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COMPARISON OF INTEGRAL AND DISCRETE OSTROWSKI'S INEQUALITIES IN THE PLANE

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Abstract. A comparison of integral and discrete Ostrowski's inequalities in the plane is considered. An integral inequality is described by Legendre's elliptic integrals. A natural discrete analogue of the inequality is also given. The main point is to find a suitable decomposition of the radius in polar coordinates.

1. Introduction

Our aim in this note is to give a comparison of integral and discrete Ostrowski's inequalities in the plane $\mathbb{R}^2$. We restrict ourselves to two-dimensional case; higher-dimensional case can also be considered. We are motivated to give our results by Ostrowski [5] as well as by Pólya [6] together with the comment on it made by R. P. Boas [7, p. 489]. For a related result, see the discrete analogue of Northcott's inequality. See Fan-Taussky-Todd [3]. It should be mentioned that "Integral Geometry" has some relevance to our subject. See Santaló [8, Section I. 4. 2], where we can find some examples of the mean distances between two points of a convex set. See also Tricot [9, Section 8. 2], where the measure of families of straight lines is discussed.

The original inequalities are as follows [5, (3)–(6)]: For a differentiable real-valued function $f(x)$ satisfying the condition that

$$\int_0^1 f(x)\,dx = 0 \quad \text{and} \quad |f'(x)| \leqslant 1 \ \ (0 < x < 1), \tag{1}$$

we have

$$|f(x)| \leqslant \left(x - \frac{1}{2}\right)^2 + \frac{1}{4} \quad (0 < x < 1); \tag{2}$$

for n real numbers $a_1, a_2, \ldots, a_n$ satisfying the condition that

$$\sum_{i=1}^{n} a_i = 0 \quad \text{and} \quad |a_i - a_{i+1}| \leqslant 1 \ \ (i = 1, 2, \ldots, n-1), \tag{3}$$

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we have

$$\left|\frac{a_k}{n}\right| \leqslant \left(\frac{k}{n}-\frac{1}{2}-\frac{1}{2n}\right)^2+\frac{1}{4}\left(1-\frac{1}{n^2}\right) \quad (k=1,2,\ldots,n). \tag{4}$$

Many extensions of the integral inequality (1)–(2) have been recently given. See Duoandikoetxea [1], Mitrinović-Pečarić-Fink [4, Chapter XV] and papers by e.g. G. A. Anastassiou, S. S. Dragomir and J. E. Pečarić. We aim at offering a new and fresh insight into the inequality. As far as we know, the results of this note have not appeared in any literature up to now.

The organization is as follows. In Section 2, we begin with notations and definitions. Our theorems are collected in Section 3. Theorems 3.1 and 3.2 correspond to the inequalities (1)–(2) and (3)–(4), respectively. Our main Theorem 3.3 gives a comparison of Theorems 3.1 and 3.2. The main point is the following: If the general point x in the plane $\mathbb{R}^2$ is denoted by its polar coordinate $x=(r,\theta)$, then a suitable decomposition of the r-space gives us annuli of equal areas. Proofs are given in Section 4. Supplementary theorems, which seem to be interesting in themselves, though elementary, are given in Section 5 without proof.

2. Notations and definitions

Let $\mathbb{R}^2$ be two-dimensional real Euclidean space. For a general point $x=(x_1,x_2)\in\mathbb{R}^2$, $|x|=(x_1^2+x_2^2)^{1/2}$ is the Euclidean norm of x in $\mathbb{R}^2$. Let $\mathbb{Z}^2$ and $(\mathbb{N}\cup\{0\})^2$ be the lattice of all points $l=(l_1,l_2)\in\mathbb{R}^2$, where the components l_1 and l_2 are integers and nonnegative integers, respectively. For $a\in\mathbb{R}^2$ and $R>0$, the ball $B(a,R)$ and the cube $Q(a,R)$ are defined by $B(a,R)=\{x\in\mathbb{R}^2; |x-a|<R\}$ and $Q(a,R)=\{x\in\mathbb{R}^2; |x_1-a_1|+|x_2-a_2|<R\}$, respectively. We write $B=B(0,1)$ and $Q=Q(0,1)$. For $N\in\mathbb{N}$, a set of lattice points $L(N)$ is given by $L(N)=\{(l_1,l_2)\in\mathbb{Z}^2; |l_1|+|l_2|\leqslant N\}$.

For $N\in\mathbb{N}$, let I_N be the index set defined by $I_N=\{(i,j)\in(\mathbb{N}\cup\{0\})^2; 1\leqslant i\leqslant N,\ 0\leqslant j\leqslant 2N-1 \text{ or } i=j=0\}$. For $(i,j)\in I_N$, let $P_{i,j}$ be a point in the ball $B(0,N^{1/2})$ with its polar coordinate $(r,\theta)=(i^{1/2},\pi j/N)$. The Euclidean distance between $P_{i,j}$ and $P_{p,q}$ is denoted by $f(i,j;p,q;N)$:

$$f(i,j;p,q;N) := [i+p-2(ip)^{1/2}\cos(\pi|j-q|/N)]^{1/2}. \tag{5}$$

The mean value over all $(i,j)\in I_N$ of the distances between $P_{i,j}$ and $P_{p,q}$ is denoted by $F(p,q;N)$:

$$F(p,q;N) := (2N^2+1)^{-1}\sum_{(i,j)\in I_N} f(i,j;p,q;N). \tag{6}$$

For $k\in(0,1)$, we denote as usual by $K(k)$ and $E(k)$ Legendre's complete elliptic integrals of the first and second kinds, respectively. Namely,

$$K(k)=\int_0^{\pi/2}\left(1-k^2\sin^2\theta\right)^{-1/2}d\theta,\ E(k)=\int_0^{\pi/2}\left(1-k^2\sin^2\theta\right)^{1/2}d\theta,\ 0<k<1.$$

3. Theorems

THEOREM 3.1. *Let f be a differentiable real-valued function in $B = B(0,1)$ such that $\int_B f(y)\,dy = 0$ and $|f'(x)| \leqslant 1$ for every $x \in B$. Then we have, for $x \in B$,*

$$|f(x)| \leqslant \frac{4}{9\pi}\left[\left(7+|x|^2\right)E\left(|x|\right) - 4\left(1-|x|^2\right)K\left(|x|\right)\right]. \tag{7}$$

By using the rotationally invariant set of points $P_{i,j}$, $(i,j) \in I_N$, and the functions $f(i,j;p,q;N)$ and $F(p,q;N)$ defined in Section 2, we can have a natural discrete analogue of integral Ostrowski's inequality. If the lattice $\mathbb{Z}^2$ was used, then difficulties occurring in "lattice-point problems" or Gauss's "circle problem" would be unavoidable. See also Remark 5.4. Our main result is the following Theorem 3.3, which gives a comparison of Theorems 3.1 and 3.2.

THEOREM 3.2. *Let $f(i,j;p,q;N)$ and $F(p,q;N)$ be defined by (5) and (6), respectively. If $2N^2+1$ real numbers $a_{i,j}$, where $(i,j) \in I_N$, satisfy the conditions that $\sum_{(i,j)\in I_N} a_{i,j} = 0$ and that $|a_{i,j} - a_{p,q}| \leqslant f(i,j;p,q;N)$ for every (i,j) and $(p,q) \in I_N$, then we have that $|a_{p,q}| \leqslant F(p,q;N)$ for every $(p,q) \in I_N$.*

THEOREM 3.3. *Let $F(p,q;N)$ be defined by (6). Then we have the following estimate (8) for $F(p,q;N)$:*

$$\left|N^{-1/2}F(p,q;N) - \frac{4}{9\pi}\left[\left(7+\frac{p}{N}\right)E\left(\left(\frac{p}{N}\right)^{1/2}\right) - 4\left(1-\frac{p}{N}\right)K\left(\left(\frac{p}{N}\right)^{1/2}\right)\right]\right| \leqslant CN^{-1}, \quad (p,q) \in I_N, \tag{8}$$

where C is a positive constant independent of p,q and N.

4. Proofs

Proof of Theorem 3.1. Fix an $x \in B \setminus \{(0,0)\}$ with $f(x) \geqslant 0$. Because $|f'(y)| \leqslant 1$ for every $y \in B$, the mean value theorem yields that $f(y) \geqslant f(x) - |y-x|$. Therefore, $0 = \int_B f(y)\,dy \geqslant \pi f(x) - \int_B |y-x|\,dy$, i.e. $f(x) \leqslant \pi^{-1}\int_B |y-x|\,dy$. We shall show that $\int_B |y-x|\,dy = \pi A(|x|)$, where $A(|x|)$ is the right side of (7):

$$A(|x|) = \frac{4}{9\pi}\left[(7+|x|^2)E(|x|) - 4(1-|x|^2)K(|x|)\right], \qquad x \in B(0,1). \tag{9}$$

Let θ be the angle between the vectors x and $y-x$. Then the polar coordinate of the point y is given by $y = x + (r\cos\theta, r\sin\theta)$, where $0 \leqslant \theta < 2\pi$ and $0 \leqslant r < r(\theta) := -|x|\cos\theta + (1-|x|^2\sin^2\theta)^{1/2}$. This range of r is determined by the equation $r(\theta)^2 + 2r(\theta)|x|\cos\theta + |x|^2 = 1$. Then we have that $\int_B |y-x|\,dy = (1/3)\int_0^{2\pi} r(\theta)^3\,d\theta$.

Therefore,

$$\begin{aligned}\int_B |y-x|\,dy &= \frac{4}{3}\int_0^{\pi/2}\left(1-|x|^2\sin^2\theta\right)^{3/2}d\theta \\ &\qquad + 4|x|^2\int_0^{\pi/2}\cos^2\theta\left(1-|x|^2\sin^2\theta\right)^{1/2}d\theta \\ &= \frac{4}{3}(1-|x|^2)\int_0^{\pi/2}\left(1-|x|^2\sin^2\theta\right)^{1/2}d\theta \\ &\qquad + \left(\frac{4}{3}+4\right)|x|^2\int_0^{\pi/2}\cos^2\theta\left(1-|x|^2\sin^2\theta\right)^{1/2}d\theta,\end{aligned}$$

the right side of which is equal to $\pi A(|x|)$, as is seen from the following formula [2, p. 301, (21), (22) and (25)]: For $k\in(0,1)$, we have

$$\int_0^{\pi/2}\cos^2\theta\left(1-k^2\sin^2\theta\right)^{1/2}d\theta = \frac{1}{3k^2}\left[\left(k^2+1\right)E(k)-\left(1-k^2\right)K(k)\right].$$

The case where $x=(0,0)$ can be treated separately. The case where $f(x)\leqslant 0$ can also be treated similarly. □

REMARK 4.1. The function $A(|x|)$, $x\in B(0,1)$, corresponds to the function $(x-1/2)^2+1/4$, $x\in(0,1)$, of the inequality (1)–(2). By using the power series expansions for $K(k)$ and $E(k)$ obtained from [2, p. 313, (5)–(8)], we can express $A(|x|)$ as

$$A(|x|) = \frac{2}{3}\left[1+\frac{3}{4}|x|^2-3\sum_{n=2}^{\infty}\left(\frac{(2n-5)!!}{(2n)!!}\right)^2(2n-3)|x|^{2n}\right],$$

where $(2n)!!=2n(2n-2)\cdots4\cdot2$, $(2n-1)!!=(2n-1)(2n-3)\cdots3\cdot1$, $0!!=(-1)!!=1$.

REMARK 4.2. If we replace $B(0,1)$ and $|f'(x)|\leqslant 1$ in Theorem 3.1 by $B(a,R)$ and $|f'(x)|\leqslant m$, respectively, then a similar inequality corresponding to [5, (1) and (2)] is obtained by "dilation and translation" argument.

REMARK 4.3. A part of the comments on [6] made by Boas [7, p. 489] runs as follows: "… theorems on functions deviating least from zero in the Chebyshev sense: the one-variable one says that if $f(a)=f(b)=0$ and $\int_a^b f(x)\,dx=L$, then the deviation of f' from 0 is greater than $4L/(b-a)^2$…."

REMARK 4.4. The discrete analogue of Northcott's inequality is as follows [3]: For $n+1$ real numbers $a_1,a_2,\ldots,a_{n+1}$ satisfying the conditions that $\sum_{i=1}^n a_i=0$, $\max_{1\leqslant i\leqslant n}|a_i|=1$, and that $a_{n+1}=a_1$, we have that the minimum of $\max_{1\leqslant i\leqslant n}|a_i-a_{i+1}|$ is $4/n$ if n is even, and $4n/(n^2-1)$ if n is odd.

The proof of Theorem 3.2 is analogous to that of the inequality (3)–(4) and is omitted.

Proof of Theorem 3.3. Because of the rotational symmetry of the set of all points $P_{i,j}$, we may assume that $q=0$, and write

$$F(p,0;N)=\frac{2}{2N^2+1}g(p)+\frac{1}{2N^2+1}h(p), \qquad 0\leqslant p\leqslant N, \tag{10}$$

where

$$\begin{cases} g(p):=\displaystyle\sum_{i=1}^{N}\sum_{j=1}^{N-1}\Big[i+p-2(ip)^{1/2}\cos(\pi j/N)\Big]^{1/2},\\ h(p):=\displaystyle\sum_{i=0}^{N}\Big(i^{1/2}+p^{1/2}\Big)+\sum_{i=1}^{N}\Big|i^{1/2}-p^{1/2}\Big|. \end{cases}$$

Since, for fixed i and p, the sequence $\{i+p-2(ip)^{1/2}\cos(\pi j/N); j=0,1,\ldots,N\}$ is monotonically increasing in j, we have

$$\begin{cases} g(p)\leqslant\displaystyle\int_1^N\left(\sum_{i=1}^{N-1}\Big[i+p-2(ip)^{1/2}\cos(\pi v/N)\Big]^{1/2}+2N^{1/2}\right)dv,\\ g(p)\geqslant\displaystyle\int_0^{N-1}\sum_{i=1}^{N}\Big[i+p-2(ip)^{1/2}\cos(\pi v/N)\Big]^{1/2}dv. \end{cases}$$

Furthermore, for fixed v and p, there exists some i_0 $(0\leqslant i_0\leqslant N)$ such that the sequence $\{i+p-2(ip)^{1/2}\cos(\pi v/N); i=0,1,\ldots,N\}$ is monotonically decreasing in i $(0\leqslant i\leqslant i_0)$ and monotonically increasing in i $(i_0+1\leqslant i\leqslant N)$. Therefore, we obtain the following estimates:

$$\begin{cases} g(p)\leqslant\displaystyle\int_1^N\left(\int_0^N\Big[u+p-2(up)^{1/2}\cos(\pi v/N)\Big]^{1/2}du+2N^{1/2}\right)dv,\\ g(p)\geqslant\displaystyle\int_0^{N-1}\left(\int_1^N\Big[u+p-2(up)^{1/2}\cos(\pi v/N)\Big]^{1/2}du\right)dv. \end{cases}$$

Making the substitution $u=r^2$ and $v=N\theta/\pi$, we obtain

$$\begin{cases} g(p)\leqslant\dfrac{2N}{\pi}\displaystyle\int_{N^{-1}\pi}^{\pi}\int_0^{N^{1/2}}\Big(r^2+p-2rp^{1/2}\cos\theta\Big)^{1/2}r\,dr\,d\theta+2N^{3/2},\\ g(p)\geqslant\dfrac{2N}{\pi}\displaystyle\int_0^{(1-N^{-1})\pi}\int_1^{N^{1/2}}\Big(r^2+p-2rp^{1/2}\cos\theta\Big)^{1/2}r\,dr\,d\theta. \end{cases} \tag{11}$$

It is easy to see that the following estimates hold for $h(p)$:

$$C_2N^{3/2}\leqslant h(p)\leqslant C_1N^{3/2}, \qquad 0\leqslant p\leqslant N, \tag{12}$$

where C_1 and C_2 are positive constants independent of N and p. Consequently, from (10)–(12), we have

$$F(p,0;N) \leqslant \frac{(C_1+4)N^{3/2}}{2N^2+1} + \frac{2N}{\pi(2N^2+1)} \int_{N^{-1}\pi}^{(2-N^{-1})\pi} \int_0^{N^{1/2}} \left(r^2+p-2rp^{1/2}\cos\theta\right)^{1/2} r\,dr\,d\theta\,; \tag{13}$$

$$F(p,0;N) \geqslant \frac{C_2N^{3/2}}{2N^2+1} + \frac{2N}{\pi(2N^2+1)} \int_{(-1+N^{-1})\pi}^{(1-N^{-1})\pi} \int_1^{N^{1/2}} \left(r^2+p-2rp^{1/2}\cos\theta\right)^{1/2} r\,dr\,d\theta. \tag{14}$$

From (13), we have

$$F(p,0;N) \leqslant \frac{2N}{\pi(2N^2+1)} \iint_{B(0,N^{1/2})} \left|x-(p^{1/2},0)\right| dx + C_3N^{-1/2},$$

where C_3 is a positive constant independent of N and p. Let $p \neq 0, N$. The case where $p=0$ or N can be treated separately. By setting $k=(p/N)^{1/2}$, we apply the same ideas as in the proof of Theorem 3.1 to obtain

$$F(p,0;N) \leqslant \frac{2N}{\pi(2N^2+1)} N^{3/2}\pi A(k) + C_3N^{-1/2} \leqslant N^{1/2}A(k) + C_3N^{-1/2},$$

where $A(k)$, $0<k<1$, is the function defined by (9).

In order to obtain the estimate for $F(p,0;N)$ from below, we note the following estimates (15) and (16):

$$\int_0^{2\pi}\int_0^1 \left(r^2+p-2rp^{1/2}\cos\theta\right)^{1/2} r\,dr\,d\theta \leqslant \pi(N^{1/2}+1)\,; \tag{15}$$

$$\int_{(1-N^{-1})\pi}^{(1+N^{-1})\pi}\int_0^{N^{1/2}} \left(r^2+p-2rp^{1/2}\cos\theta\right)^{1/2} r\,dr\,d\theta \leqslant 2\pi N^{1/2}. \tag{16}$$

From (14)–(16), we have

$$F(p,0;N) \geqslant \frac{2N}{\pi(2N^2+1)} \iint_{B(0,N^{1/2})} \left|x-(p^{1/2},0)\right| dx - C_4N^{-1/2},$$

where C_4 is a positive constant independent of N and p. By setting $k=(p/N)^{1/2}$ for $p \neq 0, N$, we arrive at the following estimate in exactly the same way:

$$F(p,0;N) \geqslant \frac{2N}{\pi(2N^2+1)} N^{3/2}\pi A(k) - C_4N^{-1/2} \geqslant N^{1/2}(1-N^{-2})A(k) - C_4N^{-1/2}.$$

Because $A(k)$ is bounded for $0<k<1$, we have

$$F(p,0;N) \geqslant N^{1/2}A(k) - C_5N^{-1/2},$$

where C_5 is a positive constant independent of N and p. □

5. Supplements: Ostrowski's inequality in the cube

Supplementary theorems, which seem to be interesting in themselves, though elementary, are given below without proof; in particular, the calculation of the integral $\int_Q |y-x|\,dy$ requires a lot of perseverance.

In order to state integral Ostrowski's inequalities in the cube $Q = Q(0,1)$, we define the functions $A_{Q,1}$ and $A_{Q,2}$ as follows: $A_{Q,1}(x) := |x|^2 - (|x_1|^3 + |x_2|^3)/3 + +2/3$, $x \in Q(0,1)$, and

$$A_{Q,2}(x) := \frac{1}{12}\Bigg[\sum_{\lambda,\mu,\nu\in\{0,1\}} x_{\lambda,\mu}x_{1-\lambda,\nu}\left(x_{\lambda,\mu}^2 + x_{1-\lambda,\nu}^2\right)^{1/2} + \sum_{\lambda,\mu,\nu\in\{0,1\}} x_{\lambda,\mu}^3 \log\left|x_{1-\lambda,\nu} + \left(x_{\lambda,\mu}^2 + x_{1-\lambda,\nu}^2\right)^{1/2}\right| - 2\sum_{\lambda,\mu\in\{0,1\}} x_{\lambda,\mu}^3 \log|x_{\lambda,\mu}|\Bigg], \quad x \in Q(0,1),$$

where we set $x_{\lambda,\mu} = 2^{-1/2}\left[1 + (-1)^\mu(x_1 + (-1)^\lambda x_2)\right]$ for $x = (x_1,x_2) \in \mathbb{R}^2$ and $\lambda,\mu \in \{0,1\}$.

THEOREM 5.1. *Let f be a differentiable real-valued function in $Q = Q(0,1)$ such that $\int_Q f(y)\,dy = 0$ and $\max\{|\partial f/\partial x_1|, |\partial f/\partial x_2|\} \leqslant 1$ in Q. Then we have that $|f(x)| \leqslant A_{Q,1}(x)$ for every $x \in Q$.*

THEOREM 5.2. *Let f be a differentiable real-valued function in $Q = Q(0,1)$ such that $\int_Q f(y)\,dy = 0$ and $|f'(x)| \leqslant 1$ for every $x \in Q$. Then we have that $|f(x)| \leqslant A_{Q,2}(x)$ for every $x \in Q$.*

We define the double sequence $\{f_L(p,q;N); (p,q) \in L(N)\}$, where $L(N) = \{(l_1,l_2) \in \mathbb{Z}^2; |l_1| + |l_2| \leqslant N\}$, and then state a discrete analogue of Theorem 5.1 as follows:

$$f_L(p,q;N) := (2N+1)\left(p^2+q^2\right) - \frac{2}{3}\left(\left|p^3-p\right| + \left|q^3-q\right|\right) + \frac{2}{3}N(N+1)(2N+1).$$

THEOREM 5.3. *Suppose that $2N^2+2N+1$ real numbers $a_{i,j}$, where $(i,j) \in L(N)$, satisfy the following conditions* (i) *and* (ii)*:*

(i) $\sum_{(i,j)\in L(N)} a_{i,j} = 0$*;*

(ii) $\left|a_{i,j} - a_{p,q}\right| \leqslant |i-p| + |j-q|$ *for every (i,j) and $(p,q) \in L(N)$.*

Then we have that $|a_{p,q}| \leqslant f_L(p,q;N)/(2N^2+2N+1)$ for every $(p,q) \in L(N)$.

The proofs are carried out by showing that $\int_Q(|y_1-x_1| + |y_2-x_2|)\,dy = 2A_{Q,1}(x)$, $\int_Q |y-x|\,dy = 2A_{Q,2}(x)$, and that $\sum_{(i,j)\in L(N)}(|i-p| + |j-q|) = f_L(p,q;N)$. We only note the following: For every $x = (x_1,x_2) \in Q$, we have that $f_L(p,q;N)/N(2N^2+2N+1) \to A_{Q,1}(x)$ as $N \to \infty$ with $\max\{|x_1 - N^{-1}p|, |x_2 - N^{-1}q|\} < N^{-1}$.

REMARK 5.4. Let F_1 and F_2 be distance-functions, which play an important role in the "Geometry of Numbers". Then a general form of our problem is to get an estimate for $f_{F_1,F_2}(y;N) := \sum_{x\in\mathbb{Z}^2, F_1(x)\leqslant N} F_2(x-y)$, where $y \in \mathbb{Z}^2$ satisfies $F_1(y) \leqslant N$. A discrete analogue of Theorem 5.2, where F_1 and F_2 are defined by L^1 and L^2 norms, respectively, is yet to be considered.

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