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SOME CHARACTERIZATIONS OF INNER PRODUCT SPACES BY INEQUALITIES

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Abstract. In this paper we present some new characterizations of inner product spaces by using inequalities. First, we explore a classical idea consisting in the transformation of the parallelogram law into an inequality. Then we give some characterizations by using convex and Wright-convex functions.

1. Introduction

In 1935, Fréchet ([10]) obtained the first characterization of inner product spaces, namely a real or complex normed space with the norm defined by an inner product.

PROPOSITION 1. (Fréchet) *A complex normed space* $(X,\ \|\cdot\|)$ *is an inner product space if and only if*

$$\|x+y+z\|^2+\|x\|^2+\|y\|^2+\|z\|^2=\|x+y\|^2+\|y+z\|^2+\|z+y\|^2,$$

for all $x,y,z\in X$.

In the same year, Jordan and von Neumann presented a new result ([13]), known as *the parallelogram law*. This result had a decisive impact on the development of this research direction.

PROPOSITION 2. (Jordan, von Neumann) *A complex normed space* $(X,\|\cdot\|)$ *is an inner product space if and only if*

$$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2,$$

for all $x,y\in X$.

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Later, many mathematicians have obtained other such kind of characterizations. An extensive collection of such results can be found in [2] or [4]. Recently, Moslehian and Rassias have proved some new results (see the references [17], [18] or [19]). The authors of the present paper have contributed to this subject with some results contained in [15] or [16]. The common point of most of these results consists in the characterizations of an inner product spaces by using the equalities.

Hence, transforming Jordan's result, a new idea appears: the characterizations of inner product space in terms of inequalities. In this direction, the first result, due to Schoenberg ([22]), is contained in the following proposition.

PROPOSITION 3. *Let $(X,\|\cdot\|)$ be a normed space. The norm $\|\cdot\|$ is deduced by an inner product if and only if*

$$\|x+y\|^2+\|x-y\|^2 \approx 2\|x\|^2+2\|y\|^2,$$

for all $x,y\in X$, where the symbol $\approx$ denotes exactly one and only one of the symbol $\leqslant$ or $\geqslant$.

Amir's book ([4]) contains similar results in this sense (see for example Proposition 6.1–6.6). For the development of this new direction, we mention the contributions of Lorch ([14]), Gurarii and Sozonov ([11]), Alonso ([1]) or Hirzallah, Kittaneh and Moslehian ([12]). Recently, Angulo, Giménéz, Moros, Nikodem ([3]) or Nikodem, Páles ([20]) have used strongly convex functions to characterize the inner product spaces.

In this paper, we prove other characterizations of the inner product spaces involving inequalities. We present two types of results. In the second section, we transform some characterizations of an inner product spaces by identities and we obtain some characterizations using inequalities. In the third section, we use convex and Wright-convex functions to prove new results. An interesting fact is that these inequalities can not be reverted under the same conditions.

2. Some results similar to Proposition 3

In this section, we explore the Schoenberg idea and we prove some similar characterizations. To avoid the repetitions, we consider $(X,\|\cdot\|)$ as a real or complex normed space. The symbol $\approx$ denotes one and only one of the symbol $\leqslant$ or $\geqslant$.

In Corollary 2.2 of [17], Moslehian and Rassias proved that X is an inner product if and only if

$$||ax+by||^2+||ay-bx||^2=(a^2+b^2)(||x||^2+||y||^2), \tag{2.1}$$

for any $x,y\in X$ and $a,b>0$. We replace the symbol $=$ and we obtain the following proposition.

PROPOSITION 4. *X is an inner product if and only if*

$$||ax+by||^2+||ay-bx||^2\approx(a^2+b^2)(||x||^2+||y||^2), \tag{2.2}$$

for every $x, y \in X$ *and* $a, b \in (0, \infty)$.

Proof. Clearly, if X is an inner product space, the equality is true. Assume that

$$||ax+by||^2 + ||ay-bx||^2 \leqslant (a^2+b^2)(||x||^2+||y||^2).$$

Replacing x with $ay-bx$ and y with $ax+by$, we obtain

$$\begin{aligned} &||a(ay-bx)+b(ax+by)||^2 + ||a(ax+by)-b(ay-bx)||^2 \\ &\leqslant (a^2+b^2)(||ay-bx||^2+||ax+by||^2). \end{aligned}$$

This is equivalent with

$$\begin{aligned} &\left\|\left(a^2+b^2\right)y\right\|^2 + \left\|\left(a^2+b^2\right)x\right\|^2 \leqslant (a^2+b^2)(||ay-bx||^2+||ax+by||^2) \\ &\Leftrightarrow (a^2+b^2)\|y\|^2 + (a^2+b^2)\|x\|^2 \leqslant ||ay-bx||^2+||ax+by||^2. \end{aligned}$$

Therefore, we obtain relation (2.1) and the conclusion follows. □

By using trigonometric manipulations and relation (2.1), it was proved (Proposition 2.1 in [15]) that X is an inner product if and only if

$$\|x\cos\alpha + y\sin\alpha\|^2 + \|y\cos\alpha - x\sin\alpha\|^2 = \|x\|^2 + \|y\|^2, \tag{2.3}$$

for any $x, y \in X$ and $\alpha \in \mathbb{R}$. We show that it is possible to relax the hypothesis (2.3) and we will obtain the following weaker version:

PROPOSITION 5. *X is an inner product if and only if*

$$\|x\cos\alpha + y\sin\alpha\|^2 + \|y\cos\alpha - x\sin\alpha\|^2 \approx \|x\|^2 + \|y\|^2, \tag{2.4}$$

for every $x, y \in X$ *and* $\alpha \in \mathbb{R}$.

Proof. If X is an inner product space, then the relation holds with equality. To prove the other implication, we assume that

$$\|x\cos\alpha + y\sin\alpha\|^2 + \|y\cos\alpha - x\sin\alpha\|^2 \geqslant \|x\| + \|y\|^2. \tag{2.5}$$

Denoting $u = x\cos\alpha + y\sin\alpha$ and $v = y\cos\alpha - x\sin\alpha$, we obtain $x = u\cos\alpha - v\sin\alpha$ and $y = u\sin\alpha + v\cos\alpha$. The inequality (2.5) becomes

$$\|u\|^2 + \|v\|^2 \geqslant \|u\cos\alpha - v\sin\alpha\|^2 + \|u\sin\alpha + v\cos\alpha\|^2,$$

for any $u, v \in X$ and $\alpha \in \mathbb{R}$. This transforms the relation (2.4) in equality and the conclusion follows. □

An improved version of the characterization contained in relation (2.3) was given in Theorem 3.1 of [15]: X is an inner product if and only if there exists $\alpha \in \mathbb{R}\backslash\pi\mathbb{Q}$ such that

$$\|x\cos\alpha + y\sin\alpha\|^2 + \|y\cos\alpha - x\sin\alpha\|^2 = \|x\|^2 + \|y\|^2,$$

for all $x, y \in X$. Similar to Proposition 5, the following result holds:

PROPOSITION 6. *X is an inner product if and only if it exists $\alpha \in \mathbb{R}\backslash\pi\mathbb{Q}$ such that*

$$\|x\cos\alpha + y\sin\alpha\|^2 + \|y\cos\alpha - x\sin\alpha\|^2 \approx \|x\|^2 + \|y\|^2,$$

for every $x, y \in X$.

We conclude this section with a classic result, known as *Stewart identity*. Recall that X is an inner product if and only if there exists $a \in (0,1)$ such that

$$\|ax + (1-a)y\|^2 = a\|x\|^2 + (1-a)\|y\|^2 - a(1-a)\|x-y\|^2, \tag{2.6}$$

for any $x, y \in X$. A generalization of (2.6) was presented in [16].

PROPOSITION 7. *X is an inner product if and only if there exists $a \in (0,1)$ such that*

$$\|ax + (1-a)y\|^2 \approx a\|x\|^2 + (1-a)\|y\|^2 - a(1-a)\|x-y\|^2, \tag{2.7}$$

for every $x, y \in X$.

Proof. If X is an inner product spaces we have equality. For the second part of the proof, we assume that the symbol $\approx$ denotes $\leqslant$. If $a = \frac{1}{2}$, we obtain the conclusion contained in Proposition 3. If $a \neq \frac{1}{2}$, we apply the Daróczy-Páles identity ([9]). Then

$$\frac{x+y}{2} = a\left[(1-a)x + a\frac{x+y}{2}\right] + (1-a)\left[(1-a)\frac{x+y}{2} + ay\right],$$

for any $x, y \in X$. By using (2.7), we obtain

$$\left\|\frac{x+y}{2}\right\|^2 \leqslant a\left\|(1-a)x + a\frac{x+y}{2}\right\|^2 + (1-a)\left\|(1-a)\frac{x+y}{2} + ay\right\|^2 - a(1-a)\left\|\frac{x-y}{2}\right\|^2, \tag{2.8}$$

Therefore

$$\left\|(1-a)x + a\frac{x+y}{2}\right\|^2 \leqslant (1-a)\|x\|^2 + a\left\|\frac{x+y}{2}\right\|^2 - a(1-a)\left\|\frac{x-y}{2}\right\|^2 \tag{2.9}$$

and

$$\left\|(1-a)\frac{x+y}{2} + ay\right\|^2 \leqslant (1-a)\left\|\frac{x+y}{2}\right\|^2 + a\|y\|^2 - a(1-a)\left\|\frac{x-y}{2}\right\|^2. \tag{2.10}$$

It follows from (2.8), (2.9) and (2.10) that

$$\left\|\frac{x+y}{2}\right\|^2 \leqslant a(1-a)\|x\|^2 + a^2\left\|\frac{x+y}{2}\right\|^2 - a^2(1-a)\left\|\frac{x-y}{2}\right\|^2 + (1-a)^2\left\|\frac{x+y}{2}\right\|^2$$
$$+a(1-a)\|y\|^2 - a(1-a)^2\left\|\frac{x-y}{2}\right\|^2 - a(1-a)\left\|\frac{x-y}{2}\right\|^2,$$

so

$$\left(1-a^2 - (1-a)^2\right)\left\|\frac{x+y}{2}\right\|^2 \leqslant a(1-a)\left(\|x\|^2+\|y\|^2\right) - a(1-a)(a+(1-a)+1)\left\|\frac{x-y}{2}\right\|^2$$
$$\Leftrightarrow 2a(1-a)\left\|\frac{x+y}{2}\right\|^2 \leqslant a(1-a)\left(\|x\|^2+\|y\|^2\right) - 2a(1-a)\left\|\frac{x-y}{2}\right\|^2$$
$$\Leftrightarrow 2\left\|\frac{x+y}{2}\right\|^2 + 2\left\|\frac{x-y}{2}\right\|^2 \leqslant \|x\|^2+\|y\|^2$$
$$\Leftrightarrow \|x+y\|^2+\|x-y\|^2 \leqslant 2\|x\|^2+2\|y\|^2.$$

Now, the conclusion follows from Proposition 3. □

3. Characterizations of inner product spaces concerning some types of the convexity

In this section, we consider a real interval I. Recall that a function $f: I \to \mathbb{R}$ is *convex* if

$$f(\alpha x + (1-\alpha)y) \leqslant \alpha f(x) + (1-\alpha) f(y),$$

for every $x, y \in I$ and for every $\alpha \in (0,1)$. We begin with the following auxiliary result.

LEMMA 1. *Let be $f : [0,\infty) \longrightarrow \mathbb{R}$ a convex function, satisfying the property $f(0) = 0$. Then it is a superadditive function, that is it satisfies*

$$f(x) + f(y) \leqslant f(x+y),$$

for all $x, y \in [0,\infty)$.

Proof. The case $x+y=0$ is trivial. We can consider $x+y>0$ and we have

$$\begin{aligned} f(x) &= f\left(\frac{y}{x+y}\cdot 0 + \frac{x}{x+y}\cdot(x+y)\right) \\ &\leqslant \frac{y}{x+y}\cdot f(0) + \frac{x}{x+y}\cdot f(x+y). \end{aligned}$$

Then

$$f(x) \leqslant \frac{x}{x+y}\cdot f(x+y). \tag{3.1}$$

In a similar way, we obtain

$$f(y) \leqslant \frac{y}{x+y}\cdot f(x+y). \tag{3.2}$$

Summing down to the relations (3.1) and (3.2), the conclusion follows. □

We are now in a position to prove our main results:

THEOREM 1. *Let be* $(X,\|\cdot\|)$ *a real or complex normed space. The following statements are equivalent:*

a) *X is an inner product spaces;*

b) *For every convex function $f:[0,\infty)\longrightarrow\mathbb{R}$ satisfying the property $f(0)=0$, for every $n\in\mathbb{N}$, $n\geqslant 2$, and for any $x_1,x_2,\ldots,x_n\in X$, the following inequality holds:*

$$2^n\left(f\left(\|x_1\|^2\right)+f\left(\|x_2\|^2\right)+\ldots+f\left(\|x_n\|^2\right)\right)\leqslant\sum f\left(\|\pm x_1\pm x_2\pm\ldots\pm x_n\|^2\right), \tag{3.3}$$

where the summation in the right hand side is considered over all 2^n combinations of the signs $\pm$;

c) *For every real number $p>2$, for every $n\in\mathbb{N}$, $n\geqslant 2$, and for every $x_1,x_2,\ldots,x_n\in X$, the following inequality holds:*

$$2^n\left(\|x_1\|^p+\|x_2\|^p+\ldots+\|x_n\|^p\right)\leqslant\sum\|\pm x_1\pm x_2\pm\ldots\pm x_n\|^p, \tag{3.4}$$

where the summation in the right hand side is over all 2^n combinations of the signs $\pm$.

Proof. We proceed to prove the sequence of implications as follows $a)\Rightarrow b)\Rightarrow c)\Rightarrow a)$.

$a)\Rightarrow b)$ From the parallelogram law, we have

$$\|x_1\|^2+\|x_2\|^2=\frac{1}{2}\left(\|x_1+x_2\|^2+\|x_1-x_2\|^2\right).$$

The function f is convex, so that

$$f\left(\|x_1\|^2+\|x_2\|^2\right)\leqslant\frac{1}{2}\left(f\left(\|x_1+x_2\|^2\right)+f\left(\|x_1-x_2\|^2\right)\right). \tag{3.5}$$

Using the superadditivity of f, we obtain

$$f\left(\|x_1\|^2\right)+f\left(\|x_2\|^2\right)\leqslant f\left(\|x_1\|^2+\|x_2\|^2\right)$$

and, together with (3.5), we get

$$2\left(f\left(\|x_1\|^2\right)+f\left(\|x_2\|^2\right)\right)\leqslant f\left(\|x_1+x_2\|^2\right)+f\left(\|x_1-x_2\|^2\right). \tag{3.6}$$

Also, we have

$$2\left(f\left(\|x_1\|^2\right)+f\left(\|x_2\|^2\right)\right)\leqslant f\left(\|-x_1-x_2\|^2\right)+f\left(\|x_2-x_1\|^2\right). \tag{3.7}$$

Summing down (3.6) and (3.7), we obtain the inequality (3.3) for $n=2$.

Further, we suppose that inequality (3.3) holds for n and we will prove it for $n+1$. Then, we may write

$$\sum f\left(\|x_1 \pm \ldots \pm x_{n+1}\|^2\right) = \sum f\left(\|\pm x_1 \pm \ldots \pm x_n + x_{n+1}\|^2\right) + \sum f\left(\|\pm x_1 \pm \ldots \pm x_n - x_{n+1}\|^2\right), \tag{3.8}$$

and now we employ (3.6). The inequality (3.8) becames

$$\begin{aligned}\sum f\left(\|\pm x_1 \pm \ldots \pm x_n \pm x_{n+1}\|^2\right) &\geqslant \sum 2\left(f\left(\|\pm x_1 \pm \ldots \pm x_n\|^2\right) + f\left(\|x_{n+1}\|^2\right)\right) \\ &= 2\left(\sum\left(f\left(\|\pm x_1 \pm \ldots \pm x_n\|^2\right) + f\left(\|x_{n+1}\|^2\right)\right)\right) \\ &= 2\sum f\left(\|\pm x_1 \pm \ldots \pm x_n\|^2\right) + 2^{n+1} f\left(\|x_{n+1}\|^2\right).\end{aligned}$$

Now we apply the induction hypothesis for n and the conclusion follows.

$b) \Rightarrow c)$ For every real number $p > 2$, let us define the function

$$f : [0,\infty) \to \mathbb{R},\ f(x) = x^{\frac{p}{2}}.$$

We have $f(0) = 0$ and

$$f''(x) = \frac{p}{2}\left(\frac{p}{2} - 1\right) x^{\frac{p}{2}-2}.$$

Function f is convex and conclusion follows if we apply (3.3).

$c) \Rightarrow a)$ For $n = 2$, we have

$$4\left(\|x_1\|^p + \|x_2\|^p\right) \leqslant \|x_1 + x_2\|^p + \|x_1 - x_2\|^p + \|-x_1 + x_2\|^p + \|-x_1 - x_2\|^p,$$

relation equivalent to

$$2\left(\|x_1\|^p + \|x_2\|^p\right) \leqslant \|x_1 + x_2\|^p + \|x_1 - x_2\|^p, \tag{3.9}$$

for every real number $p > 2$ and for every $x_1, x_2 \in X$.

Let us define the function

$$u : [2,\infty) \to \mathbb{R},\ u(p) = 2\left(\|x_1\|^p + \|x_2\|^p\right) - \|x_1 + x_2\|^p - \|x_1 - x_2\|^p.$$

It is a continuous function and $u(p) \leqslant 0$, for every $p \in (2,\infty)$. Then $u(2) = \lim_{p\to 2, p>2} u(p) \leqslant 0$ and we obtain

$$2\left(\|x_1\|^2 + \|x_2\|^2\right) \leqslant \|x_1 + x_2\|^2 + \|x_1 - x_2\|^2,$$

for every $x_1, x_2 \in X$. The conclusion follows by applying the result in Proposition 3. □

We mention that into the case $X = \mathbb{C}$ and $n = 2$, the inequality in the statement *c)* is called *van der Corput inequality* ([6]). Later, Cong ([5]) has generalized it for an inner product space. Moreover, he proved that this inequality characterizes this type of space.

Another type of convexity is given by Wright-convex functions. Recall that a function $f : I \to \mathbb{R}$ is called *Wright-convex* if

$$f(\alpha x+(1-\alpha)y)+f((1-\alpha)x+\alpha y) \leqslant f(x)+f(y), \tag{3.10}$$

for every $x,y \in I$ and for every $\alpha \in (0,1)$. A function which is satisfying the opposite inequality (3.10) is called *Wright-concave*. This definition shows us that for every Wright-convex function $f : I \to \mathbb{R}$ and for every $a,b,c,d \in I$ with $a \leqslant b \leqslant d$ and $a \leqslant c \leqslant d$ and satisfying the condition $a+d = b+c$, we have

$$f(b)+f(c) \leqslant f(a)+f(d). \tag{3.11}$$

The reverse of (3.11) holds for the Wright-concave functions. Remark that every convex function is a Wright-convex function and every concave function is Wright-concave.

THEOREM 2. *Let be $(X,\|\cdot\|)$ a real or complex normed space. The following statements are equivalent:*

a) *X is an inner product spaces;*

b) *For every Wright-convex function $f : [0,\infty) \longrightarrow \mathbb{R}$ and for every $x,y \in X$, the following inequality holds:*

$$f\left(\|x+y\|^2\right)+f\left(\|x-y\|^2\right) \leqslant f\left((\|x\|+\|y\|)^2\right)+f\left((\|x\|-\|y\|)^2\right); \tag{3.12}$$

c) *For every $p \in (2,\infty)$ and for every $x,y \in X$, the following inequality holds:*

$$\|x+y\|^p+\|x-y\|^p \leqslant |\|x\|+\|y\||^p+|\|x\|-\|y\||^p; \tag{3.13}$$

d) *For every $p \in (2,\infty)$ and for every $x,y \in X$, the following inequality holds:*

$$\|x+y\|^p+\|x-y\|^p \leqslant 2^{p-1}\left(\|x\|^p+\|y\|^p\right). \tag{3.14}$$

Proof. We will prove the implications $a) \Rightarrow b) \Rightarrow c) \Rightarrow d) \Rightarrow a)$.

$a) \Rightarrow b)$ The parallelogram law and some algebraic transformations give us the identity:

$$\|x+y\|^2+\|x-y\|^2 = (\|x\|+\|y\|)^2+(\|x\|-\|y\|)^2.$$

Using the Cauchy-Schwartz inequality, we obtain

$$-\|x\|\cdot\|y\| \leqslant \operatorname{Re}\langle x,y\rangle \leqslant \|x\|\cdot\|y\|,$$

for every $x,y \in X$, where $\langle\cdot,\cdot\rangle$ denotes the inner product in X. Then $\|x-y\|^2 \leqslant (\|x\|+\|y\|)^2$ and $\|x+y\|^2 \leqslant (\|x\|+\|y\|)^2$ for every $x,y \in X$. Similarly, $(\|x\|-\|y\|)^2 \leqslant \|x+y\|^2$ and $(\|x\|-\|y\|)^2 \leqslant \|x-y\|^2$. Applying (3.11), we obtain the conclusion.

$b) \Rightarrow c)$ For every real number $p>2,$ let us define the function

$$f:[0,\infty)\to[0,\infty),\ f(t)=t^{\frac{p}{2}}.$$

Clearly, it is a convex function, therefore a Wright-convex function. We apply inequality (3.12) and the conclusion follows.

$c) \Rightarrow d)$ For every real number $p>2,$ we define the function

$$g:[0,\infty)\to\mathbb{R}, g(t)=t^{p-1}$$

Obviously, it is continuous on $[0,\infty)$ and twice differentiable on $(0,\infty)$. We have

$$g''(t)=(p-1)(p-2)t^{p-3},$$

hence $g''(t)>0,$ for every $t\in(0,\infty)$. The function g is convex and we have

$$\frac{g(t-1)+g(t+1)}{2}>g(t),$$

for every $t\in[1,\infty)$. The last inequality is equivalent to the next:

$$(t+1)^{p-1}+(t-1)^{p-1}>2t^{p-1}. \tag{3.15}$$

Now, we define the differentiable function

$$h:[1,\infty)\to\mathbb{R}, h(t)=(t+1)^{p}+(t-1)^{p}-2t^{p}-2,$$

where $p>2.$ We have

$$h'(t)=p\left((t+1)^{p-1}+(t-1)^{p-1}-2t^{p-1}\right).$$

Inequality (3.15) implies $h'(t)>0,$ for every $t\in[1,\infty)$. Further, the function h is increasing and $h(t)\geqslant h(1),$ for every $t\in[1,\infty)$. Let be $u,v\in(0,\infty),$ $u\geqslant v.$ From $h(\frac{u}{v})\geqslant h(1),$ we obtain

$$(u+v)^{p}+(u-v)^{p}\geqslant 2\left(u^{p}+v^{p}\right). \tag{3.16}$$

Denoting $u+v=2a$ and $u-v=2b,$ the inequality (3.16) becomes

$$(a+b)^{p}+(a-b)^{p}\leqslant 2^{p-1}\left(a^{p}+b^{p}\right),$$

for every $a,b\geqslant 0,$ $a\geqslant b.$ Due to the symmetry of the previous inequality, we obtain the following general version:

$$|a+b|^{p}+|a-b|^{p}\leqslant 2^{p-1}\left(a^{p}+b^{p}\right),$$

If we choose $a=\|x\|,$ $b=\|y\|,$ then we obtain

$$|\|x\|+\|y\||^{p}+|\|x\|-\|y\||^{p}\leqslant 2^{p-1}\left(\|x\|^{p}+\|y\|^{p}\right). \tag{3.17}$$

The proof is finished if we combine the inequalities (3.13) and (3.17).

$d) \Rightarrow a)$ Let us define the function

$$v:[2,\infty)\to\mathbb{R},\quad v(p)=\|x+y\|^p+\|x-y\|^p-2^{p-1}\left(\|x\|^p+\|y\|^p\right).$$

It is a continuous function and $v(p)\leqslant 0$, for every $p\in(2,\infty)$. Then $v(2)=\lim_{p\to 2,p>2} v(p)\leqslant 0$ and we obtain

$$\|x+y\|^2+\|x-y\|^2\leqslant 2\left(\|x\|^2+\|y\|^2\right),$$

for every $x,y\in X$. Now, Proposition 3 concludes our proof. □

The equivalence $a)\Leftrightarrow d)$ can be found in [21], but we have included it in Theorem 2 to complete the chains of implications.

Further, for every $p\in(1,2)$, the function $g:[0,\infty)\to\mathbb{R}$, $g(t)=t^{p-1}$ is concave, therefore it is Wright-concave. Then, we obtain the reverse of the inequalities in Theorem 2.

THEOREM 3. *Let be $(X,\|\cdot\|)$ a real or complex normed space. The next statements are equivalent:*

a) *X is an inner product spaces;*

b) *For every Wright-concave function $f:[0,\infty)\longrightarrow\mathbb{R}$ and for every $x,y\in X$, the following inequality holds:*

$$f\left(\|x+y\|^2\right)+f\left(\|x-y\|^2\right)\geqslant f\left((\|x\|+\|y\|)^2\right)+f\left((\|x\|-\|y\|)^2\right);$$

c) *For every $p\in(1,2)$ and for every $x,y\in X$, the following inequality holds:*

$$\|x+y\|^p+\|x-y\|^p\geqslant |\|x\|+\|y\||^p+|\|x\|-\|y\||^p.$$

d) *For every $p\in(1,2)$ and for every $x,y\in X$, the following inequality holds:*

$$\|x+y\|^p+\|x-y\|^p\geqslant 2^{p-1}\left(\|x\|^p+\|y\|^p\right).$$

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