting that the constants appearing in these inequalities are also expressed in terms of the Gamma function. In addition, we show that these constants remain the best possible. Furthermore, in Section 3 we consider our main results in some particular settings and compare them with some previously known from the literature.

Techniques that will be used in the proofs are mainly based on classical real analysis. Moreover, by $\Gamma(\cdot)$ we denote the usual Gamma function defined by

$$\Gamma(a)=\int_0^\infty t^{a-1}e^{-t}dt, \qquad a>0.$$

Finally, if nothing else is explicitly stated, all integrals in this paper are assumed to converge.

2. Main results

In this section, we give our main results, especially the extension of inequality (1) for the case of an arbitrary homogeneous kernel. The corresponding inequalities will be given in both equivalent forms, as (2) and (3).

In contrast to the proof of inequality (1) (see [2]), the following inequalities will be carried out by virtue of the Hardy inequality (4). Moreover, we shall also derive appropriate complementary relations, based on the application of the so-called dual Hardy inequality.

It is interesting that the constants appearing in our extended inequalities are also expressed in terms of the Gamma function. Therefore, it is necessary to introduce the concept of rising and falling factorial powers.

The rising factorial power $x^{\overline{n}}$, where n is a non-negative integer, also known as a Pochhammer symbol, is defined by

$$x^{\overline{n}}=x(x+1)(x+2)\cdots(x+n-1),$$

while the falling factorial power $x^{\underline{n}}$ is given by

$$x^{\underline{n}}=x(x-1)(x-2)\cdots(x-n+1).$$

The rising and falling factorial powers may be expressed in terms of the usual Gamma function, i.e.

$$x^{\overline{n}}=\frac{\Gamma(x+n)}{\Gamma(x)} \quad \text{and} \quad x^{\underline{n}}=\frac{\Gamma(x+1)}{\Gamma(x-n+1)}.$$

It should be noticed here that the above relations hold for complex arguments of the Gamma function which are not negative integers (for more details, see e.g. [1] or [4]).

Now, we are ready to state and prove our main result which is an extension of inequality (1).

THEOREM 1. *Let* $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, *and let* α_1, α_2 *be real parameters such that* $\alpha_1, \alpha_2 \in (n-1, \lambda-1)$ *and* $\alpha_1+\alpha_2=\lambda-2$, *where* n *is a fixed non-negative integer and* $\lambda>n$. *If* $K_\lambda : \mathbb{R}_+^2 \to \mathbb{R}$ *is a non-negative measurable homogeneous function of degree* $-\lambda$, *then the inequalities*

$$\begin{aligned}&\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K_\lambda(x,y)f(x)g(y)dxdy\\ <&M\left[\int_{\mathbb{R}_+} x^{p(n-\alpha_1)-1}\left(\mathscr{D}_+^n f(x)\right)^p dx\right]^{\frac{1}{p}}\left[\int_{\mathbb{R}_+} y^{q(n-\alpha_2)-1}\left(\mathscr{D}_+^n g(y)\right)^q dy\right]^{\frac{1}{q}}\end{aligned} \tag{5}$$

and

$$\begin{aligned}&\left[\int_{\mathbb{R}_+} y^{(p-1)(1+q\alpha_2)}\left(\int_{\mathbb{R}_+} K_\lambda(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}}\\ <&m\left[\int_{\mathbb{R}_+} x^{p(n-\alpha_1)-1}\left(\mathscr{D}_+^n f(x)\right)^p dx\right]^{\frac{1}{p}}\end{aligned} \tag{6}$$

hold for all non-negative functions $f, g \in \Lambda_+^n$.

In addition, the constants $M=k(\alpha_2)\frac{\Gamma(\alpha_1-n+1)\Gamma(\alpha_2-n+1)}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}$ *and* $m=k(\alpha_2)\frac{\Gamma(\alpha_1-n+1)}{\Gamma(\alpha_1+1)}$ *are the best possible in the corresponding inequalities.*

Proof. Obviously, if $n=0$ inequalities (5) and (6) become respectively (2) and (3). Now, our first step is to rewrite the right-hand side of inequality (2) in a form that is more suitable for the application of the Hardy inequality. Namely, since

$$\mathscr{H}(\mathscr{D}_+ f)(x)=\int_0^x f'(t)dt=f(x)-f(0)=f(x),$$

we have that

$$\begin{aligned}&k(\alpha_2)\left[\int_{\mathbb{R}_+} x^{-p\alpha_1-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{\mathbb{R}_+} y^{-q\alpha_2-1}g^q(y)dy\right]^{\frac{1}{q}}\\ =&k(\alpha_2)\left[\int_{\mathbb{R}_+} x^{-(p\alpha_1+1)}(\mathscr{H}(\mathscr{D}_+ f)(x))^p dx\right]^{\frac{1}{p}}\\ &\times\left[\int_{\mathbb{R}_+} y^{-(q\alpha_2+1)}(\mathscr{H}(\mathscr{D}_+ g)(y))^q dy\right]^{\frac{1}{q}}.\end{aligned} \tag{7}$$

Moreover, due to the weighted Hardy inequality (4), it follows that

$$\left[\int_{\mathbb{R}_+} x^{-(p\alpha_1+1)}(\mathscr{H}(\mathscr{D}_+ f)(x))^p dx\right]^{\frac{1}{p}}<\frac{1}{\alpha_1}\left[\int_{\mathbb{R}_+} x^{p(1-\alpha_1)-1}(\mathscr{D}_+ f(x))^p dx\right]^{\frac{1}{p}}$$

and

$$\left[\int_{\mathbb{R}_+} y^{-(q\alpha_2+1)}(\mathscr{H}(\mathscr{D}_+ g)(y))^q dy\right]^{\frac{1}{q}}<\frac{1}{\alpha_2}\left[\int_{\mathbb{R}_+} y^{q(1-\alpha_2)-1}(\mathscr{D}_+ g(y))^q dy\right]^{\frac{1}{q}}.$$

In addition, applying the Hardy inequality to the right-hand sides of the last two inequalities $n-1$ times, yields relations

$$\left[\int_{\mathbb{R}_+} x^{-(p\alpha_1+1)}(\mathscr{H}(\mathscr{D}_+f)(x))^p dx\right]^{\frac{1}{p}} < \frac{1}{\alpha_1^{\underline{n}}}\left[\int_{\mathbb{R}_+} x^{p(n-\alpha_1)-1}(\mathscr{D}_+^n f(x))^p dx\right]^{\frac{1}{p}} \tag{8}$$

and

$$\left[\int_{\mathbb{R}_+} y^{-(q\alpha_2+1)}(\mathscr{H}(\mathscr{D}_+g)(y))^q dy\right]^{\frac{1}{q}} < \frac{1}{\alpha_2^{\underline{n}}}\left[\int_{\mathbb{R}_+} y^{q(n-\alpha_2)-1}(\mathscr{D}_+^n g(y))^q dy\right]^{\frac{1}{q}}. \tag{9}$$

Finally, since $\alpha_1^{\underline{n}} = \frac{\Gamma(\alpha_1+1)}{\Gamma(\alpha_1-n+1)}$ and $\alpha_2^{\underline{n}} = \frac{\Gamma(\alpha_2+1)}{\Gamma(\alpha_2-n+1)}$, the inequality (5) holds due to (2), (7), (8), and (9). In the same way the inequality (6) holds by virtue of (3) and (8).

The next step is to prove that the constants M and m, appearing on the right-hand sides of the inequalities (5) and (6), are the best possible. For this reason, suppose that there exists a positive constant C smaller than M such that the inequality

$$\begin{aligned}&\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K_\lambda(x,y)f(x)g(y)dxdy\\&<C\left[\int_{\mathbb{R}_+} x^{p(n-\alpha_1)-1}\left(\mathscr{D}_+^n f(x)\right)^p dx\right]^{\frac{1}{p}}\left[\int_{\mathbb{R}_+} y^{q(n-\alpha_2)-1}\left(\mathscr{D}_+^n g(y)\right)^q dy\right]^{\frac{1}{q}}\end{aligned} \tag{10}$$

holds for all non-negative functions $f,g:\mathbb{R}_+\to\mathbb{R}$ fulfilling conditions as in the statement of the Theorem.

Considering the above inequality with functions $\widetilde{f},\widetilde{g}:\mathbb{R}_+\to\mathbb{R}$ defined by

$$\widetilde{f}(x) = \begin{cases} 0, & 0<x<1 \\ \dfrac{\Gamma\left(1+\alpha_1-\frac{\varepsilon}{p}-n\right)}{\Gamma\left(1+\alpha_1-\frac{\varepsilon}{p}\right)} x^{\alpha_1-\frac{\varepsilon}{p}}, & x\geqslant 1\end{cases},$$

$$\widetilde{g}(y) = \begin{cases} 0, & 0<y<1 \\ \dfrac{\Gamma\left(1+\alpha_2-\frac{\varepsilon}{q}-n\right)}{\Gamma\left(1+\alpha_2-\frac{\varepsilon}{q}\right)} y^{\alpha_2-\frac{\varepsilon}{q}}, & y\geqslant 1\end{cases},$$

where $\varepsilon>0$ is a sufficiently small number, the well-known Fubini theorem and the change of variables $t=\frac{y}{x}$ imply that

$$
\begin{aligned}
&\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K_\lambda(x,y)\widetilde{f}(x)\widetilde{g}(y)dxdy\\
=&\varphi(\varepsilon)\int_1^\infty\int_1^\infty K_\lambda(x,y)x^{\alpha_1-\frac{\varepsilon}{p}}y^{\alpha_2-\frac{\varepsilon}{q}}dxdy\\
=&\varphi(\varepsilon)\int_1^\infty x^{-\varepsilon-1}\int_{\frac{1}{x}}^\infty K_\lambda(1,t)t^{\alpha_2-\frac{\varepsilon}{q}}dtdx\\
=&\frac{\varphi(\varepsilon)}{\varepsilon}\int_1^\infty K_\lambda(1,t)t^{\alpha_2-\frac{\varepsilon}{q}}dt+\varphi(\varepsilon)\int_1^\infty x^{-\varepsilon-1}\int_{\frac{1}{x}}^1 K_\lambda(1,t)t^{\alpha_2-\frac{\varepsilon}{q}}dtdx\\
=&\frac{\varphi(\varepsilon)}{\varepsilon}\int_1^\infty K_\lambda(1,t)t^{\alpha_2-\frac{\varepsilon}{q}}dt+\varphi(\varepsilon)\int_0^1 K_\lambda(1,t)t^{\alpha_2-\frac{\varepsilon}{q}}\int_{\frac{1}{t}}^\infty x^{-\varepsilon-1}dxdt\\
=&\frac{\varphi(\varepsilon)}{\varepsilon}\left(\int_1^\infty K_\lambda(1,t)t^{\alpha_2-\frac{\varepsilon}{q}}dt+\int_0^1 K_\lambda(1,t)t^{\alpha_2+\frac{\varepsilon}{p}}dt\right),
\end{aligned}
\tag{11}
$$

where $\varphi(\varepsilon)=\frac{\Gamma\left(1+\alpha_1-\frac{\varepsilon}{p}-n\right)\Gamma\left(1+\alpha_2-\frac{\varepsilon}{q}-n\right)}{\Gamma\left(1+\alpha_1-\frac{\varepsilon}{p}\right)\Gamma\left(1+\alpha_2-\frac{\varepsilon}{q}\right)}$. On the other hand, since the n-th derivative of the function $x^{\alpha_1-\frac{\varepsilon}{p}}$ is equal to $\frac{\Gamma\left(1+\alpha_1-\frac{\varepsilon}{p}\right)}{\Gamma\left(1+\alpha_1-\frac{\varepsilon}{p}-n\right)}x^{\alpha_1-\frac{\varepsilon}{p}-n}$, it follows that

$$
\mathscr{D}_+^n\widetilde{f}(x)=\begin{cases}0, & 0<x<1\\ x^{\alpha_1-\frac{\varepsilon}{p}-n}, & x>1\end{cases},\quad \mathscr{D}_+^n\widetilde{g}(y)=\begin{cases}0, & 0<y<1\\ y^{\alpha_2-\frac{\varepsilon}{q}-n}, & y>1\end{cases},
$$

and the right-hand side of (10) reduces to

$$
C\left[\int_{\mathbb{R}_+} x^{p(n-\alpha_1)-1}\left(\mathscr{D}_+^n\widetilde{f}(x)\right)^p dx\right]^{\frac{1}{p}}\left[\int_{\mathbb{R}_+} y^{q(n-\alpha_2)-1}\left(\mathscr{D}_+^n\widetilde{g}(y)\right)^q dy\right]^{\frac{1}{q}}=\frac{C}{\varepsilon}. \tag{12}
$$

Now, multiplying both sides of relation (10) by ε, and taking into account relations (11) and (12), we have that

$$
\varphi(\varepsilon)\left(\int_1^\infty K_\lambda(1,t)t^{\alpha_2-\frac{\varepsilon}{q}}dt+\int_0^1 K_\lambda(1,t)t^{\alpha_2+\frac{\varepsilon}{p}}dt\right)<C.
$$

Finally, as $\varepsilon\to 0$, it follows that $M\leqslant C$, which is in contrast to our hypothesis. Therefore, the constant M is the best possible in (5).

It remains to show that m is the best constant in (6). Similarly to above discussion, suppose that there exists a positive constant c smaller than m such that inequality

$$
\begin{aligned}
&\left[\int_{\mathbb{R}_+} y^{(p-1)(1+q\alpha_2)}\left(\int_{\mathbb{R}_+} K_\lambda(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}}\\
<&c\left[\int_{\mathbb{R}_+} x^{p(n-\alpha_1)-1}\left(\mathscr{D}_+^n f(x)\right)^p dx\right]^{\frac{1}{p}}
\end{aligned}
$$

holds for all non-negative functions $f:\mathbb{R}_+\to\mathbb{R}$ as in the statement of Theorem. Then, utilizing the well-known Hölder inequality and relation (9), we have

$$\begin{aligned}
&\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K_\lambda(x,y)f(x)g(y)dxdy\\
&=\int_{\mathbb{R}_+}\left[y^{\frac{q\alpha_2+1}{q}}\int_{\mathbb{R}_+} K_\lambda(x,y)f(x)dx\right]\cdot[y^{-\frac{q\alpha_2+1}{q}}g(y)]dy\\
&\leqslant\left[\int_{\mathbb{R}_+} y^{(p-1)(1+q\alpha_2)}\left(\int_{\mathbb{R}_+} K_\lambda(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}}\left[\int_{\mathbb{R}_+} y^{-q\alpha_2-1}g^q(y)dy\right]^{\frac{1}{q}}\\
&<c\frac{\Gamma(\alpha_2-n+1)}{\Gamma(\alpha_2+1)}\left[\int_{\mathbb{R}_+} x^{p(n-\alpha_1)-1}\left(\mathscr{D}_+^n f(x)\right)^p dx\right]^{\frac{1}{p}}\left[\int_{\mathbb{R}_+} y^{q(n-\alpha_2)-1}\left(\mathscr{D}_+^n g(y)\right)^q dy\right]^{\frac{1}{q}},
\end{aligned}$$

which results that the constant M is not the best possible in (5), since $c\frac{\Gamma(\alpha_2-n+1)}{\Gamma(\alpha_2+1)}<m\frac{\Gamma(\alpha_2-n+1)}{\Gamma(\alpha_2+1)}=M$. With this contradiction, the proof is completed. □

REMARK 1. Since for $n=0$ inequalities (5) and (6) reduce respectively to (2) and (3), Theorem 1 may be regarded as an extension of relations (2) and (3). However, if $n\geqslant 1$, the relations (5) and (6) are less precise than (2) and (3), since the right-hand sides of (2) and (3) interpolate between the left-hand side and the right-hand side of inequalities (5) and (6).

Observe that the Theorem 1 covers the case when the degree of homogeneity of the kernel, i.e. $-\lambda$ is less than $-n$, for a fixed non-negative integer n. Our next intention is to derive the corresponding relations that cover the case $0<\lambda\leqslant 1$. Such result is in some way complementary to Theorem 1 and it may be derived by virtue of the weighted dual Hardy inequality.

The dual Hardy inequality, accompanied with the dual integration operator or the dual Hardy operator

$$\mathscr{H}^* f(x)=\int_x^\infty f(t)dt,$$

asserts that

$$\int_{\mathbb{R}_+} x^{-r}\left(\mathscr{H}^* f(x)\right)^p dx<\left(\frac{p}{1-r}\right)^p\int_{\mathbb{R}_+} x^{p-r}f^p(x)dx, \tag{13}$$

holds for $p>1$ and $r<1$, provided that $0<\int_{\mathbb{R}_+} x^{p-r}f^p(x)dx<\infty$.

In order to state the next result, we define a differential operator $\mathscr{D}_\pm^n$ by

$$\mathscr{D}_\pm^n f(x)=(-1)^n f^{(n)}(x),$$

where n is a non-negative integer. Moreover, the following theorem holds for all non-negative functions $f:\mathbb{R}_+\to\mathbb{R}$ such that the n-th derivative $f^{(n)}$ exists a.e. on $\mathbb{R}_+$, $\mathscr{D}_\pm^n f(x)>0$, a.e. on $\mathbb{R}_+$, and $\lim_{x\to\infty} f^{(k)}(x)=0$ for $k=0,1,2,\dots,n-1$. This set of functions will be denoted by $\Lambda_\pm^n$.

THEOREM 2. *Let* $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, *and let* α_1, α_2 *be real parameters such that* $\alpha_1,\alpha_2 \in (-1,\lambda-1)$ *and* $\alpha_1+\alpha_2=\lambda-2$, *where* $0<\lambda\leqslant 1$. *If* $K_\lambda:\mathbb{R}_+^2\to\mathbb{R}$ *is a non-negative homogeneous function of degree* $-\lambda$, *then the inequalities*

$$\begin{aligned}&\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}K_\lambda(x,y)f(x)g(y)dxdy\\ <&M^*\left[\int_{\mathbb{R}_+}x^{p(n-\alpha_1)-1}\left(\mathscr{D}_\pm^n f(x)\right)^p dx\right]^{\frac{1}{p}}\left[\int_{\mathbb{R}_+}y^{q(n-\alpha_2)-1}\left(\mathscr{D}_\pm^n g(y)\right)^q dy\right]^{\frac{1}{q}}\end{aligned} \tag{14}$$

and

$$\begin{aligned}&\left[\int_{\mathbb{R}_+}y^{(p-1)(1+q\alpha_2)}\left(\int_{\mathbb{R}_+}K_\lambda(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}}\\ <&m^*\left[\int_{\mathbb{R}_+}x^{p(n-\alpha_1)-1}\left(\mathscr{D}_\pm^n f(x)\right)^p dx\right]^{\frac{1}{p}}\end{aligned} \tag{15}$$

hold for all non-negative functions $f,g\in\Lambda_\pm^n$, *where* n *is a fixed non-negative integer. In addition, the constants* $M^*=k(\alpha_2)\frac{\Gamma(-\alpha_1)\Gamma(-\alpha_2)}{\Gamma(n-\alpha_1)\Gamma(n-\alpha_2)}$ *and* $m^*=k(\alpha_2)\frac{\Gamma(-\alpha_1)}{\Gamma(n-\alpha_1)}$, *appearing in* (14) *and* (15), *are the best possible.*

Proof. We follow the lines as in the proof of Theorem 1, this time accompanied with the dual Hardy inequality (13). In this setting, the right-hand side of inequality (2) may be rewritten as

$$\begin{aligned}&k(\alpha_2)\left[\int_{\mathbb{R}_+}x^{-p\alpha_1-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{\mathbb{R}_+}y^{-q\alpha_2-1}g^q(y)dy\right]^{\frac{1}{q}}\\ =&k(\alpha_2)\left[\int_{\mathbb{R}_+}x^{-(p\alpha_1+1)}(\mathscr{H}^*(\mathscr{D}_\pm f)(x))^p dx\right]^{\frac{1}{p}}\\ &\times\left[\int_{\mathbb{R}_+}y^{-(q\alpha_2+1)}(\mathscr{H}^*(\mathscr{D}_\pm g)(y))^q dy\right]^{\frac{1}{q}},\end{aligned} \tag{16}$$

since

$$\mathscr{H}^*(\mathscr{D}_\pm f)(x)=-\int_x^\infty f'(t)dt=f(x).$$

Moreover, by applying the dual Hardy inequality to the expressions on right-hand side of relation (16) n times, it follows that

$$\begin{aligned}&\left[\int_{\mathbb{R}_+}x^{-(p\alpha_1+1)}(\mathscr{H}^*(\mathscr{D}_\pm f)(x))^p dx\right]^{\frac{1}{p}}\\ <&\frac{1}{(-\alpha_1)^{\overline{n}}}\left[\int_{\mathbb{R}_+}x^{p(n-\alpha_1)-1}(\mathscr{D}_\pm^n f(x))^p dx\right]^{\frac{1}{p}}\end{aligned} \tag{17}$$

and

$$\left[\int_{\mathbb{R}_+} y^{-(q\alpha_2+1)}(\mathscr{H}^*(\mathscr{D}_\pm g)(y))^q dy\right]^{\frac{1}{q}} < \frac{1}{(-\alpha_2)^{\overline{n}}}\left[\int_{\mathbb{R}_+} y^{q(n-\alpha_2)-1}(\mathscr{D}_\pm^n g(y))^q dy\right]^{\frac{1}{q}}. \tag{18}$$

Now, since $(-\alpha_1)^{\overline{n}} = \frac{\Gamma(n-\alpha_1)}{\Gamma(-\alpha_1)}$ and $(-\alpha_2)^{\overline{n}} = \frac{\Gamma(n-\alpha_2)}{\Gamma(-\alpha_2)}$, the inequality (14) holds due to (2), (16), (17), and (18). In addition, inequality (15) holds by virtue of (3) and (17).

In order to show that M^* is the best constant in (14), we suppose that there exists a positive constant C^* smaller than M^* such that the inequality

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K_\lambda(x,y)f(x)g(y)dxdy < C^*\left[\int_{\mathbb{R}_+} x^{p(n-\alpha_1)-1}\left(\mathscr{D}_\pm^n f(x)\right)^p dx\right]^{\frac{1}{p}}\left[\int_{\mathbb{R}_+} y^{q(n-\alpha_2)-1}\left(\mathscr{D}_\pm^n g(y)\right)^q dy\right]^{\frac{1}{q}} \tag{19}$$

holds for all non-negative functions $f,g \in \Lambda_\pm^n$.

Similarly to the proof of Theorem 1, we consider the above inequality with the appropriate choice of functions f and g. It is easy to see that the functions $\widetilde{f}^*, \widetilde{g}^*: \mathbb{R}_+ \to \mathbb{R}$, defined by

$$\widetilde{f}^*(x) = \begin{cases} 0, & 0 < x < 1 \\ \frac{\Gamma\left(-\alpha_1+\frac{\varepsilon}{p}\right)}{\Gamma\left(n-\alpha_1+\frac{\varepsilon}{p}\right)} x^{\alpha_1-\frac{\varepsilon}{p}}, & x \geqslant 1 \end{cases},$$

$$\widetilde{g}^*(y) = \begin{cases} 0, & 0 < y < 1 \\ \frac{\Gamma\left(-\alpha_2+\frac{\varepsilon}{q}\right)}{\Gamma\left(n-\alpha_2+\frac{\varepsilon}{q}\right)} y^{\alpha_2-\frac{\varepsilon}{q}}, & y \geqslant 1 \end{cases},$$

$\varepsilon > 0$, belong to $\Lambda_\pm^n$. With regard to functions $\widetilde{f}^*, \widetilde{g}^*$, the left-hand side of (19) may be rewritten as

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K_\lambda(x,y)\widetilde{f}^*(x)\widetilde{g}^*(y)dxdy = \frac{\varphi^*(\varepsilon)}{\varepsilon}\left(\int_1^\infty K_\lambda(1,t)t^{\alpha_2-\frac{\varepsilon}{q}}dt + \int_0^1 K_\lambda(1,t)t^{\alpha_2+\frac{\varepsilon}{p}}dt\right), \tag{20}$$

where $\varphi^*(\varepsilon) = \frac{\Gamma\left(-\alpha_1+\frac{\varepsilon}{p}\right)\Gamma\left(-\alpha_2+\frac{\varepsilon}{q}\right)}{\Gamma\left(n-\alpha_1+\frac{\varepsilon}{p}\right)\Gamma\left(n-\alpha_2+\frac{\varepsilon}{q}\right)}$. Clearly, this follows immediately from relation (11).

On the other hand, since the n-th derivative of the function $x^{\alpha_1-\frac{\varepsilon}{p}}$ is equal to $(-1)^n\frac{\Gamma\left(n-\alpha_1+\frac{\varepsilon}{p}\right)}{\Gamma\left(-\alpha_1+\frac{\varepsilon}{p}\right)}x^{\alpha_1-\frac{\varepsilon}{p}-n}$, it follows that

$$\mathscr{D}_\pm^n\widetilde{f}^*(x) = \begin{cases} 0, & 0 < x < 1 \\ x^{\alpha_1-\frac{\varepsilon}{p}-n}, & x > 1 \end{cases}, \quad \mathscr{D}_\pm^n\widetilde{g}^*(y) = \begin{cases} 0, & 0 < y < 1 \\ y^{\alpha_2-\frac{\varepsilon}{q}-n}, & y > 1 \end{cases},$$

which means that the right-hand side of inequality (19) reads

$$C^*\left[\int_{\mathbb{R}_+} x^{p(n-\alpha_1)-1}\left(\mathscr{D}_{\pm}^n \widetilde{f}^*(x)\right)^p dx\right]^{\frac{1}{p}}\left[\int_{\mathbb{R}_+} y^{q(n-\alpha_2)-1}\left(\mathscr{D}_{\pm}^n \widetilde{g}^*(y)\right)^q dy\right]^{\frac{1}{q}} \\ =\frac{C^*}{\varepsilon}. \tag{21}$$

Consequently, comparing (19), (20), and (21), it follows that

$$\varphi^*(\varepsilon)\left(\int_1^{\infty} K_\lambda(1,t)t^{\alpha_2-\frac{\varepsilon}{q}}dt+\int_0^1 K_\lambda(1,t)t^{\alpha_2+\frac{\varepsilon}{p}}dt\right)<C^*.$$

Therefore, as $\varepsilon \to 0$, it follows that $M^* \leqslant C^*$, which contradicts with our assumption. This means that the constant M^* is the best possible in (14).

To conclude the proof, we suppose that, contrary to our claim, there exists a constant $0<c^*<m^*$ such that the inequality

$$\left[\int_{\mathbb{R}_+} y^{(p-1)(1+q\alpha_2)}\left(\int_{\mathbb{R}_+} K_\lambda(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}} \\ <c^*\left[\int_{\mathbb{R}_+} x^{p(n-\alpha_1)-1}\left(\mathscr{D}_{+}^n f(x)\right)^p dx\right]^{\frac{1}{p}}$$

holds for all non-negative functions $f \in \Lambda_{\pm}^n$, as in the statement of Theorem. In addition, employing the Hölder inequality as well as relation (18), we have

$$\begin{aligned}
&\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K_\lambda(x,y)f(x)g(y)dxdy \\
&=\int_{\mathbb{R}_+}\left[y^{\frac{q\alpha_2+1}{q}}\int_{\mathbb{R}_+} K_\lambda(x,y)f(x)dx\right]\cdot[y^{-\frac{q\alpha_2+1}{q}}g(y)]dy \\
&\leqslant\left[\int_{\mathbb{R}_+} y^{(p-1)(1+q\alpha_2)}\left(\int_{\mathbb{R}_+} K_\lambda(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}}\left[\int_{\mathbb{R}_+} y^{-q\alpha_2-1}g^q(y)dy\right]^{\frac{1}{q}} \\
&<c^*\frac{\Gamma(-\alpha_2)}{\Gamma(n-\alpha_2)}\left[\int_{\mathbb{R}_+} x^{p(n-\alpha_1)-1}\left(\mathscr{D}_{\pm}^n f(x)\right)^p dx\right]^{\frac{1}{p}}\left[\int_{\mathbb{R}_+} y^{q(n-\alpha_2)-1}\left(\mathscr{D}_{\pm}^n g(y)\right)^q dy\right]^{\frac{1}{q}}.
\end{aligned}$$

Now, according to our assumption, it follows that $c^*\frac{\Gamma(-\alpha_2)}{\Gamma(n-\alpha_2)}<m^*\frac{\Gamma(-\alpha_2)}{\Gamma(n-\alpha_2)}=M^*$, which means that M^* is not the best constant in (14). This is a clear contradiction of our assumption and the proof is completed. □

REMARK 2. It should be noticed here that Theorem 2 may also be regarded as an extension of inequalities (2) and (3). Similarly to Remark 1, the relations (14) and (15), for $n \geqslant 1$, are less precise than (2) and (3), since the right-hand sides of (2) and (3) interpolate between the left-hand side and the right-hand side of inequalities (14) and (15).

3. Applications

In this section, we discuss our main results with regard to some particular choices of kernels and parameters α_1 and α_2.

3.1. First example

Our first example refers to the homogeneous kernel $K_\lambda(x,y) = (x+y)^{-\lambda}$, $\lambda > 0$. Obviously, this function is homogeneous with degree $-\lambda$, and in this case the constant $k(\alpha_2)$, appearing in inequalities (5), (6), (14), and (15) may be expressed in terms of the usual Beta function (see e.g. [1]). More precisely, we have

$$k(\alpha_2) = \int_{\mathbb{R}_+} (1+t)^{-\lambda} t^{\alpha_2} dt = B(1+\alpha_2, \lambda-1-\alpha_2) = B(\alpha_1+1, \alpha_2+1),$$

since $\alpha_1 + \alpha_2 = \lambda - 2$. Moreover, employing the well-known relationship between the Beta and the Gamma function, i.e. the formula $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, the constants M and m appearing in (5) and (6) (denoted here by M_1 and m_1, respectively) reduce to

$$M_1 = \frac{\Gamma(\alpha_1 - n + 1)\Gamma(\alpha_2 - n + 1)}{\Gamma(\lambda)}$$
$$m_1 = \frac{\Gamma(\alpha_1 - n + 1)\Gamma(\alpha_2 + 1)}{\Gamma(\lambda)},$$

where $\alpha_1, \alpha_2 \in (n-1, \lambda-1)$ and $\lambda > n$. Now, considering the parameters $\alpha_1 = \frac{\lambda}{p} - 1$ and $\alpha_2 = \frac{\lambda}{q} - 1$, where $\lambda > n\max\{p,q\}$, the above constants reduce respectively to $A = \frac{\Gamma\left(\frac{\lambda}{p}-n\right)\Gamma\left(\frac{\lambda}{q}-n\right)}{\Gamma(\lambda)}$ and $a = \frac{\Gamma\left(\frac{\lambda}{p}-n\right)\Gamma\left(\frac{\lambda}{q}\right)}{\Gamma(\lambda)}$. The constant A provides inequality (1) from the Introduction, while its equivalent form asserts that

$$\left[\int_{\mathbb{R}_+} y^{p\lambda-\lambda-1} \left(\int_{\mathbb{R}_+} \frac{f(x)}{(x+y)^\lambda} dx\right)^p dy\right]^{\frac{1}{p}} < a \left[\int_{\mathbb{R}_+} x^{p(n+1)-\lambda-1} \left(\mathscr{D}_+^n f(x)\right)^p dx\right]^{\frac{1}{p}} \tag{22}$$

holds for all non-negative functions $f \in \Lambda_+^n$.

On the other hand, the constants M^* and m^* appearing in dual inequalities (14) and (15) (denoted here by M_1^* and m_1^*, respectively) accompanied with the kernel $K_\lambda(x,y) = (x+y)^{-\lambda}$, become

$$M_1^* = \frac{\pi^2}{\sin(\alpha_1\pi)\sin(\alpha_2\pi)} \cdot \frac{1}{\Gamma(\lambda)\Gamma(n-\alpha_1)\Gamma(n-\alpha_2)}$$
$$m_1^* = -\frac{\pi}{\sin(\alpha_1\pi)} \cdot \frac{\Gamma(\alpha_2+1)}{\Gamma(\lambda)\Gamma(n-\alpha_1)}, \quad \alpha_1, \alpha_2 \in (-1, \lambda-1), 0 < \lambda \leqslant 1,$$

after applying the Euler reflection formula $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin x\pi}$. In addition, with parameters $\alpha_1 = \frac{\lambda}{p} - 1$ and $\alpha_2 = \frac{\lambda}{q} - 1$, and this time with condition $\lambda < \min\{p,q\}$, Theorem 2 yields dual forms of inequalities (1) and (22).

COROLLARY 1. *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $\lambda < \min\{p,q\}$. Then the inequalities*

$$\begin{aligned}
&\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\frac{f(x)g(y)}{(x+y)^\lambda}dxdy \\
<&A^*\left[\int_{\mathbb{R}_+} x^{p(n+1)-\lambda-1}\left(\mathscr{D}_\pm^n f(x)\right)^p dx\right]^{\frac{1}{p}}\left[\int_{\mathbb{R}_+} y^{q(n+1)-\lambda-1}\left(\mathscr{D}_\pm^n g(y)\right)^q dy\right]^{\frac{1}{q}}
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
&\left[\int_{\mathbb{R}_+} y^{p\lambda-\lambda-1}\left(\int_{\mathbb{R}_+}\frac{f(x)}{(x+y)^\lambda}dx\right)^p dy\right]^{\frac{1}{p}} \\
<&a^*\left[\int_{\mathbb{R}_+} x^{p(n+1)-\lambda-1}\left(\mathscr{D}_\pm^n f(x)\right)^p dx\right]^{\frac{1}{p}}
\end{aligned} \tag{24}$$

hold for all non-negative functions $f,g \in \Lambda_\pm^n$, where n is a non-negative integer. Moreover, the constants $A^* = \frac{\pi^2}{\sin(\frac{\lambda\pi}{p})\sin(\frac{\lambda\pi}{q})}\cdot\frac{1}{\Gamma(\lambda)\Gamma(n+1-\frac{\lambda}{p})\Gamma(n+1-\frac{\lambda}{q})}$ and $a^* = \frac{\pi}{\sin(\frac{\lambda\pi}{p})}\cdot\frac{\Gamma(\frac{\lambda}{q})}{\Gamma(\lambda)\Gamma(n+1-\frac{\lambda}{p})}$ appearing in (23) and (24) are the best possible.

3.2. Second example

For the function K_λ given on $\mathbb{R}_+^2$ by $K_\lambda(x,y) = \max\{x,y\}^{-\lambda}$, $\lambda > 0$, we have

$$\begin{aligned}
k(\alpha_2) = &\int_{\mathbb{R}_+}\max\{1,t\}^{-\lambda}t^{\alpha_2} = \frac{\lambda}{(\alpha_2+1)(\lambda-\alpha_2-1)} \\
=&\frac{\lambda}{(\alpha_1+1)(\alpha_2+1)}, \quad \alpha_1,\alpha_2 \in (-1,\lambda-1),
\end{aligned}$$

since $\alpha_1+\alpha_2 = \lambda-2$.

This time, the constants M and m on the right-hand sides of (5) and (6) (denoted here by M_2 and m_2, respectively) read

$$\begin{aligned}
M_2 &= \lambda\cdot\frac{\Gamma(\alpha_1-n+1)\Gamma(\alpha_2-n+1)}{\Gamma(\alpha_1+2)\Gamma(\alpha_2+2)} \\
m_2 &= \frac{\lambda}{\alpha_2+1}\cdot\frac{\Gamma(\alpha_1-n+1)}{\Gamma(\alpha_1+2)}, \quad \alpha_1,\alpha_2 \in (n-1,\lambda-1), \lambda > n,
\end{aligned}$$

since $\Gamma(x+1) = x\Gamma(x)$. In this setting, dual inequalities (14) and (15) include the constants

$$M_2^* = \frac{\lambda}{(\alpha_1+1)(\alpha_2+1)} \cdot \frac{\Gamma(-\alpha_1)\Gamma(-\alpha_2)}{\Gamma(n-\alpha_1)\Gamma(n-\alpha_2)}$$
$$m_2^* = \frac{\lambda}{(\alpha_1+1)(\alpha_2+1)} \cdot \frac{\Gamma(-\alpha_1)}{\Gamma(n-\alpha_1)}, \quad \alpha_1,\alpha_2 \in (-1,\lambda-1),\ 0<\lambda\leqslant 1.$$

3.3. Third example

To conclude the paper, we also consider the kernel K_λ defined on $\mathbb{R}_+^2$ by $K_\lambda(x,y) = \frac{\log y - \log y}{y-x}$. Evidently, it is homogeneous of degree -1, $k(\alpha_2)$ converges for all $\alpha_2 \in (-1,0)$ and

$$k(\alpha_2) = \int_{\mathbb{R}_+} \frac{\log t}{t-1} t^{\alpha_2} dt = \frac{\pi^2}{\sin^2 \alpha_2 \pi}$$

(for more details, see [1] and [3]). Since Theorem 1 refers to homogeneous kernels K_λ with $\lambda > n$, it can not be applied to the above kernel for the case when $n \geqslant 1$. On the other hand, the corresponding dual result follows directly from Theorem 2:

COROLLARY 2. *Let* $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, *and let* $\alpha_1,\alpha_2 \in (-1,0)$ *be real parameters such that* $\alpha_1+\alpha_2=-1$. *Then the inequalities*

$$\begin{aligned} &\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{\log y - \log y}{y-x} f(x)g(y)dxdy \\ <&M_3^* \left[\int_{\mathbb{R}_+} x^{p(n-\alpha_1)-1} \left(\mathscr{D}_\pm^n f(x)\right)^p dx\right]^{\frac{1}{p}} \left[\int_{\mathbb{R}_+} y^{q(n-\alpha_2)-1} \left(\mathscr{D}_\pm^n g(y)\right)^q dy\right]^{\frac{1}{q}} \end{aligned} \tag{25}$$

and

$$\begin{aligned} &\left[\int_{\mathbb{R}_+} y^{(p-1)(1+q\alpha_2)} \left(\int_{\mathbb{R}_+} f(x)dx\right)^p dy\right]^{\frac{1}{p}} \\ <&m_3^* \left[\int_{\mathbb{R}_+} x^{p(n-\alpha_1)-1} \left(\mathscr{D}_\pm^n f(x)\right)^p dx\right]^{\frac{1}{p}} \end{aligned} \tag{26}$$

hold for all non-negative functions $f,g \in \Lambda_\pm^n$, *where* n *is a non-negative integer. In addition, the constants* $M_3^* = -\frac{\pi^3}{\sin^3 \alpha_2\pi} \cdot \frac{1}{\Gamma(n-\alpha_1)\Gamma(n-\alpha_2)}$ *and* $m_3^* = \frac{\pi^2}{\sin^2 \alpha_2\pi} \cdot \frac{\Gamma(-\alpha_1)}{\Gamma(n-\alpha_1)}$ *are the best possible.*

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Vandanjav Adiyasuren
National University of Mongolia
Department of Mathematical Analysis
P.O. Box 46A/125, Ulaanbaatar 14201, Mongolia
e-mail: V_Adiyasuren@yahoo.com

Tserendorj Batbold
National University of Mongolia
Institute of Mathematics
P.O. Box 46A/104, Ulaanbaatar 14201, Mongolia
e-mail: tsbatbold@hotmail.com

Mario Krnić
University of Zagreb
Faculty of Electrical Engineering and Computing
Unska 3, 10000 Zagreb, Croatia
e-mail: mario.krnic@fer.hr