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THE HERMITIAN AND NONNEGATIVE DEFINITE SOLUTIONS OF $AX = B$ SUBJECT TO $CXC^* \geqslant D$

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Abstract. In this paper, we first establish some necessary and sufficient conditions for the existence of Hermitian and nonnegative definite solutions of $AX = B$ subject to $CXC^* \geqslant D$, where D is Hermitian matrix. Furthermore, general expressions for this constrained Hermitian and nonnegative definite solutions are derived, several special cases are also considered.

1. Introduction

Let $\mathbb{C}^{m\times n}$ denote the set of all $m\times n$ matrices over the complex field $\mathbb{C}$, $\mathbb{C}_H^{m\times m}$ denote the set of all $m\times m$ Hermitian matrices, $\mathbb{C}_{\geqslant}^{n\times n}$ denote the set of all $n\times n$ nonnegative definite matrices. For $A\in\mathbb{C}^{m\times n}$, its rank, conjugate transpose, any $\{1\}$-inverse and Moore-Penrose inverse will be denoted by $r(A)$, A^*, A^- and $A^\dagger$ respectively. For nonnegative definite matrix $A\geqslant 0$, its positive and negative index of inertia are symbolled by $i_+(A)$ and $i_-(A)$ respectively. For convenience, we denote $E_A = I - AA^\dagger$ and $F_A = I - A^\dagger A$.

Linear matrix equations play a very important role in matrix theory and other disciplines, such as statistics and control theory. And researches on linear matrix equations have received more and more attentions and has had lots of nice results. Recently, a challenging research topic is to solve matrix equations subject matrix inequality constraint concerning Löwner partial ordering. For example, Li et al. [3] presented some necessary and sufficient conditions for $X \geqslant (\leqslant, >, <)P$, where X is a Hermitian least squares solution to the matrix equation $AXB = C$. Zhang al. [9] derived necessary and sufficient conditions for $X \geqslant (\leqslant, >, <)P$, where X is common Hermitian least squares solution to the matrix equations $A_1XA_1^* = B_1$ and $A_2XA_2^* = B_2$. However, the general expressions weren't established for these two constrained solutions concerning inequality restrictions.

In this article, we consider the Hermitian and nonnegative definite solutions of $AX = B$ subject to $CXC^* \geqslant D$, where $A,B\in\mathbb{C}^{m\times n}$, $C\in\mathbb{C}^{p\times n}$ and $D\in\mathbb{C}_H^{p\times p}$ are given, $X\in\mathbb{C}^{n\times n}$ is variable matrix.

Before proceeding to the next section, we first introduce the following results which will come in handy in the proofs of our theorems.

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LEMMA 1.1. [2] *Let* $A, B \in \mathbb{C}^{m\times n}$.

(1) *Matrix equation* $AX = B$ *has a Hermitian solution if and only if* $AA^{\dagger}B = B$ *and* BA^* *is Hermitian. In this case, the general Hermitian solution can be written in the parametric form*

$$X = A^{\dagger}B + F_A(A^{\dagger}B)^* + F_AUF_A,$$

where $U \in \mathbb{C}_H^{n\times n}$ *is arbitrary.*

(2) *Matrix equation* $AX = B$ $(or\ AXX^* = B)$ *has a nonnegative definite solution if and only if* BA^* *is nonnegative definite and* $r(BA^*) = r(B)$. *In this case, the general nonnegative definite solution can be written in the parametric form*

$$X(or\ XX^*) = B^*(AB^*)^{\dagger}B + F_AWW^*F_A,$$

where $W \in \mathbb{C}^{n\times n}$ *is arbitrary.*

LEMMA 1.2. [5] *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}_H^{m\times m}$. *Then the matrix equation* $AXA^* = B$ *has a Hermitian solution if and only if* $AA^{\dagger}B = B$. *In this case, the general Hermitian solution can be written in the parametric form*

$$X = A^{\dagger}B(A^{\dagger})^* + F_AZ + Z^*F_A,$$

where $Z \in \mathbb{C}^{n\times n}$ *is arbitrary.*

LEMMA 1.3. [1] *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}_{\geqslant}^{m\times m}$. *Then the matrix equation* $AXA^* = B$ *has a nonnegative definite solution if and only if* $AA^{\dagger}B = B$. *In this case, the general nonnegative definite solution can be written in the parametric form*

$$X = A^{=}B(A^{=})^* + F_AUF_A \ \ with\ \ A^{=} = A^{\dagger} + F_AZ(B^{\frac{1}{2}})^-,$$

where $Z \in \mathbb{C}^{n\times m}$ *and* $U \in \mathbb{C}_{\geqslant}^{n\times n}$ *are arbitrary.*

LEMMA 1.4. [4] *Let* $A \in \mathbb{C}_H^{m\times m}$, $B \in \mathbb{C}^{m\times n}$, *and denote* $M = \begin{pmatrix} A & B \\ B^* & 0 \end{pmatrix}$. *Then*

$$i_{\pm}(M) = r(B) + i_{\pm}(E_BAE_B).$$

LEMMA 1.5. [8] *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ *and* $C \in \mathbb{C}^{l\times n}$. *Then*

$$r\begin{pmatrix} A & B \end{pmatrix} = r(A) + r(E_AB), \ \ r\begin{pmatrix} A \\ C \end{pmatrix} = r(A) + r(CF_A),$$

$$r\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = r(B) + r(C) + r(E_BAF_C).$$

2. Main results

In this section, our purpose is to investigate the Hermitian and nonnegative definite solutions of $AX = B$ subject to $CXC^* \geqslant D$. First, we present some necessary and sufficient conditions for the existence of these constrained solutions, and then establish their general expressions.

LEMMA 2.1. [7] *Let $C \in \mathbb{C}^{p\times n}$ and $D \in \mathbb{C}_H^{p\times p}$. Then $CXC^* \geqslant D$ has a Hermitian solution if and only if $E_CDE_C \leqslant 0$ and $r(E_CDE_C) = r(E_CD)$. In which case, a general expression of the Hermitian solution is given by*

$$X = C^\dagger[D - DE_C(E_CDE_C)^\dagger E_CD + WW^*](C^\dagger)^* + Y - C^\dagger CYC^\dagger C,$$

which is equivalent to

$$X = C^\dagger[D - DE_C(E_CDE_C)^\dagger E_CD + WW^*](C^\dagger)^* + F_CZ + Z^*F_C, \tag{2.1}$$

where $W, Z \in \mathbb{C}^{n\times n}$ and $Y \in \mathbb{C}_H^{n\times n}$ are arbitrary.

It is important to point out that the representation of (2.1) can be derived by Lemma 1.2 by the same method used in [Theorem 4.1, 7].

Similarly, the following result follows from Lemma 1.1 and Lemma 1.3.

LEMMA 2.2. *Let $C \in \mathbb{C}^{p\times n}$ and $D \in \mathbb{C}_{\geqslant}^{p\times p}$. Then $CXX^*C^* \geqslant D$ has a nonnegative definite solution XX^* if and only if $E_CDE_C \leqslant 0$ and $r(E_CDE_C) = r(E_CD)$. In which case, a general expression of the nonnegative definite solution is XX^* with*

$$XX^* = C^=(D + VV^*)(C^=)^* + F_CUF_C \text{ with } C^= = C^\dagger + F_CZ[(D + VV^*)^{\frac{1}{2}}]^-,$$

where $VV^* = -DE_C(E_CDE_C)^\dagger E_CD + CC^\dagger WW^*CC^\dagger$, $W, Z \in \mathbb{C}^{n\times n}$ and $U \in \mathbb{C}_{\geqslant}^{n\times n}$ are arbitrary.

Next, we will derive the main results of this paper.

THEOREM 2.1. *Let $A, B \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{p\times n}$ and $D \in \mathbb{C}_H^{p\times p}$. Then there exists a Hermitian solution of $AX = B$ subject to $CXC^* \geqslant D$ if and only if $AA^\dagger B = B$, BA^* is Hermitian, and*

$$i_+\begin{pmatrix} D & C & 0 \\ C^* & 0 & A^* \\ 0 & A & -BA^* \end{pmatrix} = r\begin{pmatrix} A \\ C \end{pmatrix}, \quad r\begin{pmatrix} D & C & 0 \\ C^* & 0 & A^* \\ 0 & A & -BA^* \end{pmatrix} = r\begin{pmatrix} D & C \\ BC^* & A \end{pmatrix} + r\begin{pmatrix} A \\ C \end{pmatrix}. \tag{2.2}$$

If the above conditions are satisfied, then a general constrained Hermitian solution is given by

$$\begin{aligned} X = {} & A^\dagger B + F_A(A^\dagger B)^* + (CF_A)^\dagger[\tilde{D} - \tilde{D}E_{CF_A}(E_{CF_A}\tilde{D}E_{CF_A})^\dagger E_{CF_A}\tilde{D} + WW^*](F_AC^*)^\dagger \\ & F_AF_{CF_A}ZF_A + F_AZ^*F_{CF_A}F_A, \end{aligned} \tag{2.3}$$

where $\tilde{D} = D - CA^\dagger BC^* - CF_A(CA^\dagger B)^*$, $W, Z \in \mathbb{C}^{n\times n}$ are arbitrary.

Proof. In view of Lemma 1.1, $AX=B$ has a Hermitian solution if and only if $AA^{\dagger}B=B$ and BA^{*} is Hermitian. In this case, the general Hermitian solution can be written in the parametric form

$$X=A^{\dagger}B+F_A(A^{\dagger}B)^{*}+F_AUF_A, \tag{2.4}$$

where $U\in\mathbb{C}_H^{n\times n}$ is arbitrary. Substituting (2.4) into $CXC^{*}\geqslant D$ produces

$$CF_AU(CF_A)^{*}\geqslant D-CA^{\dagger}BC^{*}-CF_A(CA^{\dagger}B)^{*}, \tag{2.5}$$

Applying Lemma 2.1, (2.5) is consistent if and only if

$$E_{CF_A}(D-CA^{\dagger}BC^{*}-CF_A(CA^{\dagger}B)^{*})E_{CF_A}\leqslant 0,$$

$$r[E_{CF_A}(D-CA^{\dagger}BC^{*})E_{CF_A}]=r[E_{CF_A}(D-CA^{\dagger}BC^{*})].$$

By Lemma 1.4, Lemma 1.5, and the fact that Hermitian matrix $M\leqslant 0$ if and only if $i_{+}(M)=0$, we have

$$\begin{aligned}
0&=i_{+}[E_{CF_A}(D-CA^{\dagger}BC^{*}-CF_A(CA^{\dagger}B)^{*})E_{CF_A}]\\
&=i_{+}\begin{pmatrix} D-CA^{\dagger}BC^{*}-C(A^{\dagger}B)^{*}C^{*}+CA^{\dagger}A(A^{\dagger}B)^{*}C^{*} & CF_A\\ F_AC^{*} & 0\end{pmatrix}-r(CF_A)\\
&=i_{+}\begin{pmatrix} D-CA^{\dagger}BC^{*}-C(A^{\dagger}B)^{*}C^{*}+CA^{\dagger}A(A^{\dagger}B)^{*}C^{*} & C & 0\\ C^{*} & 0 & A^{*}\\ 0 & A & 0\end{pmatrix}-r(A)-r(CF_A)\\
&=i_{+}\begin{pmatrix} D+CA^{\dagger}A(A^{\dagger}B)^{*}C^{*} & C & CB^{*}\\ C^{*} & 0 & A^{*}\\ BC^{*} & A & 0\end{pmatrix}-r\begin{pmatrix}A\\ C\end{pmatrix}\\
&=i_{+}\begin{pmatrix} D+CA^{\dagger}A(A^{\dagger}B)^{*}C^{*} & C & 0\\ C^{*} & 0 & A^{*}\\ 0 & A & -2BA^{*}\end{pmatrix}-r\begin{pmatrix}A\\ C\end{pmatrix}\\
&=i_{+}\begin{pmatrix} D & C & -\frac{1}{2}CA^{\dagger}AB^{*}\\ C^{*} & 0 & A^{*}\\ -\frac{1}{2}BA^{\dagger}AC^{*} & A & -2BA^{*}\end{pmatrix}-r\begin{pmatrix}A\\ C\end{pmatrix}\\
&=i_{+}\begin{pmatrix} D & C & 0\\ C^{*} & 0 & A^{*}\\ 0 & A & -BA^{*}\end{pmatrix}-r\begin{pmatrix}A\\ C\end{pmatrix},
\end{aligned}$$

$$
\begin{aligned}
r[E_{CF_A}(D - CA^{\dagger}BC^*)E_{CF_A}] &= r\begin{pmatrix} D - CA^{\dagger}BC^* & CF_A \\ F_AC^* & 0 \end{pmatrix} - 2r(CF_A) \\
&= r\begin{pmatrix} D - CA^{\dagger}BC^* & C & 0 \\ C^* & 0 & A^* \\ 0 & A & 0 \end{pmatrix} - 2r\begin{pmatrix} A \\ C \end{pmatrix} \\
&= r\begin{pmatrix} D & C & 0 \\ C^* & 0 & A^* \\ BC^* & A & 0 \end{pmatrix} - 2r\begin{pmatrix} A \\ C \end{pmatrix} \\
&= r\begin{pmatrix} D & C & 0 \\ C^* & 0 & A^* \\ 0 & A & -BA^* \end{pmatrix} - 2r\begin{pmatrix} A \\ C \end{pmatrix}, \\
r[E_{CF_A}(D - CA^{\dagger}BC^*)] &= r\begin{pmatrix} CF_A & D - CA^{\dagger}BC^* \end{pmatrix} - r(CF_A) \\
&= r\begin{pmatrix} C & D - CA^{\dagger}BC^* \\ A & 0 \end{pmatrix} - r(A) - r(CF_A) \\
&= r\begin{pmatrix} C & D \\ A & BC^* \end{pmatrix} - r\begin{pmatrix} A \\ C \end{pmatrix}.
\end{aligned}
$$

Therefore, (2.2) is obvious. And,

$$U = (CF_A)^{\dagger}[\tilde{D} - \tilde{D}E_{CF_A}(E_{CF_A}\tilde{D}E_{CF_A})^{\dagger}E_{CF_A}\tilde{D} + WW^*](F_AC^*)^{\dagger} + F_{CF_A}Z + Z^*F_{CF_A},$$

where $W, Z \in \mathbb{C}^{n\times n}$ are arbitrary.

Substituting U into (2.4) yields (2.3). □

Specially, taking $C = I$ in Theorem 2.1, we have the following result, which was considered by Tian [Theorem 5.3, 6].

COROLLARY 2.1. *Let $A, B \in \mathbb{C}^{m\times n}$, and $D \in \mathbb{C}_H^{p\times p}$. Then there exists a Hermitian solution of $AX = B$ subject to $X \geqslant D$ if and only if $AA^{\dagger}B = B$, BA^* is Hermitian, and*

$$ADA^* \leqslant BA^*, \quad r(ADA^* - BA^*) = r(AD - B).$$

If the above conditions are satisfied, then a general constrained Hermitian solution is given by

$$X = A^{\dagger}B + F_A(A^{\dagger}B)^* + F_A[\tilde{D} - \tilde{D}(A^{\dagger}A(D - A^{\dagger}B)A^{\dagger}A)^{\dagger}\tilde{D} + WW^*]F_A,$$

where $\tilde{D} = D - F_A(A^{\dagger}B)^*$, $W \in \mathbb{C}^{n\times n}$ are arbitrary.

THEOREM 2.2. *Let $A, B \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{p\times n}$ and $D \in \mathbb{C}_{\geqslant}^{p\times p}$. Then there exists a nonnegative definite solution of $AX = B$ subject to $CXC^* \geqslant D$ if and only if $r(BA^*) = r(B)$, BA^* is nonnegative definite, and*

$$i_+\begin{pmatrix} D & C & 0 \\ C^* & 0 & A^* \\ 0 & A & -BA^* \end{pmatrix} = r\begin{pmatrix} A \\ C \end{pmatrix}, \quad r\begin{pmatrix} D & C & 0 \\ C^* & 0 & A^* \\ 0 & A & -BA^* \end{pmatrix} = r\begin{pmatrix} D & C \\ BC^* & A \end{pmatrix} + r\begin{pmatrix} A \\ C \end{pmatrix}. \quad (2.6)$$

If the above conditions are satisfied, then a general constrained nonnegative definite solution is given by

$$X = B^*(AB^*)^\dagger B + F_A WW^* F_A, \tag{2.7}$$

with

$$WW^* = (CF_A)^=(\hat{D} + VV^*)((CF_A)^=)^* + F_{CF_A} U F_{CF_A}, \tag{2.8}$$

$$(CF_A)^= = (CF_A)^\dagger + F_{CF_A} Z[(\hat{D} + VV^*)^{\frac{1}{2}}]^-,$$

$$VV^* = -\hat{D}(E_{CF_A}\hat{D}E_{CF_A})^\dagger \hat{D} + C(CF_A)^\dagger YY^* C(CF_A)^\dagger, \tag{2.9}$$

$$\hat{D} = D - CB^*(AB^*)^\dagger BC^*,$$

$Y, Z \in \mathbb{C}^{n\times n}$ *and* $U \in \mathbb{C}_{\geqslant}^{n\times n}$ *are arbitrary.*

Proof. In view of Lemma 1.1, $AX = B$ has a nonnegative definite solution if and only if $r(BA^*) = r(B)$ and BA^* is nonnegative definite. In this case, the general nonnegative definite solution can be written in the parametric form

$$X = B^*(AB^*)^\dagger B + F_A WW^* F_A, \tag{2.10}$$

where $W \in \mathbb{C}^{n\times n}$ is arbitrary. Substituting (2.10) into $CXC^* \geqslant D$ produces

$$CF_A WW^*(CF_A)^* \geqslant D - CB^*(AB^*)^\dagger BC^*, \tag{2.11}$$

Applying Lemma 2.2, (2.11) is consistent if and only if

$$E_{CF_A}(D - CB^*(AB^*)^\dagger BC^*)E_{CF_A} \leqslant 0,$$

$$r[E_{CF_A}(D - CB^*(AB^*)^\dagger BC^*)E_{CF_A}] = r[E_{CF_A}(D - CB^*(AB^*)^\dagger BC^*)].$$

By Lemma 1.4, Lemma 1.5, we have

$$\begin{aligned} 0 &= i_+[E_{CF_A}(D - CB^*(AB^*)^\dagger BC^*)E_{CF_A}] \\ &= i_+\begin{pmatrix} D - CB^*(AB^*)^\dagger BC^* & C & 0 \\ C^* & 0 & A^* \\ 0 & A & 0 \end{pmatrix} - r(A) - r(CF_A) \\ &= i_+\begin{pmatrix} D & C & \frac{1}{2}CB^* \\ C^* & 0 & A^* \\ \frac{1}{2}BC^* & A & 0 \end{pmatrix} - r\begin{pmatrix} A \\ C \end{pmatrix} \\ &= i_+\begin{pmatrix} D & C & 0 \\ C^* & 0 & A^* \\ 0 & A & -BA^* \end{pmatrix} - r\begin{pmatrix} A \\ C \end{pmatrix}, \end{aligned}$$

$$r[E_{CF_A}(D - CB^*(AB^*)^\dagger BC^*)E_{CF_A}] = r\begin{pmatrix} D - CB^*(AB^*)^\dagger BC^* & C & 0 \\ C^* & 0 & A^* \\ 0 & A & 0 \end{pmatrix} - 2r\begin{pmatrix} A \\ C \end{pmatrix}$$
$$= r\begin{pmatrix} D & C & 0 \\ C^* & 0 & A^* \\ BC^* & A & 0 \end{pmatrix} - 2r\begin{pmatrix} A \\ C \end{pmatrix}$$
$$= r\begin{pmatrix} D & C & 0 \\ C^* & 0 & A^* \\ 0 & A & -BA^* \end{pmatrix} - 2r\begin{pmatrix} A \\ C \end{pmatrix},$$
$$r[E_{CF_A}(D - CB^*(AB^*)^\dagger BC^*)] = r\left(CF_A \; D - CB^*(AB^*)^\dagger BC^* \right) - r(CF_A)$$
$$= r\begin{pmatrix} C & D - CB^*(AB^*)^\dagger BC^* \\ A & 0 \end{pmatrix} - r(A) - r(CF_A)$$
$$= r\begin{pmatrix} C & D \\ A & BC^* \end{pmatrix} - r\begin{pmatrix} A \\ C \end{pmatrix}.$$

Hence, (2.6) is obtained. Together with Lemma 2.2, (2.10) and (2.11), we get (2.8) and (2.9). So, (2.7) is obvious. □

The following result is a direct consequence of Theorem 2.2.

COROLLARY 2.2. *Let $A, B \in \mathbb{C}^{m\times n}$, and $D \in \mathbb{C}^{n\times n}_{\geqslant}$. Then there exists a nonnegative definite solution of $AX = B$ subject to $X \geqslant D$ if and only if $r(BA^*) = r(B)$, BA^* is nonnegative definite, and $ADA^* \leqslant BA^*$, $r(ADA^* - BA^*) = r(AD - B)$.*

If the above conditions are satisfied, then a general constrained nonnegative definite solution is given by

$$X = B^*(AB^*)^\dagger B + F_A[\hat{D} - \hat{D}(A^\dagger A\hat{D}A^\dagger A)^\dagger \hat{D}]F_A + F_A YY^* F_A,$$

with $\hat{D} = D - B^*(AB^*)^\dagger B$, and $Y \in \mathbb{C}^{n\times n}$ is arbitrary.

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