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ON SOME INEQUALITIES OF CHEBYSHEV TYPE

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Abstract. We obtain some new inequalities of Chebyshev Type.

1. Introduction

Let f, $g\colon [a,b] \to \mathbb{R}$ be integrable functions, both increasing or both decreasing. Further, let $p\colon[a,b] \to \mathbb{R}_0^+$ be an integrable function. Then (see, for example, [5, Chap. IX])

$$\int_a^b p(x)f(x)g(x)dx \geqslant \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx \left(\int_a^b p(x)dx\right)^{-1}. \qquad (1)$$

If one of the functions f or g is nonincreasing and the other nondecreasing the reversed inequality is true, i.e.,

$$\int_a^b p(x)f(x)g(x)dx \leqslant \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx \left(\int_a^b p(x)dx\right)^{-1}. \qquad (2)$$

Inequalities (1) and (2) are known as Chebyshev's inequalities. These inequalities were obtained by P. L. Chebyshev [1, 2] and they attracted great interest of the researchers. So, a lot of analogues and generalizations of inequalities (1) and (2) is known. In particular, these results can be found in Chapter IX of the book [5] by D. S. Mitrinović, J. E. Pečarić and A. M. Fink which trace completely the historical and chronological developments of Chebyshev's and related inequalities (see also [4, 6]). Also we would like to recommend the article of H. P. Heinig and L. Maligranda [3], where one can found a lot of important results on Chebyshev's inequalities for strongly increasing functions, positive convex and concave functions as well as on Chebyshev's inequalities in Banach function spaces and symmetric spaces.

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In [7], these investigations were developed in the following direction: the author found necessary and sufficient conditions on the function $g\colon [a,b] \to \mathbb{R}_0^+$ and $p\colon [a,b] \to \mathbb{R}^+$ such that for any monotone function $f\colon [a,b] \to \mathbb{R}_0^+$ the relations

$$\int_a^b p(x)f(x)g(x)dx \geqslant \left(\int_a^b p^r(x)f^r(x)dx\right)^{1/r} \int_a^b p(x)g(x)dx \left(\int_a^b p^r(x)dx\right)^{-1/r} \qquad (3)$$

and

$$\int_a^b p(x)f(x)g(x)dx \leqslant \left(\int_a^b p^r(x)f^r(x)dx\right)^{1/r} \int_a^b p(x)g(x)dx \left(\int_a^b p^r(x)dx\right)^{-1/r} \qquad (4)$$

hold with r being an arbitrary positive number.

In this paper we continue the study of the inequalities of the type (1)–(4), namely, we obtain the following assertions:

THEOREM 1. *Assume that $g\colon [a,b] \to \mathbb{R}_0^+$ and $p\colon [a,b] \to \mathbb{R}^+$ are integrable functions such that the product $p \cdot g$ is also integrable on $[a,b]$ function. Let $f\colon [a,b] \to \mathbb{R}_0^+$ be a nonincreasing function. Then for any convex function $M\colon [0,\infty) \to \mathbb{R}$ such that $M(0) = 0$, the following inequality is true:*

$$\int_a^b p(x)g(x)M(f(x))dx \leqslant \sup_{s\in(a,b]} \left\{ M\left(\frac{\int_a^b p(x)f(x)dx}{\int_a^s p(x)dx}\right) \int_a^s p(x)g(x)dx \right\}, \qquad (5)$$

and for any concave function $M\colon [0,\infty) \to \mathbb{R}$ such that $M(0) = 0$, the following inequality is true:

$$\int_a^b p(x)g(x)M(f(x))dx \geqslant \inf_{s\in(a,b]} \left\{ M\left(\frac{\int_a^b p(x)f(x)dx}{\int_a^s p(x)dx}\right) \int_a^s p(x)g(x)dx \right\}. \qquad (6)$$

Putting $M(t) = t^{1/r}$, $r > 0$, from Theorem 1 we obtain the following corollaries:

COROLLARY 1. *Let $r \in (0,1]$, and let $g\colon [a,b] \to \mathbb{R}_0^+$ and $p\colon [a,b] \to \mathbb{R}^+$ be integrable functions such that for all $s \in (a,b]$,*

$$\frac{\int_a^s p(x)g(x)dx}{\left(\int_a^s p(x)dx\right)^{1/r}} \leqslant \frac{\int_a^b p(x)g(x)dx}{\left(\int_a^b p(x)dx\right)^{1/r}}. \qquad (7)$$

Then for any nonincreasing function $f\colon [a,b] \to \mathbb{R}_0^+$,

$$\int_a^b p(x)g(x)f(x)dx \leqslant \left(\int_a^b p(x)f^r(x)dx\right)^{1/r} \frac{\int_a^b p(x)g(x)dx}{\left(\int_a^b p(x)dx\right)^{1/r}}. \qquad (8)$$

COROLLARY 2. *Let $r \geqslant 1$, and let $g\colon [a,b] \to \mathbb{R}_0^+$ and $p\colon [a,b] \to \mathbb{R}^+$ be integrable functions such that for all $s \in (a,b]$,*

$$\frac{\int_a^s p(x)g(x)dx}{\left(\int_a^s p(x)dx\right)^{1/r}} \geqslant \frac{\int_a^b p(x)g(x)dx}{\left(\int_a^b p(x)dx\right)^{1/r}}. \tag{9}$$

Then for any nonincreasing function $f\colon [a,b] \to \mathbb{R}_0^+$,

$$\int_a^b p(x)g(x)f(x)dx \geqslant \left(\int_a^b p(x)f^r(x)dx\right)^{1/r} \frac{\int_a^b p(x)g(x)dx}{\left(\int_a^b p(x)dx\right)^{1/r}}. \tag{10}$$

If in corollaries 1 and 2, we put $r = 1$, then we see that relations (8) and (10) are the Chebyshev's classical inequalities (1) and (2). Furthermore, it should be noted that conditions on the functions p and g of the form (7) and (9) for validity of inequalities (1) and (2) were considered in the papers [7] and [8].

In the case, where the function $M(f(x))$ is nonincreasing and the function g is nondecreasing (or nonincreasing), we can apply the Chebyshev's classical inequalities to the integral $\int_a^b p(x)g(x)M(f(x))dx$ on the left-hand side of relations (5) (or (6)). Respectively, we obtain

$$\int_a^b p(x)g(x)M(f(x))dx \leqslant \frac{\int_a^b p(x)M(f(x))dx}{\int_a^b p(x)dx} \int_a^b p(x)g(x)dx \tag{11}$$

and

$$\int_a^b p(x)g(x)M(f(x))dx \geqslant \frac{\int_a^b p(x)M(f(x))dx}{\int_a^b p(x)dx} \int_a^b p(x)g(x)dx. \tag{12}$$

Furthermore, if exact upper (or lower) bound on the right–hand side of (5) (or (6)) is realized for $s = b$, then from relations (5) and (6) we get

$$\int_a^b p(x)g(x)M(f(x))dx \leqslant M\left(\frac{\int_a^b p(x)f(x)dx}{\int_a^b p(x)dx}\right) \int_a^b p(x)g(x)dx, \tag{13}$$

and

$$\int_a^b p(x)g(x)M(f(x))dx \geqslant M\left(\frac{\int_a^b p(x)f(x)dx}{\int_a^b p(x)dx}\right) \int_a^b p(x)g(x)dx. \tag{14}$$

Here, it should be note that by virtue of Jensen's inequality (see, for example, [5, Chap. I]), estimations (13) and (14) of the integral $\int_a^b p(x)g(x)M(f(x))dx$ are more precisely, than estimations (11) and (12).

REMARK 1. In the case, where the function f is nondecreasing, inequalities (5) and (6) have the similar form, but in these inequalities, all the integrals of the kind $\int\limits_a^s(\cdot)$ should be replaced by the integrals of the kind $\int\limits_s^b(\cdot)$.

2. Discrete analogue of Theorem 1

LEMMA 1. *Assume that* $a=\{a_k\}_{k=1}^m$, $b=\{b_k\}_{k=1}^m$ *and* $p=\{p_k\}_{k=1}^m$, $m\in\mathbb{N}$ *are nonnegative number sequences such that* $a_1\geqslant a_2\geqslant\ldots a_m$ *and* $p_k>0$. *Then for any convex function* $M\colon [0,\infty)\to\mathbb{R}$ *such that* $M(0)=0$, *the following inequality is true:*

$$\sum_{k=1}^m p_k b_k M(a_k)\leqslant \max_{s\in[1,m]}\left\{M\left(\frac{\sum_{k=1}^m p_k a_k}{\sum_{k=1}^s p_k}\right)\sum_{k=1}^s p_k b_k\right\}, \tag{15}$$

and for any concave function $M\colon [0,\infty)\to\mathbb{R}$ *such that* $M(0)=0$, *the following inequality is true:*

$$\sum_{k=1}^m p_k b_k M(a_k)\geqslant \min_{s\in[1,m]}\left\{M\left(\frac{\sum_{k=1}^m p_k a_k}{\sum_{k=1}^s p_k}\right)\sum_{k=1}^s p_k b_k\right\}. \tag{16}$$

Proof. Consider the case, where the function M is convex (in the case, where the function M is concave, the proof is similar). Let us prove by the induction on m the proposition that for any convex function $M\colon [0,\infty)\to\mathbb{R}$ such that $M(0)=0$, inequality (15) holds.

The case $m=1$ is obvious.

Also consider the case $m=2$.

Put

$$c=p_1a_1+p_2a_2,\quad x_0=p_1a_1,\quad \alpha_k=p_kb_k,\quad \beta_k=p_k^{-1},\quad k=1,2, \tag{17}$$

and consider on the interval $[0,c]$ the function

$$h(x)=\alpha_1M(\beta_1x)+\alpha_2M(\beta_2(c-x)). \tag{18}$$

Due to convexity of the function $M(t)$, the function $h(x)$ is also convex on $[0,c]$. Hence, this function attains its maximum value on any interval $[x_1,x_2]\subseteq[0,c]$ at one of its endpoints. Thus

$$h(x)\leqslant\max\{h(x_1),h(x_2)\}\quad \forall x\in[x_1,x_2]. \tag{19}$$

Setting $x_1:=\beta_2c(\beta_1+\beta_2)^{-1}$ and $x_2:=c$, we see that the number x_0 (by virtue of monotonicity of the sequence a) belongs to the interval $[x_1,x_2]$.

Therefore, in view of relations (17)–(19) and of the equality $M(0)=0$, we get

$$\begin{aligned}\sum_{k=1}^2 p_kb_kM(a_k)&=h(x_0)\leqslant\max\{h(x_1),h(x_2)\}\\&=\max\left\{M\left(\frac{p_1a_2+p_2a_2}{p_1+p_2}\right)(p_1b_2+p_2b_2),M\left(\frac{p_1a_2+p_2a_2}{p_1}\right)p_1b_1\right\}.\end{aligned}$$

Hence, for $m=2$, inequality (15) holds.

Now, assume that for $m=n-1\geqslant1$, the proposition is true.

Let us show that for $m=n$, it is also true. Let us use notations (17) and consider on the interval $[0,c]$ the function $h(x)$ of the form as in (18). Setting $x_1 := \beta_2 c(\beta_1+\beta_2)^{-1}$ and $x_2 := c - a_3/\beta_2$, we see that the number x_0 (by virtue of monotonicity of the sequence a) belongs to the interval $[x_1,x_2]$. Thus in view of relations (17)–(19),

$$\sum_{k=1}^{n} p_k b_k M(a_k) = h(x_0) + \sum_{k=3}^{n} p_k b_k M(a_k) \leqslant \max\{h(x_1),h(x_2)\} + \sum_{k=3}^{n} p_k b_k M(a_k). \quad (20)$$

Further, in the case, where $h(x_1) \geqslant h(x_2)$, we set

$$p'_k = \begin{cases} p_1+p_2, & k=1, \\ p_{k+1}, & k=\overline{2,m-1}; \end{cases} \quad b'_k = \begin{cases} (p_1b_1+p_2b_2)/(p_1+p_2), & k=1, \\ b_{k+1}, & k=\overline{2,m-1}; \end{cases} \quad (21)$$

$$a'_k = \begin{cases} (p_1a_1+p_2a_2)/(p_1+p_2), & k=1, \\ a_{k+1}, & k=\overline{2,m-1}. \end{cases} \quad (22)$$

Then by virtue of (20), we conclude that the following relation is true:

$$\sum_{k=1}^{m} p_k b_k M(a_k) \leqslant \sum_{k=1}^{m-1} p'_k b'_k M(a'_k). \quad (23)$$

In the case, where $h(x_1) < h(x_2)$, relation (23) holds for the sequences a', b' and p' of the form:

$$p'_k = \begin{cases} p_1, & k=1, \\ p_2+p_3, & k=2, \\ p_{k+1}, & k=\overline{3,m-1}; \end{cases} \quad b'_k = \begin{cases} b_1, & k=1, \\ (p_2b_2+p_3b_3)(p_2+p_3)^{-1}, & k=2, \\ b_{k+1}, & k=\overline{3,m-1}; \end{cases} \quad (24)$$

$$a'_k = \begin{cases} (p_1a_1+p_2a_2-p_2a_3)/p_1, & k=1, \\ a_{k+1}. & k=\overline{2,m-1}, \end{cases} \quad (25)$$

The sum on the right-hand side of (23) contains $n-1$ items. Furthermore, in both cases, for the sequences a', b' and p', the induction assumption is satisfied. Thus, taking into account (21)–(25), we obtain the necessary estimate (15):

$$\sum_{k=1}^{n} p_k b_k M(a_k) \leqslant \sum_{k=1}^{n-1} p'_k b'_k M(a'_k) \leqslant \sup_{s\in[1,n-1]} \left\{ M\left(\frac{\sum_{k=1}^{n-1} p'_k a'_k}{\sum_{k=1}^{s} p'_k}\right) \sum_{k=1}^{s} p'_k b'_k \right\}$$
$$\leqslant \sup_{s\in[1,n]} \left\{ M\left(\frac{\sum_{k=1}^{n} p_k a_k}{\sum_{k=1}^{s} p_k}\right) \sum_{k=1}^{s} p_k b_k \right\}.$$

3. Proof of Theorem 1

Proof. Consider the case, where the function M is convex (in the case, where the function M is concave, the proof is similar). First, let us verify that inequality (5) holds for any function f such that for a certain $m \in \mathbb{N}$,

$$f(x) = a_k, \quad x \in [l_{k-1}, l_k),\ k = 1,2,\ldots,m,$$

where $a_1 > a_2 > \ldots > a_m \geqslant 0$ and $a = l_0 < l_1 < \ldots < l_m = b$.

For any $k = 1, 2, \ldots, m$, we put

$$p_k = \int\limits_{l_{k-1}}^{l_k} p(x)dx, \quad b_k = \int\limits_{l_{k-1}}^{l_k} p(x)g(x)dx \Bigg(\int\limits_{l_{k-1}}^{l_k} p(x)dx \Bigg)^{-1}.$$

Then by virtue of Lemma 1, we get (5):

$$\begin{aligned}
\int_a^b p(x)g(x)M(f(x))dx &= \sum_{k=1}^{m} \int\limits_{l_{k-1}}^{l_k} p(x)g(x)M(f(x))dx \\
&= \sum_{k=1}^{m} p_k b_k M(a_k) \\
&\leqslant \sup_{s\in[1,m]\cap\mathbb{N}} \left\{ M\left(\frac{\sum_{k=1}^{m} p_k a_k}{\sum_{k=1}^{s} p_k} \right) \sum_{k=1}^{s} p_k b_k \right\} \\
&= \sup_{s\in[1,m]\cap\mathbb{N}} \left\{ M\left(\frac{\int_a^b p(x)f(x)dx}{\int_a^{l_s} p(x)dx} \right) \int_a^{l_s} p(x)g(x)dx \right\} \\
&\leqslant \sup_{s\in(a,b]} \left\{ M\left(\frac{\int_a^b p(x)f(x)dx}{\int_a^{s} p(x)dx} \right) \int_a^{s} p(x)g(x)\, dx \right\}.
\end{aligned}$$

Let us prove the validity of inequality (5) in general case. First, note that if the functions M and f satisfy the conditions of Theorem 1, then there exists the number $n_0 = n_0(M, f) \in \mathbb{N}$ such that for any $n > n_0$ and for all $x \in [a; b]$, the inequality $|M(f(x))| < n$ holds.

For any $n > n_0$, consider the system of points $l_0^{(n)} < l_1^{(n)} < \ldots < l_m^{(n)} = b$, defined in the following way: we put $l_0^{(n)} := a$ and for any $k \in [1; m] \cap \mathbb{N}$ the value $l_k^{(n)}$ is the greatest positive number such that $l_k^{(n)} > l_{k-1}^{(n)}$ and for all $x \in [l_{k-1}^{(n)}; l_k^{(n)})$ the following relation is true:

$$|M(f(l_{k-1}^{(n)})) - M(f(x))| \leqslant \frac{1}{n}.$$

By virtue of the conditions on the function M and f, this system of points always exists and $m \leqslant 2n^2$.

Further, consider the functions $f_n = f_n(x)$ such that

$$f_n(x) \equiv \lim_{t \to l_k^{(n)}-} f(t), \quad \text{for all } x \in [l_{k-1}^{(n)}; l_k^{(n)}),\ k = 1, 2, \ldots, m. \tag{26}$$

We see that the inequality $|M(f(x)) - M(f_n(x))| \leqslant \frac{1}{n}$ holds for all $n > n_0$ and $x \in [a, b]$. Due to integrability on $[a, b]$ of the product $p(x)g(x)$, the values

$$\int_a^b p(x)g(x)[M(f(x)) - M(f_n(x))]\, dx$$

converge to zero as $n \to \infty$. Furthermore, for any $n > n_0$, the function $f_n(x)$ is nonincreasing and it takes finitely many values on $[a,b]$. Hence, this function satisfies the conditions of the proposition proved above.

Thus, in view of (26) and continuity of the function M, we conclude that for any $\varepsilon > 0$ and for all sufficiently great n $(n > n_1(\varepsilon))$,

$$\begin{aligned}\int_a^b p(x)g(x)M(f(x))dx &= \int_a^b p(x)g(x)M(f_n(x))dx \\ &\quad + \int_a^b p(x)g(x)(M(f(x)) - M(f_n(x)))dx \\ &\leqslant \sup_{s\in(a;b]} \left\{ M\left(\frac{\int_a^b p(x)f_n(x)dx}{\int_a^s p(x)dx}\right) \int_a^s p(x)g(x)dx \right\} + \frac{\varepsilon}{2} \\ &\leqslant \sup_{s\in(a;b]} \left\{ M\left(\frac{\int_a^b p(x)f(x)dx}{\int_a^s p(x)dx}\right) \int_a^s p(x)g(x)dx \right\} + \varepsilon.\end{aligned}$$

Hence, relation (5) is true.

Analyzing the proof of Theorem 1, we see that the similar statement is also true in the case, where $b = \infty$.

THEOREM 1′. *Assume that $g\colon [a,b] \to \mathbb{R}_0^+$ and $p\colon [a,b] \to \mathbb{R}^+$ (where $b \in (a,\infty]$) are integrable functions such that the product $p \cdot g$ is also integrable on $[a,b]$ function. Let also $f\colon [a,b] \to \mathbb{R}_0^+$ be a nonincreasing function. Then for any convex (or concave) function $M\colon [0,\infty) \to \mathbb{R}$ such that $M(0) = 0$, inequality (5) (or inequality (6)) is true.*

Analogically, one can obtain the statement, similar to Lemma 1, in the case, where $n = \infty$.

LEMMA 1′. *Let $a = \{a_k\}_{k=1}^{\infty}$, $b = \{b_k\}_{k=1}^{\infty}$ and $p = \{p_k\}_{k=1}^{\infty}$ be nonnegative number sequences such that $a_1 \geqslant a_2 \geqslant \dots$, $p_k > 0$ and the series $\sum_{k=1}^{\infty} p_k b_k$ is convergent. Then for any convex function $M\colon [0,\infty) \to \mathbb{R}$ such that $M(0) = 0$, the following inequality is true:*

$$\sum_{k=1}^{\infty} p_k b_k M(a_k) \leqslant \sup_{s\in[1,\infty)} \left\{ M\left(\frac{\sum_{k=1}^{\infty} p_k a_k}{\sum_{k=1}^{s} p_k}\right) \sum_{k=1}^{s} p_k b_k \right\}, \tag{15′}$$

and for any concave function $M\colon [0,\infty) \to \mathbb{R}$ such that $M(0) = 0$, the following inequality is true:

$$\sum_{k=1}^{\infty} p_k b_k M(a_k) \geqslant \inf_{s\in[1,\infty)} \left\{ M\left(\frac{\sum_{k=1}^{\infty} p_k a_k}{\sum_{k=1}^{s} p_k}\right) \sum_{k=1}^{s} p_k b_k \right\}. \tag{16′}$$

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