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# JACKSON–STECHKIN TYPE INEQUALITY IN WEIGHTED LORENTZ SPACES

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*Abstract.* In the present work we consider the modulus of smoothness, defined by means of the Steklov operator in weighted Lorentz spaces and prove the Jackson-Stechkin type direct theorem of trigonometric approximation. In the particular case we obtain a result on the constructive characterization of the generalized Lipschitz classes defined in these spaces. Simultaneous approximation of functions is also considered.

## 1. Introduction and the main results

Jackson–Stechkin type inequalities in the normed space under consideration estimate the order of decrease of the best approximation of a function by a finite dimensional subspace in terms of some characteristic of its smoothness.

The starting point here is the classical theorem of Jackson ([8]) on the best uniform approximation of a periodic function $f$ by trigonometric polynomials of degree $\leqslant n$:

*For any $2\pi$-periodic continuous function $f$, the following inequality holds*

$$E_n(f) \leqslant C\omega\left(f, \frac{1}{n+1}\right).$$

In this inequality, $E_n(f)$ denotes the best approximation of the function $f$ by trigonometric polynomials of degree $\leqslant n$, i.e.,

$$E_n(f) := \inf_{T_n \in \mathscr{T}_n} \max_{x \in [0,2\pi]} |f(x) - T_n(x)|,$$

where $\mathscr{T}_n$ is the class of trigonometric polynomials of degree $\leqslant n$, and

$$\omega(f,\delta) := \sup_{|h| \leqslant \delta} \max_{x \in [0,2\pi]} |f(x+h) - f(x)|$$

denotes the modulus of continuity of $f$.

In [15], Stechkin proved an analog of Jackson's inequality for the Lebesgue spaces $L^p$, $1 \leqslant p \leqslant \infty$. The elegant representation of the corresponding results in the Lebesgue

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Paper MIA-18-100

spaces $L^p$, $1 \leqslant p \leqslant \infty$, can be found in [3, 16, 17]. In weighted Lebesgue spaces with weights satisfying the Muckenhoupt's condition $A_p$, $1 < p < \infty$, the direct theorem of trigonometric approximation in the following form was proved in [7].

$$E_n(f)_{L^p_w} \lesssim \omega_r\left(f,\frac{1}{n}\right)_{L^p_w} := \sup_{0 \leqslant h_i \leqslant 1/n} \left\| \prod_{i=1}^{r} \left(I - \sigma_{h_i}\right) f \right\|_{L^p_w}, \quad r \in \mathbb{N},$$

where $I$ is the identity operator on $\mathbf{T} := [-\pi, \pi)$ and

$$\sigma_h f(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(u)du, \ \ x \in \mathbf{T}.$$

The shift operator $\sigma_h$ and the modulus of smoothness $\omega_r(f,\cdot)_{L^p_w}$ are defined in this way, because the weighted Lebesgue space $L^p_w$ is not, in general, invariant under the usual shift $f(\cdot) \to f(\cdot + h)$.

Some interesting results concerning to the best polynomial approximation in weighted Lebesgue spaces were also proved in [4, 5, 11, 19]. In weighted Lorentz spaces some converse theorems were obtained in [10, 18]. The detailed information on the weighted polynomial approximation can be found in the books [6, 12].

A measurable function $w : \mathbf{T} \to [0,\infty]$ is called a weight function if the preimage $w^{-1}(\{0,\infty\})$ has Lebesgue measure zero. Let $w$ be a weight function and $f^*_w(t)$ be a decreasing rearrangement of $f : \mathbf{T} \to \mathbb{R}$ with respect to the Borel measure

$$w(e) = \int_e w(x)dx,$$

i.e.,

$$f^*_w(t) = \inf\{\tau \geqslant 0 : w(x \in \mathbf{T} : |f(x)| > \tau) \leqslant t\}.$$

Let $1 < p,q < \infty$ and let $L^{pq}_w(\mathbf{T})$ be a weighted Lorentz space, i.e., the set of all measurable functions for which

$$\|f\|_{L^{pq}_w} = \left( \int_{\mathbf{T}} (f^{**}(t))^q \, t^{\frac{q}{p}} \frac{dt}{t} \right)^{1/q} < \infty,$$

where

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*_w(u)du.$$

If $p = q$, $L^{pq}_w(\mathbf{T})$ is turn into the weighted Lebesgue space $L^p_w(\mathbf{T})$.

The weights $w$ used in the paper are those which belong to the Muckenhoupt's ([13]) class $A_p(\mathbf{T})$, i.e., they satisfy the condition

$$\sup \frac{1}{|I|} \int_I w(x)dx \left( \frac{1}{|I|} \int_I w^{1-p'}(x)dx \right)^{p-1} = C_{A_p} < \infty, \quad p' := \frac{p}{p-1}$$

where the supremum is taken with respect to all the intervals $I$ with length $\leqslant 2\pi$ and $|I|$ denotes the length of $I$. The constant $C_{A_p}$ is called the Muckenhoupt constant of $w$.

The modulus of smoothness of a function $f \in L^{pq}_w(\mathbf{T})$ is given by

$$\Omega_r(f,\delta)_{L^{pq}_w} := \sup_{0\leqslant h_i\leqslant\delta, i=1,\dots,r} \left\| \prod_{i=1}^{r} \left(I-\sigma_{h_i}\right) f \right\|_{L^{pq}_w}, \quad r \in \mathbb{N}.$$

Whenever $w \in A_p(\mathbf{T})$, $1 < p,q < \infty$, the Hardy-Littlewood maximal function of $f \in L^{pq}_w(\mathbf{T})$ belongs to $L^{pq}_w(\mathbf{T})$ ([2, Theorem 3]). Therefore the average $\sigma_h f$ belongs to $L^{pq}_w(\mathbf{T})$. Thus $\Omega_r(f,\delta)_{L^{pq}_w}$ makes sense for every $w \in A_p(\mathbf{T})$.

By $E_n(f)_{L^{pq}_w}$ we denote the best approximation of $f \in L^{pq}_w(\mathbf{T})$ by trigonometric polynomials of degree $\leqslant n$, i.e.,

$$E_n(f)_{L^{pq}_w} = \inf_{T_n\in\mathscr{T}_n} \|f - T_n\|_{L^{pq}_w}.$$

Since $L^{pq}_w(\mathbf{T}) \subset L^1(\mathbf{T})$ when $w \in A_p(\mathbf{T})$, $1 < p,q < \infty$ (see [10, the proof of Prop. 3.3]), we can define the Fourier series of $f \in L^{pq}_w(\mathbf{T})$

$$f(x) \backsim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} \left(a_k(f)\cos kx + b_k(f)\sin kx\right) \tag{1.1}$$

and the conjugate Fourier series

$$\tilde{f}(x) \backsim \sum_{k=1}^{\infty} \left(a_k(f)\sin kx - b_k(f)\cos kx\right).$$

Here $a_0(f), a_k(f), b_k(f)$, $k = 1,\dots,$ are Fourier coefficients of $f$.

The relation $\lesssim$ is defined as "$A \lesssim B \Leftrightarrow$ there exists a positive constant $C$, independent of essential parameters, such that $A \leqslant CB$."

In this work we prove the direct and simultaneous theorems of approximation theory in the weighted Lorentz spaces using the modulus of smoothness $\Omega_r(f,\cdot)_{L^{pq}_w}$.

Our new results are the following.

THEOREM 1. *Let $w \in A_p(\mathbf{T})$, $1 < p,q < \infty$, $r \in \mathbb{N}$. Then for every $f \in W^r_{pq,w}$, the inequality*

$$\|f - S_n(f)\|_{L^{pq}_w} \lesssim \frac{1}{(n+1)^r} \left\| f^{(r)} - S_n(f^{(r)}) \right\|_{L^{pq}_w}, \quad n \in \mathbb{N}$$

*holds with a positive constant depending only on $r,p,q$ and the Muckenhoupt constant $C_{A_p}$ of $w$, where $S_n(f)$ denotes the $n$.th partial sum of the Fourier series (1.1) of $f$.*

We define $W^r_{pq,w} := \left\{ g \in L^{pq}_w : g^{(r)} \in L^{pq}_w \right\}$. Theorem 1 gives the following corollary.

COROLLARY 1. *Let $w \in A_p(\mathbf{T})$, $1 < p,q < \infty$, $r,n \in \mathbb{N}$. Then for every $f \in W^r_{pq,w}$, the inequalities*

$$E_n(f)_{L^{pq}_w} \lesssim \frac{1}{(n+1)^r} E_n(f^{(r)})_{L^{pq}_w},$$

*and*

$$E_n(f)_{L^{pq}_w} \lesssim \frac{1}{(n+1)^r} \left\| f^{(r)} \right\|_{L^{pq}_w},$$

*hold with some positive constants depending only on $r,p,q$ and the Muckenhoupt constant $C_{A_p}$ of $w$.*

THEOREM 2. *Let $f \in L^{pq}_w(\mathbf{T})$, $w \in A_p(\mathbf{T})$, $1 < p,q < \infty$. Then we have the following estimate*

$$E_n(f)_{L^{pq}_w} \lesssim \Omega_r \left( f, \frac{1}{n} \right)_{L^{pq}_w}, \quad r \in \mathbb{N}, \tag{1.2}$$

*and*

$$\|f - S_n(f)\|_{L^{pq}_w} \lesssim \Omega_r \left( f, \frac{1}{n} \right)_{L^{pq}_w} \tag{1.3}$$

*for $n \in \mathbb{N}$, with some constants depending only on $r,p,q$ and the Muckenhoupt constant $C_{A_p}$ of $w$.*

We note that the sharp inverse inequality to the Jackson-Stechkin type inequality was proved in [10]. In the sequel we use a weak version of inverse estimate:

*Let $f \in L^{pq}_w(\mathbf{T})$, $w \in A_p(\mathbf{T})$, $1 < p,q < \infty$. Then*

$$\Omega_r \left( f, \frac{1}{n} \right)_{L^{pq}_w} \lesssim \frac{1}{n^{2r}} \sum_{k=0}^{n} (k+1)^{2r-1} E_k(f)_{L^{pq}_w}, \quad r \in \mathbb{N} \tag{1.4}$$

*holds for $n \in \mathbb{N}$, with some constant depending only on $r,p,q$ and the Muckenhoupt constant $C_{A_p}$ of $w$.*

From Theorem 2 and (1.4), we obtain the following Marchaud type inequality.

COROLLARY 2. *Let $f \in L^{pq}_w(\mathbf{T})$, $w \in A_p(\mathbf{T})$, $1 < p,q < \infty$. Then we have*

$$\Omega_r (f, \delta)_{L^{pq}_w} \lesssim \delta^{2r} \int_\delta^1 \frac{\Omega_{r+1} (f,u)_{L^{pq}_w}}{u^{2r}} \frac{du}{u}, \quad 0 < \delta < 1,$$

*for $r \in \mathbb{N}$.*

From Theorem 2 and (1.4), we also obtain the following estimate.

THEOREM 3. *Let $f \in L^{pq}_w(\mathbf{T})$, $w \in A_p(\mathbf{T})$, $1 < p,q < \infty$. If*

$$E_n(f)_{L^{pq}_w} \lesssim n^{-\alpha}, \quad n \in \mathbb{N}$$

*for some $\alpha > 0$, then, for a given $r \in \mathbb{N}$, we have the estimations*

$$\Omega_r (f, \delta)_{L^{pq}_w} = \begin{cases} \delta^\alpha & , r > \alpha/2; \\ \delta^{2r} \log \frac{1}{\delta} & , r = \alpha/2; \\ \delta^{2r} & , r < \alpha/2. \end{cases}$$

If we define the generalized Lipschitz class $Lip\left(\alpha, L_w^{pq}\right)$ for $\alpha > 0$ and $k := [\alpha/2]+1,\ [x] := \max\{n \in \mathbb{Z} : n \leqslant x\}$ as

$$Lip\left(\alpha, L_w^{pq}\right) := \left\{ f \in L_w^{pq} : \Omega_k(f,\delta)_{L_w^{pq}} \lesssim \delta^{\alpha}, \quad \delta > 0 \right\},$$

then by virtue of Theorems 3 and 2 we obtain the following result which gives a constructive characterization of the Lipschitz classes $Lip\left(\alpha, L_w^{pq}\right)$.

COROLLARY 3. *Let $f \in L_w^{pq}(\mathbf{T}),\ w \in A_p(\mathbf{T}),\ 1 < p,q < \infty$ and $\alpha > 0$. The following assertions are equivalent.*

$$(i)\ f \in Lip\left(\alpha, L_w^{pq}\right) \quad (ii)\ E_n(f)_{L_w^{pq}} \lesssim n^{-\alpha}, \quad n \in \mathbb{N}.$$

Jackson's second type inequality is given in the following theorem.

THEOREM 4. *Let $w \in A_p(\mathbf{T}),\ 1 < p,q < \infty$ and $r,k \in \mathbb{N}$. Then for every $f \in W^r_{pq,w}$, the inequality*

$$E_n(f)_{L_w^{pq}} \lesssim \frac{1}{(n+1)^r}\Omega_k\left(f^{(r)}, \frac{1}{n}\right)_{L_w^{pq}}, \quad n \in \mathbb{N}$$

*holds with a positive constant depending only on $r,p,q$ and the Muckenhoupt constant $C_{A_p}$ of $w$.*

Simultaneous approximation estimates are given in the next two theorems.

THEOREM 5. *Let $w \in A_p(\mathbf{T}),\ 1 < p,q < \infty$ and $r,k,l \in \mathbb{N}$. Then for every $f \in W^r_{pq,w}$ and $0 \leqslant k \leqslant r$ the inequality*

$$\left\| f^{(k)} - S_n^{(k)}(f) \right\|_{L_w^{pq}} \lesssim \frac{1}{n^{r-k}}\Omega_l\left(f^{(r)}, \frac{1}{n}\right)_{L_w^{pq}}, \quad n \in \mathbb{N}$$

*holds with a positive constant depending only on $r,p,q$ and the Muckenhoupt constant $C_{A_p}$ of $w$.*

THEOREM 6. *Let $w \in A_p(\mathbf{T}),\ 1 < p,q < \infty$ and $r,k,n \in \mathbb{N}$. Then for every $f \in W^r_{pq,w}$ and $0 \leqslant k \leqslant r$ the inequality*

$$\left\| f^{(k)} - S_n^{(k)}(f) \right\|_{L_w^{pq}} \lesssim \frac{1}{n^{r-k}} E_n\left(f^{(r)}\right)_{L_w^{pq}}$$

*holds with a positive constant depending only on $r,p,q$ and the Muckenhoupt constant $C_{A_p}$ of $w$.*

## 2. Proofs of the main results

To prove Theorem 2 we need the following lemma. If $A \lesssim B$ and $B \lesssim A$, simultaneously, we will write $A \approx B$.

For an $f \in L_w^{pq}(\mathbf{T})$ and $r \in \mathbb{N}$ the Peetre's $K$-functional is defined as

$$K\left(f,t;L_w^{pq},W_{pq,w}^r\right) := \inf_{g \in W_{pq,w}^r} \left\{ \|f-g\|_{L_w^{pq}} + t^r \left\| g^{(r)}(x) \right\|_{L_w^{pq}} \right\}$$

for $t > 0$.

LEMMA 1. *Let $f \in L_w^{pq}(T)$, $r \in \mathbb{N}$, $w \in A_p(\mathbf{T})$, $1 < p,q < \infty$ and $t,k > 0$. Then we have*

$$\Omega_r(f,t)_{L_w^{pq}} \approx K\left(f,t;L_w^{pq},W_{pq,w}^{2r}\right) \tag{2.1}$$

*and*

$$\Omega_r(f,kt)_{L_w^{pq}} \lesssim (1+[k])^{2r}\,\Omega_r(f,t)_{L_w^{pq}},$$

*with some constant depending only on $p,q$ and the Muckenhoupt constant $C_{A_p}$ of $w$.*

*Proof.* If $h \in W_{pq,w}^{2r}(\mathbf{T})$, then from subadditivity of $\Omega_r(f,\cdot)_{L_w^{pq}}$ and $\Omega_r(h,t)_{L_w^{pq}} \lesssim t^{2r}\left\|h^{(2r)}\right\|_{L_w^{pq}}$ (see Lemma 4.1 of [10])

$$\Omega_r(f,t)_{L_w^{pq}} \lesssim \|f-h\|_{L_w^{pq}} + t^{2r}\left\|h^{(2r)}\right\|_{L_w^{pq}}.$$

Taking infimum on $h$ we get $\Omega_r(f,t)_{L_w^{pq}} \lesssim K\left(f,t;L_w^{pq},W_{pq,w}^{2r}\right)$.

We define

$$(L_\delta f)(x) := 3\delta^{-3}\int_0^\delta \int_0^u \int_{-t}^t f(x+s)\,ds\,dt\,du, \quad x \in \mathbf{T}.$$

From [1, p.15]

$$\frac{d^{2r}}{dx^{2r}}L_\delta^r f = \frac{c}{\delta^{2r}}(I-\sigma_\delta)^r, \quad r \in \mathbb{N}.$$

Because of estimates

$$\|L_\delta f\|_{L_w^{pq}} \lesssim \frac{\delta^{-3}}{3}\int_0^\delta \int_0^u 2t\,\|\sigma_t f\|_{L_w^{pq}}\,dt\,du \lesssim \|f\|_{L_w^{pq}}$$

the operator $L_\delta$ is bounded in $L_w^{pq}$.

Defining $A_\delta^r := I - \left(I - L_\delta^r\right)^r$ we obtain

$$\left\|\frac{d^{2r}}{dx^{2r}}A_\delta^r f\right\|_{L_w^{pq}} \lesssim \left\|\frac{d^{2r}}{dx^{2r}}L_\delta^r f\right\|_{L_w^{pq}} = \frac{1}{\delta^{2r}}\left\|(I-\sigma_\delta)^r\right\|_{L_w^{pq}}$$

$$\lesssim \frac{1}{\delta^{2r}}\Omega_r(f,\delta)_{L_w^{pq}}$$

and hence $A_\delta^r f \in W_{pq,w}^{2r}(\mathbf{T})$. Since $L_\delta$ is bounded in $L_w^{pq}$ and $I - L_\delta^r = (I - L_\delta)\sum_{j=0}^{r-1} L_\delta^j$ we have

$$\left\|\left(I - L_\delta^r\right) g\right\|_{L_w^{pq}} \lesssim \|(I - L_\delta) g\|_{L_w^{pq}} \lesssim \delta^{-3} \int_0^\delta \int_0^u 2t \,\|(I - \sigma_t) g\|_{L_w^{pq}} \, dt du$$
$$\lesssim \sup_{0<t\leqslant\delta} \|(I - \sigma_t) g\|_{L_w^{pq}}$$

for any $g \in L_w^{pq}$.

Applying this inequality $r$ times in $\|f - A_\delta^r f\|_{L_w^{pq}} = \left\|\left(I - L_\delta^r\right)^r f\right\|_{L_w^{pq}}$ we obtain

$$\begin{aligned}\|f - A_\delta^r f\|_{L_w^{pq}} &\lesssim \sup_{0<t_1\leqslant\delta} \left\|(I - \sigma_{t_1})\left(I - L_\delta^r\right)^{r-1} f\right\|_{L_w^{pq}} \\ &\lesssim \sup_{0<t_1,t_2\leqslant\delta} \left\|(I - \sigma_{t_1})(I - \sigma_{t_2})\left(I - L_\delta^r\right)^{r-2} f\right\|_{L_w^{pq}} \\ &\leqslant \ldots \leqslant \sup_{\substack{0<t_i\leqslant\delta \\ i=1,2,\ldots,r}} \left\|\prod_{i=1}^r (I - \sigma_{t_i}) f(x)\right\|_{L_w^{pq}} = \Omega_r(f,\delta)_{L_w^{pq}}.\end{aligned}$$

Using the equivalence (2.1) we have

$$\begin{aligned}\Omega_r(f,kt)_{L_w^{pq}} &\lesssim \inf_{g\in W_{pq,w}^{2r}} \left\{\|f - g\|_{L_w^{pq}} + (kt)^{2r}\left\|g^{(2r)}(x)\right\|_{L_w^{pq}}\right\} \\ &\lesssim (1 + [k])^{2r} \inf_{g\in W_{pq,w}^{2r}} \left\{\|f - g\|_{L_w^{pq}} + t^{2r}\left\|g^{(2r)}(x)\right\|_{L_w^{pq}}\right\} \\ &\lesssim (1 + [k])^{2r}\,\Omega_r(f,t)_{L_w^{pq}}\end{aligned}$$

and the lemma is proved. □

*Proof of Theorem* 1. We know that (see [9, Theorem 6.6.2], [10])

$$\|S_n(f)\|_{L_w^{pq}} \lesssim \|f\|_{L_w^{pq}}, \quad \left\|\tilde{f}\right\|_{L_w^{pq}} \lesssim \|f\|_{L_w^{pq}},$$

$$\|f - S_n(f)\|_{L_w^{pq}} \lesssim E_n(f)_{L_w^{pq}} \text{ and } S_n\left(t, f^{(r)}\right) = S_n^{(r)}(t,f).$$

Then (see inequality 6.15 of [14])

$$f(x) - S_n(x,f) = \sum_{k=n+1}^{\infty} \frac{1}{k^r\pi} \int_{\mathbf{T}} \left(f^{(r)}(t) - S_n\left(t, f^{(r)}\right)\right) \cos\left(k(x-t) - \frac{r\pi}{2}\right) dt.$$

When $r=2l$

$$\begin{aligned} f(x)-S_n(x,f) &= (-1)^l \sum_{k=n+1}^{\infty} \frac{1}{k^r}\left[a_k\left(f^{(r)}-S_n\left(f^{(r)}\right)\right)\cos(kx)+\right. \\ &\qquad \left.+\,b_k\left(f^{(r)}-S_n\left(f^{(r)}\right)\right)\sin(kx)\right] \\ &= (-1)^l \sum_{k=n+1}^{\infty} \frac{A_k\left(f^{(r)}-S_n\left(f^{(r)}\right)\right)}{k^r} \\ &= (-1)^l \sum_{k=n+1}^{\infty} \left(k^{-r}-(k+1)^{-r}\right) S_k\left(x, f^{(r)}-S_n\left(f^{(r)}\right)\right). \end{aligned}$$

Hence

$$\begin{aligned} \|f-S_n(f)\|_{L_w^{pq}} &\leqslant \sum_{k=n+1}^{\infty} \left(k^{-r}-(k+1)^{-r}\right) \left\|S_k\left(f^{(r)}-S_n\left(f^{(r)}\right)\right)\right\|_{L_w^{pq}} \\ &\lesssim \sum_{k=n+1}^{\infty} \left(k^{-r}-(k+1)^{-r}\right) \left\|f^{(r)}-S_n\left(f^{(r)}\right)\right\|_{L_w^{pq}} \\ &\lesssim \frac{1}{(n+1)^r} \left\|f^{(r)}-S_n\left(f^{(r)}\right)\right\|_{L_w^{pq}}. \end{aligned}$$

When $r=2l+1$ we have $\cos\left(k(x-t)-\frac{\pi}{r}\right)=\sin\left(k(x-t)(-1)^l\right)$ and

$$\begin{aligned} f(x)-S_n(x,f) &= (-1)^l \sum_{k=n+1}^{\infty} \frac{1}{k^r}\left[a_k\left(f^{(r)}-S_n\left(f^{(r)}\right)\right)\sin(kx)-\right. \\ &\qquad \left.-\,b_k\left(f^{(r)}-S_n\left(f^{(r)}\right)\right)\cos(kx)\right] \\ &= (-1)^l \sum_{k=n+1}^{\infty} \frac{A_k\left(\widetilde{f^{(r)}}-S_n\left(\widetilde{f^{(r)}}\right)\right)}{k^r} \\ &= (-1)^l \sum_{k=n+1}^{\infty} \left(k^{-r}-(k+1)^{-r}\right) S_k\left(x, \widetilde{f^{(r)}}-S_n\left(\widetilde{f^{(r)}}\right)\right). \end{aligned}$$

Hence

$$\begin{aligned} \|f-S_n(f)\|_{L_w^{pq}} &\leqslant \sum_{k=n+1}^{\infty} \left(k^{-r}-(k+1)^{-r}\right) \left\|S_k\left(\widetilde{f^{(r)}}-S_n\left(\widetilde{f^{(r)}}\right)\right)\right\|_{L_w^{pq}} \\ &\lesssim \sum_{k=n+1}^{\infty} \left(k^{-r}-(k+1)^{-r}\right) \left\|\widetilde{f^{(r)}}-S_n\left(\widetilde{f^{(r)}}\right)\right\|_{L_w^{pq}} \\ &\lesssim \frac{1}{(n+1)^r} \left\|\widetilde{f^{(r)}}-S_n\left(\widetilde{f^{(r)}}\right)\right\|_{L_w^{pq}} \\ &\lesssim \frac{1}{(n+1)^r} \left\|f^{(r)}-S_n\left(f^{(r)}\right)\right\|_{L_w^{pq}}. \end{aligned}$$

and Theorem 1 is proved. $\square$

*Proof of Theorem 2.* Let $n \in \mathbb{N}$ and $f \in L_w^{pq}$ be fixed. We will use the operator $A_{1/n}^r f$. From Corollary 1 and Lemma 1

$$\begin{aligned}E_n(f)_{L_w^{pq}} &= E_n(f - A_{1/n}^r f + A_{1/n}^r f)_{L_w^{pq}} \leqslant E_n(f - A_{1/n}^r f)_{L_w^{pq}} + E_n(A_{1/n}^r f)_{L_w^{pq}} \\ &\lesssim \left\| f - A_{1/n}^r f \right\|_{L_w^{pq}} + n^{-2r} \left\| \frac{d^{2r}}{dx^{2r}} A_{1/n}^r f(x) \right\|_{L_w^{pq}} \\ &\lesssim \Omega_r \left( f, \frac{1}{n} \right)_{L_w^{pq}}.\end{aligned}$$

Hence $\|f - S_n(f)\|_{L_w^{pq}} \lesssim E_n(f)_{L_w^{pq}} \lesssim \Omega_r \left(f, \frac{1}{n}\right)_{L_w^{pq}}$. $\square$

*Proof of Theorem 3.* Let $f \in L_w^{pq}$ and

$$E_n(f)_{L_w^{pq}} \lesssim n^{-\alpha}, \quad n \in \mathbb{N}$$

for some $\alpha > 0$. We suppose that $\delta > 0$ and $n := [1/\delta]$. From (1.4) we get

$$\begin{aligned}\Omega_r(f,\delta)_{L_w^{pq}} \leqslant \Omega_r \left( f, \frac{1}{n} \right)_{L_w^{pq}} &\lesssim \frac{1}{n^{2r}} \sum_{j=0}^{n} (j+1)^{2r-1} E_j(f)_{L_w^{pq}} \\ &\lesssim \delta^{2r} \left( E_0(f)_{L_w^{pq}} + \sum_{j=1}^{n} j^{2r-1} E_j(f)_{L_w^{pq}} \right) \\ &\lesssim \delta^{2r} \left( E_0(f)_{L_w^{pq}} + \sum_{j=1}^{n} j^{2r-1-\alpha} \right).\end{aligned}$$

If $2r > \alpha$, then we get $\Omega_r(f,\delta)_{L_w^{pq}} \lesssim \delta^{\alpha}$. If $2r = \alpha$, then

$$\sum_{j=1}^{n} j^{2r-1-\alpha} = \sum_{j=1}^{n} j^{-1} \leqslant 1 + \log(1/\delta)$$

and hence $\Omega_r(f,\delta)_{L_w^{pq}} \lesssim \delta^{2r} \log(1/\delta)$. If $2r < \alpha$, then the series $\sum_{j=0}^{n} j^{2r-1-\alpha}$ is convergent and

$$\Omega_r(f,\delta)_{L_w^{pq}} \lesssim \delta^{2r} \left( E_0(f)_{L_w^{pq}} + \sum_{j=1}^{n} j^{2r-1-\alpha} \right) \lesssim \delta^{2r}$$

holds. $\square$

*Proof of Theorem 4.* Using Corollary 1 and (1.2) we find

$$\begin{aligned}E_n(f)_{L_w^{pq}} &\lesssim \frac{1}{(n+1)^r} E_n \left( f^{(r)}, \frac{1}{n} \right)_{L_w^{pq}} \\ &\lesssim \frac{1}{(n+1)^r} \Omega_k \left( f^{(r)}, \frac{1}{n} \right)_{L_w^{pq}}. \quad \square\end{aligned}$$

*Proof of Theorem* 5. For $f \in W^{r}_{pq,w}$ we have $f^{(k)} \in W^{r-k}_{pq,w}$. Using Corollary 1, $S_n^{(k)}(f) = S_n\left(f^{(k)}\right)$ and (1.3) we find

$$\left\|f^{(k)} - S_n^{(k)}(f)\right\|_{L_w^{pq}} = \left\|f^{(k)} - S_n(f^{(k)})\right\|_{L_w^{pq}} \lesssim \frac{1}{n^{r-k}}\Omega_l\left(f^{(r)}, \frac{1}{n}\right)_{L_w^{pq}}. \quad \square$$

*Proof of Theorem* 6. Let $q, t_n^* \in \mathscr{T}_n$ and $E_n\left(f^{(k)}\right)_{L_w^{pq}} = \left\|f^{(k)} - q\right\|_{L_w^{pq}}$, $E_n(f)_{L_w^{pq}} = \|f - t_n^*\|_{L_w^{pq}}$. Then using $(S_n(f,\cdot))^{(k)} = S_n\left(f^{(k)},\cdot\right)$

$$\begin{aligned}
\left\|f^{(k)} - S_n^{(k)}(f)\right\|_{L_w^{pq}} &\leqslant \left\|f^{(k)} - S_n\left(f^{(k)},\cdot\right)\right\|_{L_w^{pq}} + \left\|S_n^{(k)}(f) - (t_n^*)^{(k)}\right\|_{L_w^{pq}} \\
&\leqslant \left\|f^{(k)} - q\right\|_{L_w^{pq}} + \left\|q - S_n\left(f^{(k)},\cdot\right)\right\|_{L_w^{pq}} + \left\|(S_n(f,\cdot) - t_n^*)^{(k)}\right\|_{L_w^{pq}} \\
&\lesssim E_n\left(f^{(k)}\right)_{L_w^{pq}} + \left\|S_n\left(q - f^{(k)},\cdot\right)\right\|_{L_w^{pq}} + n^k \|S_n(f,\cdot) - t_n^*\|_{L_w^{pq}} \\
&\lesssim E_n\left(f^{(k)}\right)_{L_w^{pq}} + n^k \|S_n(f,\cdot) - S_n(t_n^*,\cdot)\|_{L_w^{pq}} \\
&\lesssim n^{k-r}E_n\left(f^{(r)}\right)_{L_w^{pq}} + n^k E_n(f)_{L_w^{pq}} \lesssim n^{k-r}E_n\left(f^{(r)}\right)_{L_w^{pq}}
\end{aligned}$$

and the proof of Theorem is completed. $\square$

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## REFERENCES


[1] R. AKGUN, *Trigonometric approximation of functions in generalized Lebesgue spaces with variable exponent*, Ukrainian Math. J., **63** (2011), no. 1 , pp. 1–26.

[2] H. M. CHANG, R. A. HUNT AND D. S. KURTZ, *The Hardy-Littlewood maximal functions on $L(p,q)$ spaces with weights*, Indiana Univ. Math. J. **31** (1982), 109–120.

[3] R. A. DEVORE, G. G. LORENTZ, *Constructive Approximation*, Springer-Verlag (1993).

[4] Z. DITZIAN, V. TOTIK, *K-functionals and best polynomial approximation in weighted $L^p(R)$*, J. Approx. Theory, (1986), no. 1, 38–41.

[5] Z. DITZIAN, V. TOTIK, *K-functionals and weighted moduli of smoothness*, J. Approx. Theory, **63** (1990), no. 1, 3–29.

[6] Z. DITZIAN, V. TOTIK, *Moduli of Smoothness*, Springer-Verlag, 1987.

[7] E. A. GADJIEVA, *Investigation the Properties of Functions with Quasimonotone Fourier Coefficients in Generalized Nikolskii-Besov Spaces*, (Russian), Authors Summary of Candidates Dissertation, Tbilisi (1986).

[8] D. JACKSON, *Uber die Genauigkeit der Annaherung stetiger Funktionen durch ganzer rationaler Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung*, Dissertation, Gottingen, 1911.

[9] V. KOKILASHVILI AND M. KRBEC, *Weighted inequalities in Lorentz and Orlicz spaces*, World Scientific Publishing Co. Inc. River Edge, NJ, 1991.

[10] V. M. KOKILASHVILI, Y. E. YILDIRIR, *On the approximation by trigonometric polynomials in weighted Lorentz spaces*, J. Funct. Spaces Appl., **8** (2010), No. 1, 67–86.

[11] V. M. KOKILASHVILI, Y. E. YILDIRIR, *On the approximation in weighted Lebesgue spaces*, Proc. A. Razmadze Math. Inst., **143** (2007), 103–113.

[12] H. N. MHASKAR, *Introduction to the Theory of Weighted Polynomial Approximation*, Series in Approximation and Decompisitions **7**, World Scientific Publishing Co., Inc., River Edge, NJ 1996.

[13] B. MUCKENHOUPT, *Weighted Norm Inequalities for the Hardy Maximal Function*, Trans. Amer. Math. Soc. 165 (1972), 207–226.

[14] I. I. SHARAPUDINOV, *Approximation of functions in $L_{2\pi}^{p(x)}$ by trigonometric polynomials*, Izv. RAN. Ser. Mat., **77** (2013), no. 2, 197–224.

[15] S. B. STECKIN, *On the order of the best approximation of continuous functions*, Izv. Akad. Nauk SSSR Ser. Mat. **15** (1951), 219–242. (Russian) MR 13, 29.

[16] A. I. STEPANETS, *Classification and Approximation of Periodic Functions*, English translation 1995, Kluwer Academic Publishers, Russian original published in Kiev by Naukova Dumka 1987.

[17] A. F. TIMAN, *Theory of Approximaton of Functions of a Real Variable*, English translation 1963, Pergamon Press, The MacMillan Co., Russian original published in Moscow by Fizmatgiz 1960.

[18] Y. E. YILDIRIR, D. M. ISRAFILOV, *Approximation Theorems in weighted Lorentz spaces*, Carpathian J. Math., 26 (2010), No. 1, 108–119.

[19] Y. E. YILDIRIR, D. M. ISRAFILOV, *Simultaneous and converse approximation theorems in weighted Lebesgue spaces*, Math. Ineq. & Appl., **14**, 2, (2011), 359–371.

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