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WEIGHTED NORM INEQUALITIES FOR BILINEAR FOURIER MULTIPLIER OPERATORS

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Abstract. In this paper, by kernel estimates of the bilinear Fourier multiple operator and the weighted theory for the bilinear singular integral operators, some weighted norm inequality with general weights are established for the biilinear Fourier multiplier operators.

1. Introduction

In their seminal works [3, 4], Coifman and Meyer considered the mapping properties of the multilinear multiplier operators. Let $\sigma \in L^{\infty}(\mathbb{R}^{mn})$. Define the m-linear Fourier multiplier operator T_σ by

$$T_\sigma(f_1,\dots,f_m)(x) = \int_{\mathbb{R}^{mn}} \exp(2\pi i x(\xi_1+\dots+\xi_m))\sigma(\xi_1,\dots,\xi_m)\prod_{k=1}^{m}\mathscr{F}f_k(\xi_m)\mathrm{d}\vec{\xi} \quad (1.1)$$

initially for $f_1,\dots,f_m \in \mathscr{S}(\mathbb{R}^n)$ (the space of Schwartz functions), where and in the following, $\mathrm{d}\vec{\xi} = \mathrm{d}\xi_1\dots\mathrm{d}\xi_m$, and for a function $f \in \mathscr{S}(\mathbb{R}^n)$, $\mathscr{F}f$ denotes its Fourier transform. Coifman and Meyer [4] proved that if $\sigma \in C^s(\mathbb{R}^{nm}\setminus\{0\})$ satisfying that

$$|\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_m}^{\alpha_m}\sigma(\xi_1,\dots,\xi_m)| \leqslant C_{\alpha_1,\dots\alpha_m}(|\xi_1|+\dots+|\xi_m|)^{-(|\alpha_1|+\dots+|\alpha_m|)} \quad (1.2)$$

for all $|\alpha_1|+\dots+|\alpha_m| \leqslant s$ with s a positive integer large enough, then T_σ is bounded from $L^{p_1}(\mathbb{R}^n)\times\dots\times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p_1,\dots,p_m, p < \infty$, with $1/p = \sum_{1\leqslant k\leqslant m} 1/p_k$. By the multilinear Calderón-Zygmund operator theory developed in [10], we know that if σ satisfies (1.2) for some integer $s \geqslant nm+1$, then T_σ is bounded from $L^{p_1}(\mathbb{R}^n)\times\dots\times L^{p_m}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$ for all $1 \leqslant p_1,\dots,p_m \leqslant \infty$, $1/m \leqslant p < \infty$ with $1/p = \sum_{1\leqslant k\leqslant m} 1/p_k$, and is bounded from $L^{p_1}(\mathbb{R}^n)\times\dots\times L^{p_m}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$ for all $1 < p_1,\dots,p_m \leqslant \infty$. In recent years, there has been significant progress in the study of the behavior on function spaces for the multilinear multiplier operator when the multiplier satisfies certain Sobolev regularity. Let $\Phi \in \mathscr{S}(\mathbb{R}^{nm})$ such that $\mathrm{supp}\,\Phi \subset \{(\xi_1,\dots,\xi_m) : 1/2 \leqslant |\xi_1|+\dots+|\xi_m| \leqslant 2\}$ and

$$\sum_{l\in\mathbb{Z}} \Phi(2^{-l}\xi_1,\dots,2^{-l}\xi_m) = 1, \text{ for } (\xi_1,\dots,\xi_m) \in \mathbb{R}^{mn}\setminus\{0\}.$$

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Set

$$\sigma_l(\xi_1,\dots,\xi_m)=\Phi(\xi_1,\dots,\xi_m)\sigma(2^l\xi_1,\dots,2^l\xi_m), \tag{1.3}$$

and

$$\|\sigma_l\|_{W^s(\mathbb{R}^{mn})}=\Big(\int_{\mathbb{R}^{mn}}(1+|\xi_1|^2+\ldots+|\xi_m|^2)^s|\mathscr{F}\sigma_l(\xi_1,\dots,\xi_m)|^2\mathrm{d}\vec{\xi}\Big)^{1/2}.$$

Tomita [18] proved that if

$$\sup_{l\in\mathbb{Z}}\|\sigma_l\|_{W^s(\mathbb{R}^{mn})}<\infty, \tag{1.4}$$

for some $s\in(mn/2,mn]$, then T_σ is bounded from $L^{p_1}(\mathbb{R}^n)\times\ldots\times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ provided that $p_1,\dots,p_m,p\in(1,\infty)$ and $1/p=\sum_{1\leqslant k\leqslant m}1/p_k$. Using the L^r-based Sobolev space ($1<r\leqslant 2$), Grafakos and Si [9] considered the mapping properties from $L^{p_1}(\mathbb{R}^n)\times\ldots\times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for T_σ when $p\leqslant 1$. Fairly recently, Miyachi and Tomita [15] considered the problem to find the minimal smoothness conditions for the boundedness of T_σ when $m=2$ (note that the arguments in [15] also apply to the case of $m>2$) under the Sobolev regularity that

$$D_{s_1,s_2}(\sigma):=\sup_{l\in\mathbb{Z}}\|\sigma_l\|_{W^{s_1,s_2}(\mathbb{R}^{2n})}<\infty,\ s_1,s_2>n/2, \tag{1.5}$$

where and in the following, for a suitable function f on $\mathbb{R}^{2n}$ and $s_1,s_2\in(0,\infty)$,

$$\|f\|_{W^{s_1,s_2}(\mathbb{R}^{2n})}=\Big(\int_{\mathbb{R}^{2n}}\langle\xi_1\rangle^{2s_1}\langle\xi_2\rangle^{2s_2}|\mathscr{F}f(\xi_1,\xi_2)|^2\mathrm{d}\vec{\xi}\Big),$$

with $\langle\xi_k\rangle=(1+|\xi_k|^2)^{1/2}$. For the mapping properties of T_σ when σ satisfies (1.5), see also [5, 8, 11] and the references therein.

The weighted estimates for the multilinear Fourier multiplier operators are also of interest. When σ satisfies (1.2) for some $s\geqslant mn+1$, it is well known that T_σ is a standard multilinear Calderón-Zygmund operator (see [10]), and so T_σ enjoys the weighted estimates with multiple $A_{\vec{P}}(\mathbb{R}^{mn})$ weights as the multilinear Calderón-Zygmund operators established in [14]. By a suitable kernel estimate and the theory of multilinear singular integral operators, Bui and Duong [1] established the weighted estimates with multiple $A_{\vec{P}}(\mathbb{R}^{2n})$ weights for T_σ when σ satisfies (1.2) for $m=2$ and $n+1\leqslant s\leqslant 2n$. For the weighted estimates with A_p weights of T_σ when σ satisfies (1.5) with $s_1,\dots,s_m\in(n/2,n]$, see [5] and [11].

The main purpose of this paper is to consider the weighted estimates with general weights for the multilinear Fourier multiplier operator when the multiplier σ enjoys the Sobolev regularity (1.5). For the sake of simplicity, we only consider the case of $m=2$. For a weight w and $p\in(0,\infty)$, let $L^p(\mathbb{R}^n,w)$ and $L^{p,\infty}(\mathbb{R}^n,w)$ be the weighted $L^p(\mathbb{R}^n)$ space, the weighted weak $L^p(\mathbb{R}^n)$ spaces with weight w respectively. For a cube $Q\subset\mathbb{R}^n$, $\delta\in[0,\infty)$ and a suitable function f, set

$$\|f\|_{L(\log L)^\delta,Q}=\inf\Big\{\lambda>0:\frac{1}{|Q|}\int_Q\frac{|f(x)|}{\lambda}\log^\delta\Big(\mathrm{e}+\frac{|f(x)|}{\lambda}\Big)\mathrm{d}x\leqslant 1\Big\}.$$

Define the maximal operator $M_{L(\log L)^\delta}$ by

$$M_{L(\log L)^\delta} f(x) = \sup_{Q \ni x} \|f\|_{L(\log L)^\delta, Q},$$

where the supremum is taken over all cubes containing x. It is easy to see that for $\delta = 0$, $M_{L(\log L)^\delta}$ is just M, the standard Hardy-Littlewood maximal operator.

THEOREM 1.1. *Let σ be a bilinear multiplier satisfying (1.5) for some $s_1, s_2 \in (n/2, n]$, and T_σ be the multiplier operator defined by (1.1). Let w be a weight and $\delta > 0$. Set $t_k = n/s_k$.*

(i) *If $p_k \in (t_k, \infty]$, $1 \leqslant k \leqslant 2$, $p \in (1, \infty)$ such that $1/p = 1/p_1 + 1/p_2$, then*

$$\|T_\sigma(f_1, f_2)\|_{L^p(\mathbb{R}^n, w)} \lesssim \prod_{k=1}^{2} \|f_k\|_{L^{p_k}(\mathbb{R}^n, M_{L(\log L)^{p+\delta}} w)}, \tag{1.6}$$

where and in the following, for $p_1 = \infty$ and a weight u, $\|f_1\|_{L^{p_1}(\mathbb{R}^n, u)}$ should be replaced by $\|f_1\|_{L^\infty(\mathbb{R}^n)}$;

(ii) *if $p_1 \in (n/(2s_1 - n), \infty]$, $p_2 \in (t_2, \infty)$, $p \in (1/2, \infty)$ such that $1/p = 1/p_1 + 1/p_2$, then*

$$\|T_\sigma(f_1, f_2)\|_{L^p(\mathbb{R}^n, w)} \lesssim \|f_1\|_{L^{p_1}(\mathbb{R}^n, Mw)} \|f_2\|_{L^{p_2}(\mathbb{R}^n, M_{L(\log L)^{p_2+\delta}} w)}. \tag{1.7}$$

THEOREM 1.2. *Let σ be an bilinear multiplier satisfying (1.5) for some $s_1 \in (n/2, n]$ and $s_2 > n$, T_σ be the multiplier operator defined by (1.1). Let w be a weight and $\delta > 0$. Then for $p_1 \in (n/(2s_1 - n), \infty]$, $p_2 \in (1, \infty)$ $p \in (1/2, \infty)$ with $1/p = 1/p_1 + 1/p_2$,*

$$\|T_\sigma(f_1, f_2)\|_{L^p(\mathbb{R}^n, w)} \lesssim \|f_1\|_{L^{p_1}(\mathbb{R}^n, Mw)} \|f_2\|_{L^{p_2}(\mathbb{R}^n, M_{L(\log L)^{p_2-1+\delta}} w)}. \tag{1.8}$$

REMARK 1.1. To prove Theorem 1.1 and 1.2, we will introduce a class of multilinear singular integral operators, and establish the weighted norm inequality with general weights for these operators. By establishing the kernel estimates for the multiplier operator with multiplier satisfying the assumptions in our theorems, we will see that this multiplier operator can be regarded as multilinear singular integral operators which enjoy the properties (1.6)-(1.8).

Throughout this paper, C always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We use the symbol $A \lesssim B$ to denote that there exists a positive constant C such that $A \leqslant CB$. For a ball B, let $S_0(B) = B$ and $S_l(B) = 2^l B \setminus 2^{l-1} B$ for positive integer. For any set $E \subset \mathbb{R}^n$, χ_E denotes its characteristic function. For $p \in [1, \infty]$, p' denotes the dual exponent of p, namely, $1/p + 1/p' = 1$. For a locally integrable function f, $M^\sharp f$ denote the Fefferman-Stein sharp function of f, that is,

$$M^\sharp f(x) = \sup_{Q \ni x} \frac{1}{|B|} \int_B |f(x) - V_B(f)| \, \mathrm{d}x,$$

where the sup is taken over all balls containing x, and $V_B f$ is the mean value of f on B. For $r \in (0, 1)$, $M_r^\sharp(f)$ is the function defined by

$$M_r^\sharp(f)(x) = \big(M^\sharp(|f|^r)(x)\big)^{1/r}.$$

2. Estimates for multilinear singular integrals

In this section, we will establish some estimates for the multilinear singular integral operators, which will be used in the proof of Theorem 1.1 and 1.2, and are of independent interest.

Let $K(x; y_1, y_2)$ be a locally integrable function defined away from the diagonal $x = y_1 = y_2$ in $\mathbb{R}^{3n}$. An operator T, defined on $\mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n)$ and taking values in the space of tempered distributions, is said to be a bilinear operator with kernel K if T is bilinear, and satisfies that

$$T(f_1, f_2)(x) = \int_{\mathbb{R}^{mn}} K(x; y_1, f_2) f(y_1) f(y_2) \mathrm{d}y_1 \mathrm{d}y_2, \tag{2.1}$$

for bounded functions f_1, f_2 with compact supports, and $x \in \mathbb{R}^n \setminus \cap_{j=1}^2 \mathrm{supp}\, f_j$. For the mapping properties on various function spaces of the operator T defined by (2.1), see [7, 10, 1, 11, 14] and the references therein.

THEOREM 2.1. *Let T be a bilinear singular integral operator with associated kernel K in the sense of (2.1). For $y, y', y_1, y_2 \in \mathbb{R}^n$, let*

$$U_0(y, y_1, y_2; y') = |K(y; y_1, y_2) - K(y'; y_1, y_2)|.$$

Suppose that

(i) *for bounded functions f_1, f_2 with compact supports,*

$$\|T(f_1, f_2)\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim \|f_1\|_{L^\infty(\mathbb{R}^n)} \|f_2\|_{L^1(\mathbb{R}^n)},$$

(ii) *there exists a constant $\rho > 0$, such that for any ball B with radius R and any integer $j_2 \geqslant 4$,*

$$\int_B \int_B \int_{\mathbb{R}^n} U_0(y, y_1, y_2; y') \mathrm{d}y_1 \mathrm{d}y \mathrm{d}y' \chi_{S_{j_2}(B)}(y_2) \lesssim |B|^2 \frac{R^\rho}{|2^{j_2} B|^{1+\rho/n}}.$$

Then for any $\gamma \in (0, 1)$ and bounded functions f_1, f_2 with compact supports,

$$M_\gamma^\sharp(T(f_1, f_2))(x) \lesssim \|f_1\|_{L^\infty(\mathbb{R}^n)} M f_2(x).$$

Proof. Without loss of generality, we may assume that $\|f_1\|_{L^\infty(\mathbb{R}^n)} = 1$. For each fixed $x \in \mathbb{R}^n$, ball B containing x and bounded function f_2 with compact support, decompose f_2 as $f_2 = f_2^1 + f_2^2$ with

$$f_2^1(y) = f_2(y) \chi_{8B}(y), \;\; f_2^2(y) = f_2(y) \chi_{\mathbb{R}^n \setminus 8B}(y).$$

Observe that for any $y' \in B$,

$$\inf_{c\in\mathbb{C}}\left(\frac{1}{|B|}\int_B \big||T(f_1,f_2)(y)|^\gamma - c\big|\mathrm{d}y\right)^{\frac{1}{\gamma}} \lesssim \left(\frac{1}{|B|}\int_B \left|T(f_1,f_2^1)(y) - T(f_1,f_2^2)(y')\right|^\gamma \mathrm{d}y\right)^{\frac{1}{\gamma}} + \left(\frac{1}{|B|}\int_B |T(f_1,f_2^1)(y)|^\gamma \mathrm{d}y\right)^{\frac{1}{\gamma}}.$$

Thus,

$$\inf_{c\in\mathbb{C}}\left(\frac{1}{|B|}\int_B \big||T(f_1,f_2)(y)|^\gamma - c\big|\mathrm{d}y\right)^{\frac{1}{\gamma}} \lesssim \frac{1}{|B|^2}\int_B\int_B \left|T(f_1,f_2^2)(y) - T(f_1,f_2^2)(y')\right| \mathrm{d}y\mathrm{d}y' + \left(\frac{1}{|B|}\int_B |T_1(f_1,f_2^1)(y)|^\gamma \mathrm{d}y\right)^{\frac{1}{\gamma}}.$$

Recall that $\gamma \in (0,1)$, and T is bounded from $L^1(\mathbb{R}^n)\times L^\infty(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$, an argument similar to that used in the proof of Kolomogrov inequality then leads to that

$$\left(\frac{1}{|B|}\int_B |T(f_1,f_2^1)(y)|^\gamma \mathrm{d}y\right)^{1/\gamma} \lesssim \|f_1\|_{L^\infty(\mathbb{R}^n)}\frac{1}{|B|}\int_{8B}|f_2(y_2)|\,\mathrm{d}y_2 \lesssim Mf_2(x).$$

For each fixed $y, y' \in B$, it is easy to verify that

$$\int_B\int_B \left|T(f_1,f_2^2)(y) - T(f_1,f_2^2)(y')\right|\mathrm{d}y\mathrm{d}y' \lesssim \sum_{j_2=4}^{\infty}\int_{S_{j_2}(B)}|f_2(y_2)|A(y,y',y_2)\,\mathrm{d}y_2,$$

with

$$A(y,y',y_2) = \int_B\int_B\int_{\mathbb{R}^n}|U_0(y,y_1,y_2;y')|\mathrm{d}y_1\mathrm{d}y\mathrm{d}y'.$$

This, together with our assumption (ii), then gives us that

$$\frac{1}{|B|^2}\int_B\int_B \left|T(f_1,f_2^2)(y) - T(f_1,f_2^2)(y')\right|\mathrm{d}y\mathrm{d}y' \lesssim Mf_2(x)$$

and then completes the proof of Theorem 2.1. □

THEOREM 2.2. *Let T be a bilinear singular integral operator with associated kernel K in the sense of (2.1). Let $r_1, r_2 \in (1,\infty)$. Suppose that*

(i) *for any weight w and some $p_2 \in (r_2,\infty)$, T satisfies the estimate that*

$$\|T(f_1,f_2)\|_{L^{p_2,\infty}(\mathbb{R}^n,w)} \lesssim \|f_1\|_{L^\infty(\mathbb{R}^n)}\|f_2\|_{L^{p_2}(\mathbb{R}^n,\mathcal{N}w)},$$

where $\mathcal{N}$ is an operator such that $\mathcal{N}h(x) \geqslant Mh(x)$ for any function h on $\mathbb{R}^n$ and any $x \in \mathbb{R}^n$;

(ii) *For each $R>0$, there exists a function $\mathrm{H}_{1,R}$ such that for any ball B with radius R, function f_1 with* $\operatorname{supp} f_1 \subset B$, *and* $\int_{\mathbb{R}^n} f_1(y_1)\,\mathrm{d}y_1 = 0$, $y \in \mathbb{R}^n \setminus 8B$, $y_1' \in B$,

$$|T(f_1, f_2)(y)| \lesssim M_{r_2} f_2(y) \int_{\mathbb{R}^n} |f_1(y_1)| \mathrm{H}_{1,R}(y, y_1, y_1')\mathrm{d}y_1,$$

and for any $j_1 \geqslant 4$,

$$\Big(\int_B |\mathrm{H}_{1,R}(y, y_1, y_1')|^{r_1'}\,\mathrm{d}y_1\Big)^{1/r_1'} \chi_{S_{j_1}(B)}(y) \lesssim \frac{R^{n-n/r_1+\rho}}{|2^{j_1}B|^{1+\rho/n}}$$

Then for any $p_1 \in (r_1, \infty)$, *any weight* w,

$$\|T(f_1, f_2)\|_{L^{p,\infty}(\mathbb{R}^n, w)} \lesssim \|f_1\|_{L^{p_1}(\mathbb{R}^n, Mw)} \|f_2\|_{L^{p_2}(\mathbb{R}^n, \mathcal{N} w)},$$

with $1/p = 1/p_1 + 1/p_2$.

Proof. Let f_1, f_2 be bounded functions with compact supports, such that

$$\|f_1\|_{L^{p_1}(\mathbb{R}^n, Mw)} = \|f_2\|_{L^{p_1}(\mathbb{R}^n, \mathcal{N} w)} = 1.$$

Our aim is to prove that for any $\lambda > 0$,

$$w(\{x \in \mathbb{R}^n : |T(f_1, f_2)(x)| > \lambda\}) \lesssim \lambda^{-p}.$$

To do this, we apply the Calderón-Zygmund decomposition to $|f_1|^{p_1}$ at level λ^p, and then obtain a sequence of non-overlapping cubes $\{Q_1^i\}_i$, such that

$$\lambda^p < \frac{1}{|Q_1^i|} \int_{Q_1^i} |f_1(y)|^{p_1} dy \lesssim \lambda^p,$$

and

$$|f_1(x)| \leqslant \lambda^{p/p_1}, \text{ a.e.} x \in \mathbb{R}^n \setminus \cup_i Q_1^i.$$

Set

$$g_1(y) = f_1(y)\chi_{\mathbb{R}^n \setminus \cup_i Q_1^i}(y) + \sum_i V_{Q_1^i}(f_1)\chi_{Q_1^i},$$

$$b_1(y) = f_1(x) - g_1(x) = \sum_i b_1^i(y), \text{ with } b_1^i(y) = (f_1(y) - V_{Q_1^i}(f_1))\chi_{Q_1^i}(y),$$

with $V_{Q_1^i}(f_1)$ the mean value of f_1 on Q_1^i. For each fixed i, let y_1^i and $\ell(Q_1^i)$ be the center and the side length of Q_1^i, and B_1^i be the ball which is centered at y_1^i and having radius $R_1^i = 8\sqrt{n}\ell(Q_1^i)$. Set $\Omega = \cup_i 4B_1^i$. It is obvious that

$$w(\Omega) \leqslant \sum_i w(Q_1^i) \lesssim \lambda^{-p} \sum_i \int_{Q_1^i} |f_1(x)|^{p_1}\,dx \inf_{y \in Q_1^i} Mw(y) \lesssim \lambda^{-p}.$$

The fact that $\|g_1\|_{L^\infty(\mathbb{R}^n)} \lesssim \lambda^{p/p_1}$, and T is bounded from $L^\infty(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n, \mathcal{N} w)$ to $L^{p_2,\infty}(\mathbb{R}^n, w)$, implies that

$$w(\{x \in \mathbb{R}^n : |T(g_1, f_2)(x)| > \lambda/2\}) \lesssim \lambda^{-p_2} \|g_1\|_{L^\infty(\mathbb{R}^n)}^{p_2} \|f_2\|_{L^{p_2}(\mathbb{R}^n, \mathcal{N} w)}^{p_2} \lesssim \lambda^{-p}.$$

On the other hand, for $x \in \mathbb{R}^n \setminus \Omega$, a straightforward computation leads to that

$$\begin{aligned}|T(b_1^i, f_2)(x)| &\lesssim \int_{Q_1^i} |b_1^i(y_1)| \mathrm{H}_{1,R_1^i}(x; y_1, y_1^i)\, \mathrm{d}y_1 M_{r_2} f_2(x)\\ &= \mathrm{L}_i(x) M_{r_2} f_2(x),\end{aligned}$$

where $M_{r_2} f_2(x) = \big\{M(|f_2|^{r_2})(x)\big\}^{1/r_2}$, and

$$\mathrm{L}_i(x) = \|b_1^i\|_{L^{r_1}(\mathbb{R}^n)} \Big(\int_{Q_1^i} |\mathrm{H}_{1,R_1^i}(x, y_1, y_1^i)|^{r_1'}\, \mathrm{d}y_1 \Big)^{1/r_1'}.$$

It follows from assumption (ii) that

$$\begin{aligned}
&w(\{x \in \mathbb{R}^n \setminus \Omega : \sum_i \mathrm{L}_i(x) > \lambda^{p/p_1}\})\\
&\lesssim \lambda^{-p/p_1} \sum_i \int_{\mathbb{R}^n \setminus \Omega} \mathrm{L}_i(x) w(x)\, \mathrm{d}x\\
&\lesssim \lambda^{-p/p_1} \sum_i \|b_1^i\|_{L^{r_1}(\mathbb{R}^n)} \sum_{j_1=4}^{\infty} \int_{S_{j_1}(B_1^i)} \Big(\int_{Q_1^i} \big\{\mathrm{H}_{1,R_1^i}(x, y_1, y_1^i)\big\}^{r_1'} \mathrm{d}y_1 \Big)^{\frac{1}{r_1'}} w(x) \mathrm{d}x\\
&\lesssim \lambda^{-p/p_1} \sum_i \|b_1^i\|_{L^{p_1}(\mathbb{R}^n)} |Q_1^i|^{\frac{1}{r_1} - \frac{1}{p_1}} \{\ell(Q_1^i)\}^{n-n/r_1+\rho} \inf_{z \in Q_1^i} Mw(z) \sum_{j_1=4}^{\infty} \frac{1}{|2^{j_1} B_1^i|^{\rho/n}}\\
&\lesssim \sum_i |Q_1^i| \inf_{z \in Q_1^i} Mw(z) \lesssim \lambda^{-p}.
\end{aligned}$$

This, together with the fact that

$$w\Big(\Big\{x \in \mathbb{R}^n : M_{r_2} f_2(x) > \frac{1}{2} \lambda^{p/p_2}\Big\}\Big) \lesssim \lambda^{-p} \int_{\mathbb{R}^n} |f_2(x)|^{p_2} Mw(x)\, dx \lesssim \lambda^{-p},$$

then leads to that

$$\begin{aligned}
w\Big(\Big\{x \in \mathbb{R}^n : \big|T(b_1, f_2)(x)\big| > \frac{\lambda}{2}\Big\}\Big) &\lesssim w\big(\big\{x \in \mathbb{R}^n : M_{r_2} f_2(x) > \lambda^{p/p_2}\big\}\big)\\
&\quad + w\big(\big\{x \in \mathbb{R}^n : \sum_i \mathrm{L}_i(x) > \frac{1}{2} \lambda^{p/p_1}\big\}\big)\\
&\lesssim \lambda^{-p},
\end{aligned}$$

and then completes the proof of Theorem 2.2. □

3. Proof of Theorems

To prove Theorem 1.1 and 1.2, we will employ some preliminary lemmas. For $\sigma \in L^\infty(\mathbb{R}^{2n})$, let σ_l be the same as in (1.3).

LEMMA 3.1. *Let $q_1, q_2 \in [2,\infty)$, and $s_1, s_2 \geqslant 0$. Then*

$$\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|\widehat{\sigma}_\kappa(\xi_1,\xi_2)|^{q_1}\langle\xi_1\rangle^{s_1}\mathrm{d}\xi_1\right)^{q_2/q_1}\langle\xi_2\rangle^{s_2}\mathrm{d}\xi_2\right)^{1/q_2} \lesssim \|\sigma_\kappa\|_{W^{s_1/q_1,s_2/q_2}(\mathbb{R}^{2n})}.$$

For the proof of Lemma 3.1, see Appendix A in [5].

LEMMA 3.2. *Let $s_1, s_2 \in [0,\infty)$, and $\alpha_1, \alpha_2 \in \mathbb{Z}_+^n$ be multi-indices. For $\kappa \in \mathbb{Z}$, set*

$$\zeta_\kappa^{\alpha_1,\alpha_2}(\xi_1,\xi_2) := \xi_1^{\alpha_1}\xi_2^{\alpha_2}\sigma_\kappa(\xi_1,\xi_2).$$

Then

$$\|\zeta_\kappa^{\alpha_1,\alpha_2}\|_{W^{s_1,s_2}(\mathbb{R}^{2n})} \lesssim \sup_{l\in\mathbb{Z}}\|\sigma_l\|_{W^{s_1,s_2}(\mathbb{R}^{2n})}.$$

This lemma was given in [15, Remark 2.5].

Let $\Phi \in \mathscr{S}(\mathbb{R}^{2n})$ be the same as in Section 1. For $l \in \mathbb{Z}$, set

$$\widetilde{\sigma}_l(\xi_1,\xi_2) = \sigma(\xi_1,\xi_2)\Phi(2^{-l}\xi_1, 2^{-l}\xi_2),$$

and

$$K_l(x;y_1,y_2) = \mathscr{F}^{-1}\widetilde{\sigma}_l(x-y_1,x-y_2).$$

For $y, y, y_1, y_2 \in \mathbb{R}^n$, let

$$U_{0,l}(y,y_1,y_2;y') = K_l(y;y_1,y_2) - K_l(y';y_1,y_2),$$

$$U_{1,l}(y,y_1,y_2;y_1') = K_l(y;y_1,y_2) - K_l(y;y_1',y_2).$$

For $r_1, r_2 \in (1,\infty)$, $l \in \mathbb{Z}$ and ball B with radius R, set

$$A_{l,1}^{j_1,j_2}(y,y_1') = \left(\int_B\left(\int_{\mathscr{O}_{j_2}^R(y)}|U_{1,l}(y,y_1,y_2;y_1')|^{r_2'}\mathrm{d}y_2\right)^{\frac{r_1'}{r_2'}}\mathrm{d}y_1\right)^{1/r_1'}\chi_{S_{j_1}(B)}(y), \tag{3.1}$$

where $\mathscr{O}_0^R(x) = B(x,R)$ and $\mathscr{O}_j^R(x) = 2^jB(x,R)\setminus 2^{j-1}B(x,R)$ for $j \in \mathbb{N}$.

LEMMA 3.3. *Let σ be a bilinear multiplier satisfying (1.5) for some $s_1, s_2 > n/2$.*

(i) *If $r_1, r_2 \in (1,2]$ and B is a ball with radius R, $y_1' \in B$ and $l \in \mathbb{Z}$ such that $2^lR \leqslant 1$, then for nonnegative integers $j_1 \geqslant 4$ and $j_2 \geqslant 0$, the inequality*

$$A_{l,1}^{j_1,j_2}(y,y_1') \lesssim D_{s_1,s_2}(\sigma)\frac{R2^{-l(s_1+s_2-n/r_1-n/r_2-1)}}{|2^{j_1}B|^{s_1/n}|2^{j_2}B|^{s_2/n}}; \tag{3.2}$$

(ii) *for each* $l \in \mathbb{Z}$, *there exists a function* $\mathrm{H}_{1,l}$ *on* $\mathbb{R}^n \times \mathbb{R}^n$ *such that, for any ball* B *with radius* R, *any function* f_1 *with* $\operatorname{supp} f_1 \subset B$, *and any* $y \in \mathbb{R}^n \setminus 8B$,

$$\int_{\mathbb{R}^{2n}} |K_l(y,y_1,y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2 \lesssim \int_{\mathbb{R}^n} |f_1(y_1)|\mathrm{H}_{1,l}(y,y_1)\mathrm{d}y_1 M_{r_2}f_2(y), \tag{3.3}$$

and for $j_1 \geqslant 3$ *and* $r_1 \in (1,2]$

$$\Big(\int_B |\mathrm{H}_{1,l}(y,y_1)|^{r_1'}\mathrm{d}y_1\Big)^{\frac{1}{r_1'}} \chi_{S_{j_1}(B)}(y) \lesssim D_{s_1,s_2}(\sigma)\frac{2^{-l(s_1-n/r_1)}}{(2^{j_1}R)^{s_1}}. \tag{3.4}$$

Proof. For simplicity, we assume that $D_{s_1,s_2}(\sigma)=1$. We will employ the ideas used in the proof of Lemma 3.3 and Lemma 3.4 of [11].

We first consider conclusion (i). For the fixed ball B with radisu R, let $B_R = B(0,R)$. We consider the following two cases

Case 1. $j_2 \geqslant 1$. Write

$$\begin{aligned}
&|\mathscr{F}^{-1}\widetilde{\sigma}_l(z_1,z_2) - \mathscr{F}^{-1}\widetilde{\sigma}_l(z_1+y_1-y_1',z_2)| \\
&\quad = 2^{2\kappa n}\Big|\mathscr{F}^{-1}\sigma_\kappa(2^l z_1, 2^l z_2) - \mathscr{F}^{-1}\sigma_l(2^l z_1 + 2^l(y_1-y_1'), 2^l z_2)\Big| \\
&\quad \leqslant 2^{2ln}\sum_{|\alpha|=1}|2^l(y_1-y_1')|^\alpha \int_0^1 |\partial^{\alpha,0}\mathscr{F}^{-1}\sigma_l(2^l(z_1+\theta(y_1-y_1')),2^l z_2)|\mathrm{d}\theta.
\end{aligned}$$

By the Minkowsky inequality, Lemmas 3.1 and 3.2,

$$\begin{aligned}
A_{l,1}^{j_1,j_2}(x) &\lesssim \sum_{|\alpha|=1}\Big(\int_B\Big(\int_{S_{j_2}(B_R)}\Big(\int_0^1|\phi_{l,y_1-y_1'}(\theta;z_1,z_2)|\mathrm{d}\theta\Big)^{r_2'}\mathrm{d}z_2\Big)^{\frac{r_1'}{r_2'}}\mathrm{d}z_1\Big)^{\frac{1}{r_1'}}2^{2ln}2^l R \\
&\lesssim \sum_{|\alpha|=1}\Big(\int_{C_{j_1}^R}\Big(\int_{S_{j_2}(B_R)}\Big|\partial^{\alpha,0}\mathscr{F}^{-1}\sigma_l(2^l z_1,2^l z_2)\Big|^{r_2'}\mathrm{d}z_2\Big)^{\frac{r_1'}{r_2'}}\mathrm{d}z_1\Big)^{\frac{1}{r_1'}}2^{2ln}2^l R \\
&\lesssim \sum_{|\alpha|=1}\Big(\int_{C_{j_1}}\Big(\int_{S_{j_2}(B_R)}\Big|\mathscr{F}^{-1}(\xi_1^\alpha\sigma_l)(2^\kappa z_1,,2^l z_2)\Big|^{r_2'} \\
&\qquad\qquad \times|z_2|^{r_2's_m}\mathrm{d}z_2\Big)^{\frac{r_1'}{r_2'}}|z_1|^{r_1's_1}\mathrm{d}z_1\Big)^{\frac{1}{r_1'}}\frac{2^l R 2^{2ln}}{(2^{j_1}R)^{s_1}(2^{j_2}R)^{s_2}} \\
&\lesssim \sum_{|\alpha|=1}\|\xi_1^\alpha\sigma_l\|_{W^{s_1,s_2}(\mathbb{R}^{2n})}\frac{2^l R}{(2^{j_1}R)^{s_1}(2^{j_2}R)^{s_2}}2^{-l(s_1+s_2-n/r_1-n/r_2)} \\
&\lesssim \frac{R}{(2^{j_1}R)^{s_1}(2^{j_2}R)^{s_2}}2^{-l(s_1+s_2-n/r_1-n/r_2-1)},
\end{aligned}$$

where $C_{j_1}^R = \{z : 2^{j_1-2}R \leqslant |z| \leqslant 2^{j_1+2}R\}$. So (3.2) holds in this case.

Case 2. $j_2=0$. For index $\alpha\in\mathbb{Z}_+^n$ and $2^lR<1$,

$$
\begin{aligned}
&\Big(\int_{C_{j_1}^R}\Big(\int_{\mathbb{R}^n}\Big|\partial^{\alpha,0}\mathscr{F}^{-1}\sigma_l(2^lz_1,2^lz_2)\Big|^{r_2'}\mathrm{d}z_2\Big)^{\frac{r_1'}{r_2'}}\mathrm{d}z_1\Big)^{\frac{1}{r_1'}}2^{2ln}2^lR\\
&\quad\lesssim\Big(\int_{C_{j_1}^R}\Big(\int_{\mathbb{R}^n}\Big|\mathscr{F}^{-1}(\xi_1^\alpha\sigma_l)(2^lz_1,2^lz_2)\Big|^{r_2'}\mathrm{d}z_2\Big)^{\frac{r_1'}{r_2'}}|z_1|^{r_1's_1}\mathrm{d}z_1\Big)^{\frac{1}{r_1'}}\frac{2^lR2^{2ln}}{(2^{j_1}R)^{s_1}}\\
&\quad\lesssim 2^lR(2^{j_1}R)^{-s_1}2^{-l(s_1-n/r_1-n/r_2)}\\
&\quad\lesssim 2^lR(2^{j_1}R)^{-s_1}R^{-s_2}2^{-l(s_1+s_2-n/r_1-n/r_2)}.
\end{aligned}
$$

Our desired conclusion (3.2) then follows directly.

To prove conclusion (ii), let $r_2\in(1,2]$ and $r_2>n/s_2$, and set

$$
\mathrm{H}_{1,l}(y,y_1)=2^{ln}\Big(\int_{\mathbb{R}^n}|\mathscr{F}^{-1}\sigma_l(2^\kappa(y-y_1),2^ly-y_2)|^{r_2'}\langle 2^ly-y_2\rangle^{r_2's_2}dy_2\Big)^{\frac{1}{r_2'}}.
$$

For $y\in S_{j_1}(B)$ with $j_1\geqslant 4$, we deduce from Lemma 3.1 that

$$
\begin{aligned}
&\Big(\int_B|\mathrm{H}_{1,l}(y,y_1)|^{r_1'}\mathrm{d}y_1\Big)^{\frac{1}{r_1'}}\\
&\quad=2^{ln}\Big(\int_B\Big(\int_{\mathbb{R}^n}|\mathscr{F}^{-1}\sigma_l(2^l(y-y_1),2^ly-y_2)|^{r_2'}\langle 2^ly-y_2\rangle^{r_2's_2}\mathrm{d}y_2\Big)^{\frac{r_1'}{r_2'}}\\
&\qquad\times|2^l(y-y_1)|^{r_1's_1}\mathrm{d}y_1\Big)^{\frac{1}{r_1'}}(2^l2^{j_1}R)^{-s_1}\\
&\quad\lesssim 2^{ln}\Big(\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}|\mathscr{F}^{-1}\sigma_l(z_1,z_2)|^{r_2'}\langle z_2\rangle^{r_2's_2}dz_2\Big)^{\frac{r_1'}{r_2'}}\langle z_1\rangle^{r_1's_1}dz_1\Big)^{\frac{1}{r_1'}}2^{-ln/r_1'}(2^l2^{j_1}R)^{-s_1}\\
&\quad\lesssim 2^{-l(s_1-n/r_1)}(2^{j_1}R)^{-s_1}.
\end{aligned}
$$

The Hölder inequality now gives us that

$$
\begin{aligned}
&\int_{\mathbb{R}^{2n}}|\mathscr{F}^{-1}\widetilde{\sigma}_l(y-y_1,y-y_2)||f_1(y_1)f_2(y_2)|\mathrm{d}\vec{y}\\
&\quad=2^{2ln}\int_{\mathbb{R}^{2n}}|\mathscr{F}^{-1}\sigma_l(2^l(y-y_1),\,2^l(y-y_2))||f_1(y_1)f_2(y_2)|\mathrm{d}\vec{y}\\
&\quad\lesssim 2^{2ln}\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}|\mathscr{F}^{-1}\sigma_l(2^l(y-y_1),\,2^l(y-y_2))|^{r_2'}\langle 2^l(x-y_2)\rangle^{r_2's_2}\mathrm{d}y_2\Big)^{\frac{1}{r_2'}}|f_1(y_1)|\mathrm{d}y_1\\
&\qquad\times\Big(\int_{\mathbb{R}^n}\frac{|f_2(y_2)|^{r_2}}{\langle 2^l(y-y_2)\rangle^{r_2s_2}}\mathrm{d}y_2\Big)^{1/r_2}\\
&\quad\lesssim\int_{\mathbb{R}^n}\mathrm{H}_{1,l}(y,y_1)|f_1(y_1)|\,\mathrm{d}y_1M_{r_2}f_2(y).
\end{aligned}
$$

This completes the proof of Lemma 3.3. □

LEMMA 3.4. *Let σ be a multiplier satisfying (1.5) for some $s_1, s_2 > 0$, $r_1, r_2 \in (1,2]$, B be a ball with radius R. Let $j_1 \geqslant 0$, $j_2 \geqslant 4$ and l be integers. Then*

(i) *the inequality*

$$\begin{aligned}\int_B \int_B \Big(\int_{S_{j_1}(B)} |U_{0,l}(y, y_1, y_2; y')|^{r_1'} \mathrm{d}y_1 \Big)^{\frac{1}{r_1'}} \mathrm{d}y \mathrm{d}y' \chi_{S_{j_2}(B)}(y_2) \\ \lesssim |B|^{1+\frac{1}{r_2}} R 2^{-l(s_1+s_2-n/r_1-n/r_2-1)} (2^{j_1}R)^{-s_1} (2^{j_2}R)^{-s_2}, \qquad (3.5)\end{aligned}$$

holds if $2^l R \leqslant 1$;

(ii)

$$\int_B \int_{\mathbb{R}^n} |\mathscr{F}^{-1} \widetilde{\sigma}_l (y - y_1, y - y_2)| \mathrm{d}y_1 \mathrm{d}y \chi_{S_{j_2}(B)}(y_2) \lesssim |B|^{\frac{1}{r_2}} \frac{2^{-l(s_2-n/r_2)}}{(2^{j_2}R)^{s_2}}. \qquad (3.6)$$

Proof. For each fixed $y, y_1, y_2, y' \in \mathbb{R}^n$, set

$$\mathrm{J}_1(y, y_1, y_2; y') = \mathscr{F}^{-1} \widetilde{\sigma}_l (y - y_1, y - y_2) - \mathscr{F}^{-1} \widetilde{\sigma}_l (y - y_1, y' - y_2).$$

and write

$$\begin{aligned}| \ \mathscr{F}^{-1} \sigma_l (2^l z_1, 2^l (y - y_2)) - \mathscr{F}^{-1} \sigma_l (2^l z_1, 2^l (y' - y_2))| \\ \lesssim 2^l R \sum_{|\alpha|=1} \int_0^1 \big| \partial^{0,\alpha} \mathscr{F}^{-1} \sigma_l \big(2^l z_1, 2^l (y_2 + \theta(y - y'))\big) \big| \mathrm{d}\theta.\end{aligned}$$

If $j_1 \geqslant 1$, it then follows that for any $y_2 \in S_{j_2}(B)$ with $j_2 \geqslant 4$, and $y' \in B$,

$$\begin{aligned}&\int_B \Big(\int_{S_{j_1}(B)} |\mathrm{J}_1(y, y_1, y_2; y')|^{r_1'} \mathrm{d}y_1 \Big)^{\frac{1}{r_1'}} \mathrm{d}y \\ &\lesssim 2^{2ln} 2^l R \sum_{|\alpha|=1} \int_0^1 \int_B \Big(\int_{C_{j_1}^R} \big| \partial^{0,\alpha} \mathscr{F}^{-1} \sigma_l \big(2^l z_1, 2^l (y_2 + \theta(y - y'))\big) \big|^{r_1'} \mathrm{d}z_1 \Big)^{\frac{1}{r_1'}} \mathrm{d}y \mathrm{d}\theta \\ &\lesssim \sum_{|\alpha|=1} \int_0^1 \Big(\int_B \Big(\int_{C_{j_1}^R} \big| \partial^{0,\alpha} \mathscr{F}^{-1} \sigma_l \big(2^l z_1, 2^l (y_2 + \theta(y - y'))\big) \big|^{r_1'} \mathrm{d}z_1 \Big)^{\frac{r_2'}{r_1'}} \mathrm{d}y \Big)^{\frac{1}{r_2'}} \mathrm{d}\theta \\ &\qquad \times 2^{2ln} 2^l R |B|^{1/r_2} \\ &\lesssim 2^{2ln} 2^l R |B|^{1/r_2} \sum_{|\alpha|=1} \Big(\int_{C_{j_2}^R} \Big(\int_{C_{j_1}^R} \big| \partial^{0,\alpha} \mathscr{F}^{-1} \sigma_l \big(2^l z_1, 2^l z_2\big) \big|^{r_1'} \mathrm{d}z_1 \Big)^{\frac{r_2'}{r_1'}} \mathrm{d}z_2 \Big)^{\frac{1}{r_2'}} \\ &\lesssim |B|^{1/r_2} \frac{R 2^{-l(s_1+s_2-n/r_1-n/r_2-1)}}{(2^{j_1}R)^{s_1} (2^{j_2}R)^{s_2}}.\end{aligned}$$

On the other hand, if $j_1=0$, a trivial computation leads to that

$$\begin{aligned}
&\int_B\Big(\int_{S_{j_1}(B)}|\mathrm{J}_1(y,y_1,y_2;y')|^{r_1'}\mathrm{d}y_1\Big)^{\frac{1}{r_1'}}\mathrm{d}y\\
&\lesssim 2^{2ln}2^lR|B|^{1/r_2}\sum_{|\alpha|=1}\Big(\int_{C_{j_2}^R}\Big(\int_{\mathbb{R}^n}\big|\partial^{0,\alpha}\mathscr{F}^{-1}\sigma_l\big(2^lz_1,2^lz_2)\big)\big|^{r_1'}\mathrm{d}z_1\Big)^{\frac{r_2'}{r_1'}}\mathrm{d}z_2\Big)^{\frac{1}{r_2'}}\\
&\lesssim|B|^{1/r_2}\frac{R2^{-l(s_2-n/r_1-n/r_2-1)}}{(2^{j_2}R)^{s_2}}\\
&\lesssim|B|^{1/r_2}\frac{R2^{-l(s_1+s_2-n/r_1-n/r_2-1)}}{(2^{j_1}R)^{s_1}(2^{j_2}R)^{s_2}},
\end{aligned}$$

since $2^lR<1$. Similarly, if we set

$$\mathrm{J}_2(y,y_1,y_2;y')=\mathscr{F}^{-1}\widetilde{\sigma}_l(y-y_1,y'-y_2)-\mathscr{F}^{-1}\widetilde{\sigma}_l(y'-y_1,y'-y_2),$$

it is easy to verify that, for any $y\in B$ and $y_2\in S_{j_2}(B)$ with $j_2\geqslant 4$,

$$\begin{aligned}
&\int_B\Big(\int_{S_{j_1}(B)}|\mathrm{J}_2(y,y_1,y_2;y')|^{r_1'}\mathrm{d}y_1\Big)^{\frac{1}{r_1'}}\mathrm{d}y'\\
&\qquad\lesssim|B|^{1/r_2}\frac{R2^{-l(s_1+s_2-n/r_1-n/r_2-1)}}{(2^{j_1}R)^{s_1}(2^{j_2}R)^{s_2}}.
\end{aligned}$$

Note that

$$\mathscr{F}^{-1}\widetilde{\sigma}_l(y-y_1,y-y_2)-\mathscr{F}^{-1}\widetilde{\sigma}_l(y'-y_1,y'-y_2)=\mathrm{J}_1(y,y_1,y_2;y')+\mathrm{J}_2(y,y_1,y_2;y').$$

The inequality (3.5) now follows directly.

To prove (3.6), let $r_1>n/s_1$. For each fixed $y_2\in S_{j_2}(B)$ with $j_2\geqslant 4$, a straightforward computation leads to that

$$\begin{aligned}
&2^{2ln}\int_B\int_{\mathbb{R}^n}|\mathscr{F}^{-1}\sigma_l(2^l(y-y_1),2^l(y-y_2))|\mathrm{d}y_1\mathrm{d}y\\
&\lesssim 2^{2ln}\int_B\Big(\int_{\mathbb{R}^n}|\mathscr{F}^{-1}\sigma_l(2^l(y-y_1),2^l(y-y_2))|^{r_1'}\langle 2^l(x-y_1)\rangle^{r_1's_1}\mathrm{d}y_1\Big)^{1/r_1'}\\
&\qquad\times\Big(\int_{\mathbb{R}^n}\frac{1}{\langle 2^l(y-y_1)\rangle^{s_1r_1}}\mathrm{d}y_1\Big)^{1/r_1}\mathrm{d}y\\
&\lesssim 2^{ln}\Big(\int_B\Big(\int_{\mathbb{R}^n}|\mathscr{F}^{-1}\sigma_l(2^ly-y_1),2^l(y-y_2))|^{r_1'}\langle 2^ly-y_1\rangle^{r_1's_1}\mathrm{d}y_1\Big)^{\frac{r_2'}{r_1'}}\mathrm{d}y\Big)^{\frac{1}{r_2'}}|B|^{\frac{1}{r_2}}\\
&\lesssim 2^{ln}\Big(\int_{C_{j_2}^R}\Big(\int_{\mathbb{R}^n}|\mathscr{F}^{-1}\sigma_l(z_1,2^lz_2)|^{r_1'}\langle z_1\rangle^{r_1's_1}\mathrm{d}z_1\Big)^{\frac{r_2'}{r_1'}}|z_2|^{s_2r_2'}\mathrm{d}z_2\Big)^{\frac{1}{r_2'}}(2^{j_2}R)^{-s_2}|B|^{\frac{1}{r_2}}\\
&\lesssim\frac{2^{-l(s_2-n/r_2)}}{(2^{j_2}R)^{s_2}}|B|^{\frac{1}{r_2}}.
\end{aligned}$$

This finishes the proof of Lemma 3.4. □

LEMMA 3.5. *Let* $s_{k,j} > n/2$ *and* $p_{k,j} \in (0,\infty]$ $(k,j = 1.2)$. *For* $\theta \in (0,1)$, *let*

$$1/p_k^\theta = \theta/p_{k,1} + (1-\theta)/p_{k,2},\ s_k^\theta = \theta s_{k,1} + (1-\theta)s_{k,2},\ k = 1,2.$$

Suppose that for three weights u, w_1 *and* w_2,

$$\|T(f_1,f_2)\|_{L^{p^1}(\mathbb{R}^n,u)} \lesssim \sup_{\kappa\in\mathbb{Z}} \|\sigma_\kappa\|_{W^{s_{1,1},s_{2,1}}(\mathbb{R}^{2n})} \prod_{k=1}^{2} \|f_k\|_{L^{p_{k,1}}(\mathbb{R}^n,w_k)},$$

and

$$\|T(f_1,f_2)\|_{L^{p^2}(\mathbb{R}^n,u)} \lesssim \sup_{\kappa\in\mathbb{Z}} \|\sigma_\kappa\|_{W^{s_{1,2},s_{2,2}}(\mathbb{R}^{2n})} \prod_{k=1}^{2} \|f_k\|_{L^{p_{k,2}}(\mathbb{R}^n,w_k)},$$

Then for any $\theta \in (0,1)$,

$$\|T(f_1,f_2)\|_{L^{p^\theta}(\mathbb{R}^n,u)} \lesssim \sup_{\kappa\in\mathbb{Z}} \|\sigma_\kappa\|_{W^{s_1^\theta,s_2^\theta}(\mathbb{R}^{2n})} \prod_{k=1}^{2} \|f_k\|_{L^{p_k^\theta}(\mathbb{R}^n,w_k)},$$

with $1/p^\theta = 1/p_1^\theta + 1/p_2^\theta$.

By the interpolation theorem for analytic families of operators, Lemma 4.1 can be proved by an argument similar to Step 1 in the proof of Theorem 6.1 in [8]. We omit the details for brevity.

Proof of Theorem 1.1. For positive integer N, set

$$K^N(y,y_1,y_2) = \sum_{l\in\mathbb{Z},\,|l|\leqslant N} \mathscr{F}^{-1}\widetilde{\sigma}_l(y,y_1,y_2)$$

and let $T_{\sigma,N}$ be the bilinear integral operator with associated kernel K^N in the sense of (2.1). As it was pointed out in [12], [11], it suffices to prove that the conclusions of Theorem 1.1 are true for the operator $T_{\sigma,N}$ with bounds independent of N.

We first prove that $T_{\sigma,N}$ satisfies (1.6). By Lemma 3.5 and Lemma 3.6 in [11], a standard argument shows that for any $\gamma_2 > t_2$

$$M^\sharp\big(T_{\sigma,N}(f_1,f_2)\big)(x) \lesssim D_{s_1,s_2}(\sigma)\|f_1\|_{L^\infty(\mathbb{R}^n)} M_{\gamma_2} f_2(x). \tag{3.7}$$

For each fixed $p_2 \in (t_2,\infty)$, we can take $\gamma_2 \in (t_2,p_2)$. By the clever idea of Lerner (see [13]) and the inequality (3.7), we know that for each $h \in L^{p_2'}(\mathbb{R}^n, w^{1-p_2'})$ with $\|h\|_{L^{p_2'}(\mathbb{R}^n,w^{1-p_2'})} \leqslant 1$,

$$\begin{aligned}\int_{\mathbb{R}^n} |T_{\sigma,N}f(x)h(x)|\,\mathrm{d}x &\lesssim \int_{\mathbb{R}^n} M^\sharp(T_{\sigma,N}(f_1,f_2))(x)Mh(x)\,\mathrm{d}x\\ &\lesssim D_{s_1,s_2}(\sigma)\|f_1\|_{L^\infty(\mathbb{R}^n)}\|M_{\gamma_2}f_2\|_{L^{p_2}(\mathbb{R}^n,M_{L(\log L)^{p_2-1+\delta}}w)}\\ &\quad\times\|Mh\|_{L^{p_2'}\left(\mathbb{R}^n,(M_{L(\log L)^{p_2-1+\delta}}w)^{1-p_2'}\right)}\\ &\lesssim D_{s_1,s_2}(\sigma)\|f_1\|_{L^\infty(\mathbb{R}^n)}\|f_2\|_{L^{p_2}(\mathbb{R}^n,M_{L(\log L)^{p_2+\delta}}w)},\end{aligned}$$

where the last inequality follows from the fact that

$$M\big(M_{L(\log L)^{p_2-1+\delta}}w\big)(x)\lesssim M_{L(\log L)^{p_2+\delta}}w(x),$$

see [2], and that for any weight u and $p\in(1,\infty)$,

$$\|Mf\|_{L^{p'}(\mathbb{R}^n,(M_{L(\log L)^{p-1+\delta}}u)^{1-p'}}\lesssim\|f\|_{L^p(\mathbb{R}^n,u^{1-p'})},$$

see [17]. Therefore, for $p_2\in(t_2,\infty)$,

$$\|T_{\sigma,N}(f_1,f_2)\|_{L^{p_1}(\mathbb{R}^n,w)}\lesssim D_{s_1,s_2}(\sigma)\|f_1\|_{L^\infty(\mathbb{R}^n)}\|f_2\|_{L^{p_2}(\mathbb{R}^n,M_{L(\log L)^{p_2+\delta}}w)}. \tag{3.8}$$

Similarly, we have that when $p_1\in(t_1,\infty)$,

$$\|T_{\sigma,N}(f_1,f_2)\|_{L^{p_1}(\mathbb{R}^n,w)}\lesssim D_{s_1,s_2}(\sigma)\|f_1\|_{L^{p_1}(\mathbb{R}^n,M_{L(\log L)^{p_1+\delta}}w)}\|f_2\|_{L^\infty(\mathbb{R}^n)}. \tag{3.9}$$

The inequality (1.6) now follows directly, if we apply the bilinear Riesz-Thorin interpolation theorem ([6, p. 72]) to (3.8) and (3.9).

We now show that $T_{\sigma,N}$ satisfies (1.7). Set

$$U_1^N(y,y_1,y_2;y_1')=\sum_{l\in\mathbb{Z}:\,|l|\leqslant N}U_{1,l}(y,y_1,y_2;y_1').$$

Let $\mathrm{H}_{1,l}$ be the same as in Lemm 3.3. For each fixed $R>0$, set

$$\begin{aligned}\mathrm{H}^N_{1,R}(y,y_1,y_1')=&\sum_{|l|\leqslant N,2^lR>1}|\mathrm{H}_{1,l}(y,y_1)|\\&+\sum_{|l|\leqslant N,2^lR\leqslant 1}\sum_{j_2=0}^{\infty}\Big(\int_{\mathscr{O}^R_{j_2}(y)}|U_{1,l}(y,y_1,y_2;y_1')|^{r_2'}\mathrm{d}y_2\Big)^{\frac{1}{r_2'}}(2^{j_2}R)^{n/r_2}.\end{aligned}$$

Then for ball B with radius R, function f_1 with $\operatorname{supp} f_1\subset B$ and $\int_{\mathbb{R}^n}f_1(y_1)\mathrm{d}y_1=0$, $y\in\mathbb{R}^n\setminus 8B$ and $y_1'\in B$,

$$|T_{\sigma,N}(f_1,f_2)(y)|\lesssim\int_{\mathbb{R}^n}|f_1(y_1)|\mathrm{H}^N_{1,R}(y,y_1,y_1')\mathrm{d}y_1M_{r_2}f_2(y).$$

On the other hand, it follows from Lemma 3.3 that for $j_1\geqslant 4$,

$$\begin{aligned}&\Big(\int_B|\mathrm{H}^N_{1,R}(y,y_1,y_1')|^{r_1'}\mathrm{d}y_1\Big)^{\frac{1}{r_1'}}\chi_{S_{j_1}(B)}(y)\\&\lesssim\sum_{|l|\leqslant N,2^lR>1}\Big(\int_B|\mathrm{H}_{1,l}(y,y_1,y_1')|^{r_1'}\mathrm{d}y_1\Big)^{\frac{1}{r_1'}}\chi_{S_{j_1}(B)}(y)\\&\quad+\sum_{l:2^lR<1}\sum_{j_2=0}^{\infty}(2^{j_2}R)^{n/r_2}\Big(\int_B\Big(\int_{\mathscr{O}^R_{j_2}}|U_{1,l}(y,y_1,y_2;y_1')|^{r_2'}dy_2\Big)^{\frac{r_1'}{r_2'}}\mathrm{d}y_1\Big)^{\frac{1}{r_1'}}\chi_{S_{j_1}(B)}(y)\\&\lesssim D_{s_1,s_2}(\sigma)\frac{R^{s_1-n/r_1}}{|2^{j_1}B|^{s_1/n}},\end{aligned}$$

if

$$r_k \in (1,2],\ r_k > t_k\ (k=1,2), \text{ such that } s_1+s_2-n/r_1-n/r_2-1<0. \tag{3.10}$$

We know from (3.8) and Theorem 2.2 that if $s_1 > n$, $s_2 \in (n/2, n]$, then for weight w and any $\delta > 0$,

$$\|T_{\sigma,N}(f_1,f_2)\|_{L^{p,\infty}(\mathbb{R}^n,w)} \lesssim \mathrm{D}_{s_1,s_2}(\sigma)\|f_1\|_{L^{p_1}(\mathbb{R}^n,Mw)}\|f_2\|_{L^{p_2}(\mathbb{R}^n,M_{L(\log L)^{p_2+\delta}}w)} \tag{3.11}$$

when $p_k \in (r_k,\infty)$ $(k=1,2)$ with r_1, r_2 satisfying (3.10). Note that when $p_1 \in (1,\infty)$ and $p_2 \in (t_2,\infty)$, we can choose r_1, r_2 satisfies (3.10) and $p_k \in (r_k,\infty)$. Therfore, (3.11) still holds true if $p_1 \in (1,\infty)$ and $p_2 \in (t_2,\infty)$. On the other hand, for each fixed $p_1 \in (1,\infty)$, $p_2 \in (t_2,\infty)$, and fixed $\delta > 0$, we can choose p_{11}, p_{12}, p_{13}; p_{21}, p_{22}, p_{23}; and $\delta_1, \delta_2, \delta_3 \in (0,\delta)$, such that

(a) $p_{kj} \in (1,\infty)$ for $k=1,2$ and $j=1,2,3$; $p_{1j} < p_1$, or $p_{1j} > p_1$ but $p_{1j}+\delta_j < p_1+\delta$ for $j=1,2,3$;

(b) $(1/p_1, 1/p_2, 1/p)$ is in the open convex hull of the points

$$(1/p_{11}, 1/p_{21}, 1/p^1);\ (1/p_{12}, 1/p_{22}, 1/p^2);\ (1/p_{13}, 1/p_{23}, 1/p^3)$$

where $1/p^j = \sum_{k=1}^{2} 1/p_{kj}$.

We know from (3.11) that for each $j=1,2,3$,

$$\|T_{\sigma,N}(f_1,f_2)\|_{L^{p_j,\infty}(\mathbb{R}^n,w)} \lesssim \mathrm{D}_{s_1,s_2}(\sigma)\|f_1\|_{L^{p_{1j}}(\mathbb{R}^n,Mw)}\|f_2\|_{L^{p_{2j}}(\mathbb{R}^n,M_{L(\log L)^{p_{2j}+\delta_j}}w)}.$$

This, via the multilinear Marcinkiewicz interpolation (see [6, p. 72]), shows that when $s_1 > n$ and $s_2 > n/2$,

$$\|T_{\sigma,N}(f_1,f_2)\|_{L^{p}(\mathbb{R}^n,w)} \lesssim \mathrm{D}_{s_1,s_2}(\sigma)\|f_1\|_{L^{p_1}(\mathbb{R}^n,Mw)}\|f_2\|_{L^{p_2}(\mathbb{R}^n,M_{L(\log L)^{p_2+\delta}}w)}, \tag{3.12}$$

provided that $p_1 \in (1,\infty)$, $p_2 \in (t_2,\infty)$. By the inequalities (3.8) and (3.12), an application of Lemma 3.5 then yields our desired conclusion (1.7). □

Proof of Theorem 1.2. Let $\breve{\sigma}$ be the multiplier defined by $\breve{\sigma}(\xi_1,\xi_2) = \sigma(\xi_1,-\xi_2)$, and $T_{\breve{\sigma},N}$ be the multiplier operator with associated multiplier $\sum_{l\in\mathbb{Z}:|l|\leqslant N}\breve{\sigma}_l$. It is obvious that $T_{\breve{\sigma},N}$ is the second transpose of $T_{\sigma,N}$ in the sense that for Schwartz functions h, f_1, f_2,

$$\int_{\mathbb{R}^n} T_{\sigma,N}(f_1,f_2)(x)h(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} T_{\breve{\sigma},N}(f_1,h)(x)f_2(x)\,\mathrm{d}x.$$

Note that σ satisfies (1.5) for some $s_1, s_2 > n/2$ implies that $\breve{\sigma}$ satisfies (1.5). Again by Theorem 1.2 in [11] (see also [8]), we know that $T_{\breve{\sigma},N}$ is bounded from $L^\infty(\mathbb{R}^n)\times L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. For $y, y_1, y_2, y' \in \mathbb{R}^n$, let

$$U_{0,l}^{*}(y,y_1,y_2;y') = \left|\mathscr{F}\breve{\sigma}_l(y-y_1,y-y_2) - \breve{\sigma}_l(y'-y_1,y'-y_2)\right|$$

and

$$U_0^{N,*}(y, y_1, y_2; y') = \sum_{l\in\mathbb{Z}:\,|l|\leqslant N} U_{0,l}^*(y, y_1, y_2; y').$$

We have by Lemma 3.4 that, for any ball B with radius R and $y_2 \in S_{j2}(B)$ with $j_2 \geqslant 4$,

$$\begin{aligned}
&\int_B\int_B\int_{\mathbb{R}^n} |U_0^{N,*}(y, y_1, y_2; y')|\mathrm{d}y_1\mathrm{d}y\mathrm{d}y' \\
&\lesssim \sum_{l:\,2^lR\leqslant 1}\sum_{j_1=0}^{\infty}\int_B\int_B\int_{S_{j_1}(B)} |U_{0,l}^*(y, y_1, y_2; y')|\mathrm{d}y_1\mathrm{d}y\mathrm{d}y' \\
&\quad + \sum_{l:\,2^lR>1}\int_B\int_B\int_{\mathbb{R}^n} |U_{0,l}^*(y, y_1, y_2; y')|\mathrm{d}y_1\mathrm{d}y\mathrm{d}y' \\
&\lesssim |B|^2\frac{R^{s_2-n}}{(2^{j_2}R)^{s_2}},
\end{aligned}$$

if we choose $r_k > n/s_k$ closely to n/s_k $(k = 1, 2)$ sufficiently such that $s_1+s_2-n/r_1-n/r_2-1 < 0$. By Theorem 2.1, we know that if $s_1 > n/2$ and $s_2 > n$,

$$M_\gamma^\sharp\big(T_{\breve{\sigma},N}(f_1, f_2)\big)(x) \lesssim Mf_2(x)\|f_1\|_{L^\infty(\mathbb{R}^n)}.$$

This, via the argument used in the proof of Theorem 1.1 in [16], tells us that if $s_1 > n/2$, $s_2 > n$, then for any $p \in (1,\infty)$, weight w and any $\delta > 0$,

$$\|T_{\sigma,N}(f_1, f_2)\|_{L^p(\mathbb{R}^n,w)} \lesssim \|f_1\|_{L^\infty(\mathbb{R}^n)}\|f_2\|_{L^p(\mathbb{R}^n,M_{L(\log L)^{p-1+\delta}}w)}. \tag{3.12}$$

Lemma 3.3, along with Theorem 2.2 and the multilinear Marcinkiewicz interpolation theorem, now states that for $s_1 > n$, $s_2 > n$ and $p_1, p_2 \in (1,\infty)$, $p \in (1/2,\infty)$ with $1/p = 1/p_1 + 1/p_2$,

$$\|T_{\sigma,N}(f_1, f_2)\|_{L^p(\mathbb{R}^n,w)} \lesssim \|f_1\|_{L^{p_1}(\mathbb{R}^n Mw)}\|f_2\|_{L^p(\mathbb{R}^n,M_{L(\log L)^{p-1+\delta}}w)}. \tag{3.13}$$

By (3.11) and (3.12), another application of Lemma 3.5 then gives us the estimate (1.8). □

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REFERENCES

[1] T. A. BUI AND X. T. DUONG, *Weighted norm inequalities for multilinear operators and applications to multilinear Fourier multipliers*, Bull. Sci. Math. **137** (2013), 63–75.

[2] M. CARROZZA AND A. PASSARELLI DI NAPOLI, *Composition of maximal operators*, Publ. Mat. **40** (1996), 397–409.

[3] R. R. COIFMAN AND Y. MEYER, *On commutators of singular integrals and bilinear singular integrals*, Trans. Amer. Math. Soc. **212** (1975), 315–331.

[4] R. R. COIFMAN AND Y. MEYER, *Au delà des opérateurs pseudo-différentiels*, Astériaque **57** (1978), 1–185.

[5] M. FUJITA AND N. TOMITA, *Weighted norm inequalities for multilinear Fourier multipliers*, Trans Amer Math Soc. **364** (2012), 6335–6353.

[6] L. GRAFAKOS, *Classical Fourier analysis*, GTM249, 2nd Edition, Springer, New York, 2008.

[7] L. GRAFAKOS AND N. KALTON, *Multilinear Calderón-Zygmund operators on Hardy spaces*, Collect. Math. **52** (2001), 169–179.

[8] L. GRAFAKOS, A. MIYACHI AND N. TOMITA, *On multilinear Fourier multipliers of limited smoothness*, Canad. J. Math. **65** (2013), 299–330.

[9] L. GRAFAKOS AND Z. SI, *The Hörmander multiplier theorem for multilinear operators*, J. Reine Angewandte Math. **668** (2012), 133–147.

[10] L. GRAFAKOS AND R. TORRES, *Multilinear Calderón-Zygmund theory*, Adv. Math., **165** (2002), 124–164.

[11] G. HU. AND C. LIN, *Weighted norm inequalities for multilinear singular integral operators and applications*, Anal. Appl., to appear, arXiv:1208.6346.

[12] D. S. KURTZ AND R. L. WHEEDEN, *Results on weighted norm inequalities for multipliers*, Trans. Amer. Math. Soc. **255** (1979), 343–362.

[13] A. LERNER, *Weighted norm inequalities for the local sharp maximal function*, J. Fourier Anal. Appl. **10** (2004), 645–674.

[14] A. LERNER, S. OMBROSSI, C. PEREZ, R. H. TORRES AND R. TROJILLO-GONZALEZ, *New maximal functions and multiple weights for the multilinear Calderón-Zygmund theorey*, Adv. Math. **220** (2009), 1222–1264.

[15] A. MIYACHI AND N. TOMITA, *Minimal smoothness conditions for bilinear Fourier multiplier*, Rev Mat Iberomericana, to appear.

[16] C. PÉREZ, *Weighted norm inequalities for singular integral operators*, J. London Math. Soc. **49** (1994), 296–308.

[17] C. PÉREZ, *On sufficient conditions for the boundedness of the Hardy-Littlewood maximal operator between weighted L^p-spaces with different weights*, Proc. London Math. Soc. **49** (1995), 135–157.

[18] N. TOMITA, *A Hörmander type multiplier theorem for multilinear operator*, J. Func. Anal. **259** (2010), 2028–2044.

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