Mathematical
Inequalities
& Applications
Volume 18, Number 4 (2015), 1329–1340 doi:10.7153/mia-18-104

SHARP FORM OF MODIFIED WEIGHTED HARDY INEQUALITIES OF TRACE TYPE

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(*Communicated by L. E. Persson*)

Abstract. The extremal function and the best constant for a modified weighted Hardy inequality of trace type are investigated by using the Bliss' argument. Computational difficulties of the self-contained proof are overcome by the techniques taken from a paper by Talenti.

1. Background and the main theorem

Modified weighted Hardy inequalities of trace type

$$\left[\int_0^\infty |u(r)|^q r^{n-1} dr\right]^{1/q} \leqslant C\left[\int_0^\infty |u'(r)|^p r^n dr\right]^{1/p}, \quad ' \equiv \frac{d}{dr} \tag{1.1}$$

are considered. Here u is a differentiable function of one variable vanishing at infinity and the exponents p $(1 < p < n)$ and q satisfy a certain relation (presented in (2.1)). The trace type Hardy inequality (1.1) originates from the classical Sobolev trace inequalities:

$$\left(\int_{\mathbb{R}^n} |f(x)|^q dx\right)^{1/q} \leqslant A_{p,q}\left(\int_{\mathbb{R}^{n+1}_+} |\nabla u(x,y)|^p dxdy\right)^{1/p}, \tag{1.2}$$

whereas the classical Hardy inequality came from a new and elementary proof for the Hilbert's double integral theorem by G. H. Hardy [7]. For the last couple of decades, the classical Hardy inequality has been generalized, refined and applied in analysis and in the theory of differential equations. See, for example, [3, 4, 5, 8, 10, 11, 14, 15, 16, 17, 19] and the references cited therein. The trace type Hardy inequality (1.1) is a very special case of the generalized weighted Hardy inequalities some of whose best constants have been specified. However, the best constant of the trace type Hardy inequality (1.1) is not reported yet.

This paper is mainly devoted to identify the sharp constant and the extremal functions of inequality (1.1). This sharp constant is worthy of notice in effort to give some clues to the best constant for Sobolev trace inequalities (1.2) which is still an important open problem. Our main result is presented as follows:

Mathematics subject classification (2010): 26D10, 41A44, 46E35, 47B38, 42Axx, 42C99.
Keywords and phrases: Hardy inequality, Sobolev trace inequality, extremal function, best constant.
The first author was supported by the research fund of Dankook University in 2013.

Paper MIA-18-104

THEOREM 1. *The best constant of the modified weighted Hardy inequality of trace type (1.1) constrained by (2.1) below is*

$$n^{-\frac{1}{p}}\left[\frac{p-1}{n+1-p}\right]^{\frac{q-1}{q}}\left[\frac{\Gamma(n)\Gamma(\frac{n}{p-1}-1)}{\Gamma(\frac{np}{p-1})}\right]^{\frac{1}{q}-\frac{1}{p}}$$

with the extremal functions of the form

$$(a+br)^{1-\frac{n}{p-1}} \qquad \textit{for } a,b>0.$$

The proof is not short and is comprised of several stages. The basic strategies of the proof are to turn the problem into a Lagrange one, and then to analyze it as the Lagrange flows by the Bliss' argument [3]. However, the computations of this problem are not straightforward. In fact, there are some serious computational difficulties to complete the arguments. The novelty of this report is to get over the difficulties both directly and indirectly by the techniques taken largely from the paper by Talenti [20].

2. The arguments

1. We first note by the dilation argument that the exponents p and q should satisfy the relation

$$\frac{1}{q}=\frac{n+1}{n}\,\frac{1}{p}-\frac{1}{n}. \tag{2.1}$$

(The same index relation holds in the classical Sobolev trace inequalities (1.2).) The rearrangement argument can be used to restrict the functions to be positive and monotonically decreasing.

2. In order to find the extremal functions and the best constant of (1.1), we try to look for the extremal functions of the functional $J(u)$

$$J(u)=\frac{\left[\int_0^\infty |u(r)|^q r^{n-1}dr\right]^{1/q}}{\left[\int_0^\infty |u'(r)|^p\, r^n dr\right]^{1/p}},$$

where functions u are in some admissible collection. In fact, the argument above leads to consider the (admissible) collection $\mathscr{A}$ of functions which are sufficiently smooth, non-negative real valued functions decreasing on the interval $[0,\infty)$ satisfying the conditions

$$\int_0^\infty |u'(r)|^p r^n dr<\infty \quad\text{and}\quad u(r)\to 0 \ \text{ as } \ r\to\infty.$$

We now derive the (Gateaux) differential of the functional $J(u)$:

$$\frac{1}{J(u)}J'(u)(v)=\frac{\int_0^\infty(|u|^{q-1}\mathrm{sgn}u)\,r^{n-1}v\,dr}{\int_0^\infty |u|^q r^{n-1}dr}+\frac{\int_0^\infty(|u'|^{p-1}\mathrm{sgn}\,u'r^n)'v\,dr}{\int_0^\infty |u'|^p r^n dr}.$$

Then the extremals of J are the solutions in $\mathscr{A}$ of the differential equation of the form

$$(|u'|^{p-1}r^n \operatorname{sgn} u')' + Cr^{n-1}|u|^{q-1} \operatorname{sgn} u = 0, \tag{2.2}$$

where C is a positive constant. Conversely, every solution in $\mathscr{A}$ of any differential equation of the form (2.2) is an extremal function of J.

3. We will look for a solution of (2.2) of the form $(a+br^s)^{-\alpha}$ with a,b,s and α being positive. Setting $\phi(r) = (a+br^s)^{-\alpha}$, we will first determine the exponent s and α using (2.2). The equation (2.2) now becomes:

$$\begin{aligned} &C(a+br^s)^{-\alpha(q-1)}r^{n-1} \\ &\quad = (\alpha s b)^{p-1}(a+br^s)^{-(\alpha+1)(p-1)-1}r^{(s-1)(p-1)+n-1} \\ &\qquad \times [a\{(s-1)(p-1)+n\}+br^s\{-(\alpha+1)(p-1)s+(s-1)(p-1)+n\}]. \end{aligned}$$

This leaves us a few different choices for s and α. It can be computed that the only possibility is of the form $\phi(r) = (a+br)^{1-\frac{n}{p-1}}$ for positive constants a and b. It is easy to check that ϕ is a solution of (2.2) in $\mathscr{A}$, and the constant C in (2.2) turns out to be

$$C = nab^{p-1}\left(\frac{n+1-p}{p-1}\right)^{p-1}.$$

So far we have found a two-parameter family of extremal functions of J. Such extremals are positive decreasing functions. A slight (equivalent) change can be made to represent these extremals in the form

$$\varphi(r) = a(1+br)^{1-\frac{n}{p-1}} \quad \text{for } a,b>0.$$

Then equation (2.2) can be written as

$$\left(|\varphi'|^{p-1}\operatorname{sgn}\varphi' r^n\right)' + Cr^{n-1}|\varphi|^{q-1}\operatorname{sgn}\varphi = 0,$$

which is equivalent to saying that

$$na^{p-q}b^{p-1}\left(\frac{n+1-p}{p-1}\right)^{p-1}|\varphi|^{q-1}r^{n-1} = \left(r^n|\varphi'|^{p-1}\right)'. \tag{2.3}$$

4. Our goal is to show that the extremals we found in the previous discussion actually give the maximum, which is the essential part of the proof. To do it, we are now going to look at the following equivalent Lagrange problem.

Letting $M := \int_0^\infty |u'|^p r^n dr$, we have $\int_0^\infty \left|\frac{u'}{M^{1/p}}\right|^p r^n dr = 1$, $\int_0^\infty \left|\frac{u}{M^{1/p}}\right|^q r^{n-1} dr = \frac{1}{M^{q/p}}\int_0^\infty |u|^q r^{n-1} dr$ and $J(u) = \frac{1}{M^{1/p}}\left(\int_0^\infty |u|^q r^{n-1} dr\right)^{1/q}$, which in turn implies that

$J(u)=J\left(\frac{u}{M^{1/p}}\right)$. By this observation, we can put our problem in the form of a Lagrange problem: maximize

$$\int_0^\infty r^{n-1}|u_1(r)|^q dr \tag{2.4}$$

subject to

$$\begin{aligned} &u_2'(r)=r^n|u_1'(r)|^p\\ &u_2(0)=0\\ &u_2(r)\to 1 \text{ and } u_1(r)\to 0 \text{ as } r\to\infty. \end{aligned} \tag{2.5}$$

It is clear that maximizing J is equivalent to the Lagrange problem (2.4). From the constraints (2.5), we have $\int_0^\infty r^n|u_1'(r)|^p dr=1$, and so we point out that

$$u_1(r)=o(r^{1-\frac{n+1}{p}}) \text{ as } r\to 0 \text{ or } \infty. \tag{2.6}$$

5. We consider the two parameter family of extremals

$$\begin{aligned} \varphi_1(r) &= a(1+br)^{1-\frac{n}{p-1}}\\ \varphi_2(r) &= \int_0^r t^n|\varphi_1'(t)|^p dt=\varphi_1(r)^p r^{n+1-p} f\left(\frac{br}{1+br}\right), \end{aligned}$$

where we put

$$f(\xi):=\left(\frac{n+1-p}{p-1}\right)^p \xi^p\int_0^1(1-t)^n(1-\xi t)^{-\frac{np}{p-1}}dt. \tag{2.7}$$

Then it is a *Mayer field* in the first octant $\{(r,u_1,u_2)\in\mathbb{R}^3|r>0,u_1,u_2>0\}:=\mathbb{R}^3_+$. In other words, the paths $\alpha(r)=(r,\varphi_1(r),\varphi_2(r))$ $(r>0)$ are the trajectories of a *smooth vector field* X defined on $\mathbb{R}^3_+$. Thus exactly one such path passes through any point in $\mathbb{R}^3_+$, and $X(\alpha(r))$ is the slope of the path passing through this point at the point $\alpha(r)$, that is,

$$\frac{d}{dr}\alpha(r)=X(\alpha(r)):=(X_0,X_1,X_2) \tag{2.8}$$

or equivalently to say, the vector field X is defined as follows: for any $(r,u_1,u_2)\in\mathbb{R}^3_+$, $(r,u_1,u_2)=(r,\varphi_1(r),\varphi_2(r))$ for some $\alpha(r)=(r,\varphi_1(r),\varphi_2(r))$, and so

$$\begin{aligned} X_0(r,u_1,u_2) &= \frac{d}{dr}(r)=1\\ X_1(r,u_1,u_2) &= \frac{d}{dr}\{\varphi_1(r)\}=-\left(\frac{n+1-p}{p-1}\right)\frac{u_1}{r}\xi\\ X_2(r,u_1,u_2) &= \frac{d}{dr}\{\varphi_2(r)\}=r^n|X_1(r,u_1,u_2)|^p, \end{aligned} \tag{2.9}$$

where ξ is the root of the equation

$$f(\xi) = r^{p-(n+1)} u_1^{-p} u_2 \quad \text{for } 0 < \xi < 1. \tag{2.10}$$

6. Note that the equation (2.10) has exactly one solution ξ as long as $r, u_1 > 0$ and $u_2 \geqslant 0$ due to the following Lemma:

LEMMA 1. *The function* $f : [0,1) \to [0,\infty)$ *is a monotone increasing bijection. Moreover, we have the following asymptotic behaviors:*

$$\xi^{-p} f(\xi) \to \left(\frac{n+1-p}{p-1}\right)^p \frac{1}{n+1} \quad \text{as } r \to 0, \tag{2.11}$$

$$\frac{\xi^{n+1-p} f(\xi)}{(1-\xi)^{1-\frac{n}{p-1}}} \to \left(\frac{n+1-p}{p-1}\right)^p B\left(n+1, \frac{n}{p-1} - 1\right) \quad \text{as } r \to \infty, \tag{2.12}$$

where B *represents the beta function.*

Proof. We differentiate (2.7) to get

$$\begin{aligned} f'(\xi) = \left(\frac{n+1-p}{p-1}\right)^p \Bigg[p\xi^{p-1} \int_0^1 (1-t)^n (1-\xi t)^{-p'n} dt \\ + \xi^p \int_0^1 (1-t)^n (-p'n)(1-\xi t)^{-p'n-1}(-t) dt \Bigg] \\ = \left(\frac{n+1-p}{p-1}\right)^p \int_0^1 (1-t)^n (1-\xi t)^{-p'n-1} \xi^{p-1} [p - p\xi t + \xi p' n t] dt, \end{aligned}$$

where p' represents the Hölder conjugate $\frac{p}{p-1}$ of p. Since

$$p - p\xi t + \xi p' n t = p + t\xi(p'n - p) = p + \xi t p \frac{n-p+1}{p-1} > 0,$$

we have $f'(\xi) > 0$ if $0 < \xi < 1$. It has been proved that the function f is increasing.

In order to demonstrate $f(\xi) \to 0$ as $\xi \to 0$, we first observe that

$$\begin{aligned} \xi^{-p} f(\xi) &= \left(\frac{n+1-p}{p-1}\right)^p \int_0^1 (1-t)^n (1-\xi t)^{-p'n} dt \\ &\to \left(\frac{n+1-p}{p-1}\right)^p \int_0^1 (1-t)^n dt = \left(\frac{n+1-p}{p-1}\right)^p \frac{1}{n+1} \end{aligned}$$

as $\xi \to 0$. So we get that $f(\xi) \to 0$ as $\xi \to 0$. From the calculation that

$$\begin{aligned}\int_0^1 (1-t)^n(1-\xi t)^{-np'}dt &= \int_0^1 \left(\frac{1-t}{1-\xi t}\right)^{np'} (1-t)^{n-np'}dt \\ &= \int_0^1 s^{np'} \left\{\frac{s(1-\xi)}{1-\xi s}\right\}^{-\frac{n}{p-1}} \frac{1-\xi}{(1-\xi s)^2}ds \\ &= (1-\xi)^{-\frac{n}{p-1}+1} \int_0^1 \frac{t^n}{(1-\xi t)^{-\frac{n}{p-1}+2}}dt \\ &= (1-\xi)^{-\frac{n}{p-1}+1}\xi^{-(n+1)} \int_0^{\xi} t^n(1-t)^{\frac{n}{p-1}-2}dt, \end{aligned} \quad (2.13)$$

we have

$$(1-\xi)^{\frac{n}{p-1}-1} f(\xi) \left(\frac{n+1-p}{p-1}\right)^{-p} = \xi^{p-(n+1)} \textstyle\int_0^{\xi} t^n(1-t)^{\left(\frac{n}{p-1}-1dt\right)-1}. \quad (2.14)$$

Therefore as $\xi \to 1^-$, we have

$$\xi^{n+1-p}(1-\xi)^{\frac{n}{p-1}-1} f(\xi) \to \left(\tfrac{n+1-p}{p-1}\right)^p B\left(n+1, \tfrac{n}{p-1}-1\right).$$

This implies the fact that $\lim_{\xi \to 1^-} f(\xi) = \infty$. $\square$

Before we proceed, we would like to point out two facts. First, differentiating both sides of (2.14), we have a representation for the derivative of f:

$$f'(\xi) = \left(\frac{n+1-p}{p-1}\right)^p \frac{\xi^{p-1}}{1-\xi} + \left(\frac{n+1-p}{p-1}\right) \frac{\xi p - p+1}{\xi(1-\xi)} f(\xi). \quad (2.15)$$

Next, concerning the function $\xi(r, u_1, u_2)$, we see that from (2.3)

$$(r^n|X_1|^{p-1})' = n\left[\frac{n+1-p}{p-1}\right]^{p-1} r^{n-p}u_1^{p-1}\xi^{p-1}(1-\xi). \quad (2.16)$$

7. In the following we will show that there exists an exact differential dW satisfying the following: for any path $c: (0,\infty) \to \mathbb{R}^3_+$ of the form $c(r) = (r, u_1(r), u_2(r))$ with $u_2'(r) = r^n|u_1'(r)|^p$, we have

$$\int_c dW \geqslant \int_0^{\infty} r^{n-1}|u_1(r)|^q dr,$$

and equality holds when the path is an extremal as follows:

$$u_1(r) = a(1+br)^{1-\frac{n}{p-1}}, \quad \text{and} \quad u_2(r) = \int_0^r t^n|u_1'(t)|^p dt.$$

8. We look tentatively at a twice differentiable real valued function W defined on $\mathbb{R}^3_+$ satisfying the following property: for any $(r,u_1,u_2) \in \mathbb{R}^3_+$, the function $\Phi : M \to \mathbb{R}$ defined by

$$\Phi(\xi_0,\xi_1,\xi_2) = r^{n-1}u_1^q\xi_0 - \frac{\partial W}{\partial r}(r,u_1,u_2)\xi_0 - \frac{\partial W}{\partial u_1}(r,u_1,u_2)\xi_1 - \frac{\partial W}{\partial u_2}(r,u_1,u_2)\xi_2$$

has a critical point at $X(r,u_1,u_2)$ on M, where

$$M = \left\{ (\xi_0,\xi_1,\xi_2) \,\middle|\, \xi_0 > 0,\ \xi_0^{p-1}\xi_2 = r^n|\xi_1|^p \right\}$$

is the cone of all direction issuing from the point (r,u_1,u_2). Note that by the way of defining M, we have $\xi_2 > 0$, and also

$$\left(\frac{\partial \Phi}{\partial \xi_0}, \frac{\partial \Phi}{\partial \xi_1}, \frac{\partial \Phi}{\partial \xi_2}\right) = \left(r^{n-1}u_1^q - \frac{\partial W}{\partial r}, -\frac{\partial W}{\partial u_1}, -\frac{\partial W}{\partial u_2}\right).$$

Define $G(\xi_0,\xi_1,\xi_2) := \xi_0^{p-1}\xi_2 - r^n|\xi_1|^p$. Then simple computations yield that

$$\begin{aligned}
\frac{\partial G}{\partial \xi_0} &= (p-1)\xi_0^{p-2}\xi_2\,\big|_{(X_0,X_1,X_2)} = (p-1)r^n|X_1|^p \\
\frac{\partial G}{\partial \xi_1} &= -r^n p|\xi_1|^{p-1}\text{sgn}\xi_1\,\big|_{(X_0,X_1,X_2)} = -r^n p|X_1|^{p-1}\text{sgn}X_1 = pr^n|X_1|^{p-1} \\
\frac{\partial G}{\partial \xi_2} &= \xi_0^{p-1}\,\Big|_{(X_0,X_1,X_2)} = X_0^{p-1} = 1.
\end{aligned}$$

at $X(r,u_1,u_2) := (X_0,X_1,X_2)$ (see (2.9)). Lagrange's multipliers rule gives the following relation at the critical point $X(r,u_1,u_2) = (X_0,X_1,X_2)$

$$\nabla\Phi = \lambda\nabla G,$$

where $\lambda(r,u_1,u_2)$ is a differentiable function to be determined. That is,

$$\begin{aligned}
\frac{\partial W}{\partial r} &= r^{n-1}u_1^q - \lambda(p-1)r^n|X_1|^p \\
\frac{\partial W}{\partial u_1} &= -\lambda p r^n |X_1|^{p-1} \\
\frac{\partial W}{\partial u_2} &= -\lambda.
\end{aligned} \tag{2.17}$$

We notice that the directional derivative $\nabla_X W$ of W in the direction of the vector field X is $\nabla_X W := \left[\frac{\partial}{\partial r} + X_1\frac{\partial}{\partial u_1} + r^n|X_1|^p\frac{\partial}{\partial u_2}\right]W = r^{n-1}u_1^q$.

The compatibility conditions: $\frac{\partial}{\partial r}\left(\frac{\partial W}{\partial u_2}\right) = \frac{\partial}{\partial u_2}\left(\frac{\partial W}{\partial r}\right)$, $\frac{\partial}{\partial u_1}\left(\frac{\partial W}{\partial u_2}\right) = \frac{\partial}{\partial u_2}\left(\frac{\partial W}{\partial u_1}\right)$, $\frac{\partial}{\partial r}\left(\frac{\partial W}{\partial u_1}\right) = \frac{\partial}{\partial u_1}\left(\frac{\partial W}{\partial r}\right)$ can be arranged in the form of an overdetermined system of linear partial differential equations of the first order in the unknown $\nabla\lambda$:

$$\begin{pmatrix} 1 & 0 & a_1 \\ 0 & 1 & a_2 \\ a_2 & -a_1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial\lambda}{\partial r} \\ \frac{\partial\lambda}{\partial u_1} \\ \frac{\partial\lambda}{\partial u_2} \end{pmatrix} = \begin{pmatrix} -p\lambda(X_1\frac{\partial}{\partial u_2})(r^n|X_1|^{p-1}) \\ p\lambda\frac{\partial}{\partial u_2}(r^n|X_1|^{p-1}) \\ qr^{n-1}u_1^{q-1} + \lambda p\left[\frac{\partial}{\partial r} + X_1\frac{\partial}{\partial u_1}\right](r^n|X_1|^{p-1}) \end{pmatrix},$$

where we set $a_1 = -(p-1)r^n|X_1|^p$, $a_2 = -pr^n|X_1|^{p-1}$. Consider the eigenvectors for A^T, where A is the 3×3 matrix in the argument. Some computation gives us that the eigenvalues λ of A^T are 0 and 1. It also turns out that

$$\vec{v} = (-a_2, a_1, 1) = (pr^n|X_1|^{p-1}, -(p-1)r^n|X_1|^p, 1)$$

is an eigenvector corresponding to the eigenvalue 0, which implies that

$$\vec{b}\cdot\vec{v} = A\vec{x}\cdot\vec{v} = \vec{x}\cdot A^T\vec{v} = 0,$$

where $\vec{x} = \left(\frac{\partial\lambda}{\partial r}, \frac{\partial\lambda}{\partial u_1}, \frac{\partial\lambda}{\partial u_2}\right)^T$. From this observation, we obtain that

$$qr^{n-1}u_1^{q-1} + \lambda p\nabla_X(r^n|X_1|^{p-1}) = 0. \tag{2.18}$$

The identity (2.16) can be rewritten as

$$\begin{aligned} \nabla_X(r^n|X_1|^{p-1}) &= X\cdot\nabla(r^n|X_1|^{p-1}) \\ &= (r^n|X_1|^{p-1})' \\ &= n\left[\frac{n+1-p}{p-1}\right]^{p-1} r^{n-p}u_1^{p-1}\xi^{p-1}(1-\xi), \end{aligned}$$

where ξ is the root of the equation (2.10). Applying this into (2.18), we get

$$\lambda = -\frac{(p-1)^{p-1}}{(n+1-p)^p}\frac{r^{p-1}u_1^{q-p}}{\xi^{p-1}(1-\xi)}. \tag{2.19}$$

It can be proved that the function λ defined by (2.19) is actually a solution to the system (2.17). The procedure described above gives us a smooth solution W to the system when we use λ as defined in (2.19).

9. Now we want to show that the differential dW maximizes the functional J. Let

$$\begin{aligned} E(r,u_1,u_2;\xi_0,\xi_1,\xi_2) := r^{n-1}u_1^q\xi_0 &- \frac{\partial W}{\partial r}(r,u_1,u_2)\xi_0 \\ &- \frac{\partial W}{\partial u_1}(r,u_1,u_2)\xi_1 - \frac{\partial W}{\partial u_2}(r,u_1,u_2)\xi_2, \end{aligned}$$

under the constraints: $\xi_0 > 0$ and $\xi_0^{p-1}\xi_2 = r^n|\xi_1|^p$. From (2.17), we have

$$\begin{aligned} &E(r,u_1,u_2;\xi_0,\xi_1,\xi_2) \\ &= \left(\frac{\partial W}{\partial r} + (p-1)r^n|X_1|^p\lambda\right)\xi_0 - \frac{\partial W}{\partial r}\xi_0 - \frac{\partial W}{\partial u_1}\xi_1 - \frac{\partial W}{\partial u_2}\xi_2 \\ &= \xi_0 r^n\lambda(r,u_1,u_2)\left[(p-1)|X_1|^p + p\frac{\xi_1}{\xi_0}|X_1|^{p-1} + \left|\frac{\xi_1}{\xi_0}\right|^p\right]. \end{aligned}$$

The last equation follows from the relation $\frac{\xi_2}{\xi_0} = r^n \left|\frac{\xi_1}{\xi_0}\right|^p$. It can be shown (via elementary calculus) that the expression in the bracket is always nonnegative and vanishes if and only if (ξ_0, ξ_1, ξ_2) is parallel to $X(r,u_1,u_2)$. Overall, we have

$$E(r,u_1,u_2;\xi_0,\xi_1,\xi_2) \leqslant 0 \tag{2.20}$$

since $\xi_0 r^n \lambda < 0$ and $E = 0$ only when (ξ_0,ξ_1,ξ_2) is parallel to $X(r,u_1,u_2)$.

10. Now an explicit expression for the function W can be derived, which is needed to determine the boundary behavior of W. Put λ in equation (2.17) and we have

$$\frac{\partial W}{\partial r} = \frac{r^{n-1}u_1^q}{1-\xi}, \tag{2.21}$$

$$\frac{\partial W}{\partial u_1} = \frac{p}{n+1-p}\frac{r^n u_1^{q-1}}{1-\xi} = \frac{q}{n}\frac{r^n u_1^{q-1}}{1-\xi}, \tag{2.22}$$

$$\frac{\partial W}{\partial u_2} = \frac{(p-1)^{p-1}}{(n+1-p)^p}\frac{r^{p-1}u_1^{q-p}}{\xi^{p-1}(1-\xi)}, \tag{2.23}$$

where ξ is the root of the equation (2.10). Denoting $\bar{W}(r,u_1) := W(r,u_1,0)$, we note that (2.21) and (2.22) imply that

$$\frac{\partial \bar{W}}{\partial r}(r,u_1) = r^{n-1}u_1^q \quad \text{and} \quad \frac{\partial \bar{W}}{\partial u_1}(r,u_1) = \frac{q}{n}r^n u_1^{q-1},$$

which, in turn, leads to

$$W(r,u_1,0) = \bar{W}(r,u_1) = \frac{1}{n}r^n u_1^q + C_0 \tag{2.24}$$

for some constant C_0. Hence from (2.23) together with (2.24), we obtain that

$$\begin{aligned} W(r,u_1,u_2) &= W(r,u_1,0) + \frac{(p-1)^{p-1}}{(n+1-p)^p}\int_0^{u_2} \frac{r^{p-1}u_1^{q-p}}{\xi^{p-1}(1-\xi)}du_2 \\ &= \frac{1}{n}r^n u_1^q + \frac{(p-1)^{p-1}}{(n+1-p)^p}\int_0^{u_2} \frac{r^{p-1}u_1^{q-p}}{\xi^{p-1}(1-\xi)}du_2 + C_0. \end{aligned} \tag{2.25}$$

We now consider $f(\xi) = r^{p-(n+1)}u_1^{-p}u_2$ in order to find the integral

$$\int_0^{u_2} \frac{r^{p-1}u_1^{q-p}}{\xi^{p-1}(1-\xi)}du_2 = r^n u_1^q \int_0^{\xi} \frac{f'(\xi)}{\xi^{p-1}(1-\xi)}d\xi. \tag{2.26}$$

Integration by parts together with the asymptotic behavior (2.11) and the formula (2.15) successively implies

$$\begin{aligned}
&\int_0^\xi \frac{f'(\xi)}{\xi^{p-1}(1-\xi)}d\xi \\
&= \frac{f(\xi)}{\xi^{p-1}(1-\xi)}\bigg|_0^\xi - \int_0^\xi f(\xi)\frac{d}{d\xi}\left(\frac{1}{\xi^{p-1}(1-\xi)}\right)d\xi \\
&= \frac{f(\xi)}{\xi^{p-1}(1-\xi)} - \int_0^\xi f(\xi)\frac{\xi p-(p-1)}{\xi^p(1-\xi)^2}d\xi \\
&= \frac{f(\xi)}{\xi^{p-1}(1-\xi)} - \int_0^\xi \left[\frac{p-1}{n+1-p}\frac{f'(\xi)}{\xi^{p-1}(1-\xi)} - \left(\frac{n+1-p}{p-1}\right)^{p-1}\frac{1}{(1-\xi)^2}\right]d\xi.
\end{aligned}$$

Hence we have

$$\left(\int_0^\xi \frac{f'(\xi)}{\xi^{p-1}(1-\xi)}d\xi\right)\left(1+\frac{p-1}{n+1-p}\right) = f(\xi)\frac{1}{\xi^{p-1}(1-\xi)} + \left(\frac{n+1-p}{p-1}\right)^{p-1}\frac{\xi}{1-\xi}.$$

Therefore we get

$$\int_0^\xi \frac{f'(\xi)}{\xi^{p-1}(1-\xi)}d\xi = \frac{(n+1-p)f(\xi)}{n\xi^{p-1}(1-\xi)} + \frac{n+1-p}{n}\left(\frac{n+1-p}{p-1}\right)^{p-1}\frac{\xi}{1-\xi}.$$

Applying this into (2.26) and (2.25), W can be written as

$$W(r,u_1,u_2) = \frac{r^n u_1^q}{n(1-\xi)} + \frac{1}{n}\left(\frac{p-1}{n+1-p}\right)^{p-1}\frac{r^{p-1}u_1^{q-p}u_2}{\xi^{p-1}(1-\xi)} + C_0. \tag{2.27}$$

11. Let $(u_1(r),u_2(r))$ be a pair satisfying the conditions (2.5). By looking at asymptotic behaviors of W, we first demonstrate that $\lim_{r\to 0} W(r,u_1(r),u_2(r))$ exists, and it is precisely C_0 by means of finding that the first and second terms in (2.27) vanish as r goes to zero. We recall (2.6): $u_1(r) = o(r^{1-\frac{n+1}{p}})$ as $r\to 0$. Hence the first term in the right side of (2.27) tends to 0:

$$\frac{r^n u_1^q}{n}\frac{1}{1-\xi} = \left(\frac{u_1}{r^{1-\frac{n+1}{p}}}\right)^q \frac{1}{1-\xi} \to 0$$

as $r\to 0$ (so $\xi\to 0$). Also the asymptotic behavior (2.11) explains the diminishment of the second term in (2.27):

$$\frac{r^{p-1}u_1^{q-p}u_2}{\xi^{p-1}(1-\xi)} = \frac{r^n u_1^q}{1-\xi}\frac{f(\xi)}{\xi^{p-1}} \to 0$$

as $r\to 0$ ($\xi\to 0$). Next we consider $\lim_{r\to\infty} W(r,u_1(r),u_2(r))$. We can rewrite the terms in (2.27):

$$\frac{r^n u_1^q}{n}\frac{1}{1-\xi} = \frac{u_2^{\frac{q}{p}}}{n}\xi^n\left(\frac{\xi^{n+1-p}f(\xi)}{(1-\xi)^{1-\frac{n}{p-1}}}\right)^{-\frac{q}{p}}(1-\xi)^{\frac{n}{p-1}-1} \to 0$$

as $r \to \infty$ (so $\xi \to 1$) by the asymptotic behavior (2.12), and also

$$\begin{aligned}\frac{r^{p-1}u_1^{q-p}u_2}{\xi^{p-1}(1-\xi)} &= u_2^{\frac{q}{p}}\left(\frac{\xi^{n+1-p}f(\xi)}{(1-\xi)^{1-\frac{n}{p-1}}}\right)^{1-\frac{q}{p}} \\ &\to \left[\left(\frac{n+1-p}{p-1}\right)^p B\left(n+1,\frac{n}{p-1}-1\right)\right]^{1-\frac{q}{p}} \\ &= \left(\frac{n+1-p}{p-1}\right)^{\frac{-p(p-1)}{n+1-p}}\left[B\left(n+1,\frac{n}{p-1}-1\right)\right]^{\frac{-(p-1)}{n+1-p}}\end{aligned}$$

as $r \to \infty$. Hence we conclude

$$\begin{aligned}&\lim_{r\to\infty} W(r,u_1(r),u_2(r)) \\ &= C_0 + \frac{1}{n}\left(\frac{p-1}{n+1-p}\right)^{\frac{(n+1)(p-1)}{n+1-p}}\left[B\left(n+1,\frac{n+1-p}{p-1}\right)\right]^{-\frac{p-1}{n+1-p}}. \qquad (2.28)\end{aligned}$$

12. We know from (2.20) that

$$\begin{aligned}\int_0^\infty r^{n-1}|u_1(r)|^q dr &\leqslant \int_0^\infty \nabla W(r,u_1(r),u_2(r))\cdot(1,u_1'(r),u_2'(r))dr \left(=\int_c dW\right) \\ &= \lim_{r\to\infty} W(r,u_1(r),u_2(r)) - \lim_{r\to 0} W(r,u_1(r),u_2(r)),\end{aligned}$$

which is independent of the path by the above observation. Combine these estimates together with (2.28) and the equality condition of the estimate (2.20) to have that

$$\int_0^\infty r^{n-1}|u_1(r)|^q dr \leqslant \frac{1}{n}\left(\frac{p-1}{n+1-p}\right)^{\frac{(n+1)(p-1)}{n+1-p}}\left[B\left(n+1,\frac{n+1-p}{p-1}\right)\right]^{-\frac{p-1}{n+1-p}}$$

and the equality holds if (and only if) $(1,u_1',u_2')$ is parallel to $X(r,u_1,u_2)$. Therefore we can conclude that

$$\begin{aligned}&\left(\int_0^\infty r^{n-1}|u_1(r)|^q dr\right)^{1/q} \\ &\leqslant \left(\frac{1}{n}\left(\frac{p-1}{n+1-p}\right)^{\frac{(n+1)(p-1)}{n+1-p}}\left[B\left(n+1,\frac{n+1-p}{p-1}\right)\right]^{-\frac{p-1}{n+1-p}}\right)^{1/q} \\ &= n^{-\frac{1}{q}}\left(\frac{p-1}{n+1-p}\right)^{\frac{q-1}{q}}\left[B\left(n+1,\frac{n+1-p}{p-1}\right)\right]^{\frac{1}{q}-\frac{1}{p}} \\ &= n^{-\frac{1}{p}}\left(\frac{p-1}{n+1-p}\right)^{\frac{q-1}{q}}\left[\frac{\Gamma(n)\Gamma(\frac{n}{p-1}-1)}{\Gamma(\frac{np}{p-1})}\right]^{\frac{1}{q}-\frac{1}{p}}\left[\int_0^\infty |u'(t)|^p t^n dt\right]^{1/p}.\end{aligned}$$

The proof is now completed. $\square$

Acknowledgement. The authors were informed at the acceptance of the paper that the results discussed here would be available shortly in [18] with a more general setting as well as with a completely different approach.

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(Received August 17, 2014)

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