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# INEQUALITIES FOR THE BETA FUNCTION

LOÏC GRENIÉ AND GIUSEPPE MOLTENI

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*Abstract.* Let $g(x) := (e/x)^x\Gamma(x+1)$ and $F(x,y) := g(x)g(y)/g(x+y)$. Let $D^{(k)}_{x,y}$ be the $k$th differential in Taylor's expansion of $\log F(x,y)$. We prove that when $(x,y) \in \mathbb{R}^2_+$ one has $(-1)^{k-1}D^{(k)}_{x,y}(X,Y) > 0$ for every $X,Y \in \mathbb{R}_+$, and that when $k$ is even the conclusion holds for every $X,Y \in \mathbb{R}$ with $(X,Y) \neq (0,0)$. This implies that Taylor's polynomials for $\log F$ provide upper and lower bounds for $\log F$ according to the parity of their degree. The formula connecting the Beta function to the Gamma function shows that the bounds for $F$ are actually bounds for Beta.

## Notation

We will denote

| | |
|---|---|
| $\lfloor x\rfloor, \{x\}$ | the integral and fractional parts of $x$, respectively; |
| $B_k(x)$ | the $k$th Bernoulli polynomial; |
| $B_k := B_k(0)$ | the $k$th Bernoulli number; |
| $\theta$ | a number in $[-1,1]$ whose value may change in each occurrence; |
| $\zeta(s,x)$ | Hurwitz's zeta function: $\zeta(s,x) := \sum_{j=0}^{\infty}(j+x)^{-s}$ for $\mathrm{Re}(s) > 1$, $x > 0$. |

## 1. Motivations and main result

Functions $f\colon (0,+\infty) \to \mathbb{R}$ having derivatives of all orders and satisfying the inequalities

$$(-1)^k f^{(k)}(x) \geqslant 0 \qquad \forall x > 0, \quad k = 0,1,2,\dots$$

are called *completely*, or even *totally*, monotone. This notion is the analogue for functions on $(0,+\infty)$ of the *totally* monotone sequences introduced by Hausdorff [27,28,29] in connection to his solution of the moment problem in the case of a compact interval. It has quickly gained importance when Bernstein proved [15] that $f$ is completely monotone if and only if there exists a nonnegative Borel measure $\mathrm{d}\nu$ on $[0,+\infty)$ such that

$$f(x) = \int_{0^-}^{+\infty} e^{-xt}\,\mathrm{d}\nu(t),$$

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and that $f$ has a finite limit in $0$ if and only if $\nu(\mathbb{R}_+)$ is finite (see [41, p. 161] and [11, Ths. 6.13 and 6.14]). Several functions, defined in terms of Gamma and other special functions, are completely monotone [1,2,6,7,9,13,30,39]: this allows to derive many inequalities [10, 14, 17, 19, 23, 24, 25, 31, 34, 36] with applications in probability theory [16,21,33], in potential theory [12], and numerical and asymptotic analysis [23, 24, 25, 42].

A very general inequality was found by Kimberling [33, Th. 3] as an almost immediate consequence of Bernstein's representation. It states that

$$\frac{f(x+y)}{f(x)f(y)} \geqslant 1 \qquad \forall x,y>0 \tag{1}$$

whenever $f\colon (0,+\infty) \to (0,1]$ is completely monotone.

Let $\psi(x) := \Gamma'(x)/\Gamma(x)$ be the digamma function. Using the representation

$$h(x) := \frac{1}{x} - \log x + \psi(x) = \int_0^{+\infty} e^{-xt}\varphi(t)\,\mathrm{d}t$$

with $\varphi(t) := 1+t^{-1}-(1-e^{-t})^{-1}$ (see [8, Th. 1.6.3]) and Bernstein's criterion, one proves that $h(x)$ is completely monotone. Let

$$H(x) := x - x\log x + \log(\Gamma(x+1)),$$

a nonnegative primitive of $h(x)$ in $[0,+\infty)$. Then $\exp(-H(x))$ is completely monotone as well (see [21, p. 441]). As a consequence, from (1) one gets that

$$\frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+1)} \geqslant \frac{x^x y^y}{(x+y)^{x+y}} \qquad \forall x,y>0. \tag{2}$$

This is essentially a lower bound for Euler's Beta function

$$B(x,y) := \int_0^1 z^{x-1}(1-z)^{y-1}\,\mathrm{d}z \qquad \forall x,y>0,$$

since it can be written as

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \qquad \forall x,y>0, \tag{3}$$

(see [8, Ch. 1, Th. 1.1.4]), so that (2) means that

$$B(x,y) \geqslant \frac{x^{x-1}y^{y-1}}{(x+y)^{x+y-1}} \qquad \forall x,y>0. \tag{4}$$

In this paper we are interested mainly in explicit upper and lower bounds for $B(x,y)$, and the inequality in (4) is the typical result we would like to improve. In fact Stirling's formula, if necessary in one on its versions with explicit remainder term [35, 38], gives bounds for $B(x,y)$ which are well tuned for the case where one of the arguments diverges, but that are not optimal in the opposite scenario where the arguments are kept close to a fixed point. For example, in [40, p. 263 Ex. 45] it is reported a formula which implies immediately the bound

$$B(x,y) > \sqrt{2\pi}\,\frac{x^{x-1/2}\,y^{y-1/2}}{(x+y)^{x+y-1/2}} \qquad \forall x,y>0.$$

The right hand side equals what we get from the first two terms of Stirling's expansion of $\Gamma(x)$. Observe that it is poorer than (4) when $x^{-1}+y^{-1} \geqslant 2\pi$. Using a different method, Alzer proved a bound for the region $x,y \geqslant 1$ in [3].

Recently bounds in the square $(0,1]\times(0,1]$ have been proved by Alzer, among other results, in [4]. These bounds have been improved by Ivády [32]. Another set of bounds for Beta is proved in [5].

An interesting survey of the study of Beta and Gamma functions is provided in [20].

In essence, we prove a different set of approximations for $B(x,y)$ coming from a kind of complete monotonicity for the function $H(x+y)-H(x)-H(y)$. In order to formulate our result more easily it is convenient to introduce the functions

$$g(x) := \exp(H(x)) = (e/x)^x\,\Gamma(x+1)$$

and

$$F(x,y) := \frac{g(x)g(y)}{g(x+y)} = \frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+1)}\cdot\frac{(x+y)^{x+y}}{x^x\,y^y}.$$

In this way every bound for $F(x,y)$ is actually a bound for $B(x,y)$, taken apart the factor $(x+y)^{x+y-1}x^{1-x}y^{1-y}$ which we can consider as essentially elementary; for example, (2) simply says that $F(x,y) \geqslant F(0^+,0^+) = 1$.

Let Taylor's series for the logarithm of $F$ at the point $x,y>0$ be written as

$$\log F(x+X,y+Y) = H(x+X+y+Y)-H(x+X)-H(y+Y) = \sum_{k=0}^{+\infty}\frac{1}{k!}D^{(k)}_{x,y}(X,Y),$$

where each $D^{(k)}_{x,y}(X,Y)$ is the $k$th differential of $F$ in $X$ and $Y$. For positive $k$ it can be written in terms of $g$ as

$$D^{(k)}_{x,y}(X,Y) = -\Big(\frac{g'}{g}\Big)^{(k-1)}(x+y)(X+Y)^k + \Big(\frac{g'}{g}\Big)^{(k-1)}(x)X^k + \Big(\frac{g'}{g}\Big)^{(k-1)}(y)Y^k. \quad (5)$$

The representation

$$-\frac{g'}{g}(x) = \gamma + \sum_{j=1}^{+\infty}\Big(\frac{1}{j+x}-\frac{1}{j}\Big) + \log x$$

(see [8], p. 13) gives

$$-\Big(\frac{g'}{g}\Big)^{(k-1)}(x) = (-1)^k(k-1)!\Big(\frac{1}{(k-1)x^k} - \sum_{j=1}^{+\infty}\frac{1}{(j+x)^k}\Big)$$

when $k \geqslant 2$, therefore the coefficients of every differential can be explicitly computed in terms of Hurwitz's zeta function $\zeta(s,x) := \sum_{j=0}^{+\infty}(j+x)^{-s}$. Consider the following claim:

$(-1)^{k-1}D^{(k)}_{x,y}(X,Y)$ *is positive for* $X,Y>0$ *and if* $k$ *is even then it is positive definite in* $X,Y$. (∗)

Some numeric experiments suggest the validity of (∗) for every $k$, independently of the point $x,y$ where it is computed; our main result is the proof of this conjecture.

THEOREM. (∗) *holds for every* $k \geqslant 1$ *and every point* $x,y>0$.

The theorem produces lower and upper bounds for $F(x,y)$ and then for $B(x,y)$ simply by taking the truncated Taylor approximations of any even or odd order in (any) point. It is particularly useful when an explicit bound is needed in a neighborhood of a given point. For example, we can use it to bound $F(x,y)$ in terms of $F(\lfloor x\rfloor,\lfloor y\rfloor)$ and the fractional parts $\{x\}$, $\{y\}$. Here is a concrete example: for $x,y\in[1,2)$ the first three differentials give the bounds

$$F(x,y) \geqslant F(1,1)=2, \quad F(x,y) \leqslant 2\exp\big((\log 2-\tfrac{1}{2})(\{x\}+\{y\})\big),$$
$$F(x,y) \geqslant 2\exp\big((\log 2-\tfrac{1}{2})(\{x\}+\{y\})-\tfrac{1}{8}(\{x\}^2-2(7-4\zeta(2))\{x\}\{y\}+\{y\}^2)\big).$$

These bounds improve those coming from [3] in a wide subset of the square, but not everywhere.

The theorem implies that $-\log F(x,y)$ is completely monotone along each ray of the first quadrant. Hence for every base point $x,y>0$ and every $\vartheta\in[0,\pi/2]$ there is a nonnegative Borel measure $\nu_{x,y,\vartheta}$ such that

$$-\log F(x+\rho\cos\vartheta, y+\rho\sin\vartheta) = \int_0^{+\infty} e^{-\rho t}\,\mathrm{d}\nu_{x,y,\vartheta}(t);$$

such a formula could probably be deduced from Binet's identity (see [8, Th. 1.6.3]). However, $-\log F(x,y)$ is not completely monotone as a bivariate function (see [11, Ex. 6.27 p. 140]) since the sign of its mixed derivatives does not depend only on their total order ($\partial_x^2 \log F$ is negative while $\partial_x\partial_y \log F$ is positive). This is probably the main cause of our difficulties in the proof of the theorem.

Lastly, we remark that $F(x,y)$ is the case $d=2$ of the family of functions

$$F_d(x_1,x_2,\ldots,x_d) := \frac{\prod_{j=1}^d g(x_j)}{g(x_1+x_2+\cdots+x_d)}.$$

The result extends immediately to each $F_d$ via the recursive identity

$$F_d(x_1,x_2,\ldots,x_d) = F_{d-1}(x_1,x_2,\ldots,x_{d-1})F(x_1+\cdots+x_{d-1},x_d).$$

## 2. Proof

The proof is a patchwork of several techniques, each one proving ($*$) in a proper subset of the space of values for the parameters $x$, $y$ and $k$.

We denote by $G$ the derivative of $-g'/g$, thus

$$G(x) := \frac{1}{x} - \sum_{j=1}^{\infty} \frac{1}{(j+x)^2}. \tag{6}$$

The alternative representation

$$G(x) = \int_{0^+}^{+\infty} \frac{\mathrm{d}\{t\}}{(t+x)^2}$$

gives

$$(-1)^k G^{(k)}(x) = (k+1)! \int_{0^+}^{+\infty} \frac{\mathrm{d}\{t\}}{(t+x)^{k+2}} = (k+2)! \int_{0^+}^{+\infty} \frac{\{t\}\,\mathrm{d}t}{(t+x)^{k+3}} > 0 \tag{7}$$

for every $k$, proving that $G$ is completely monotone.

By (5) the first differential is

$$D^{(1)}_{x,y}(X,Y) = \Big(-\frac{g'}{g}(x+y) + \frac{g'}{g}(x)\Big)X + \Big(-\frac{g'}{g}(x+y) + \frac{g'}{g}(y)\Big)Y,$$

thus its positivity for $X,Y > 0$ and for every fixed $x,y \in \mathbb{R}^2_+$ can be obtained showing that $-\frac{g'}{g}$ is a strictly increasing function. This is true since its derivative is $G$, which is positive. This proves the case $k = 1$ of the theorem.

From now on we discuss only the other differentials, so that we assume $k \geqslant 2$. In terms of $G$, Formula (5) reads

$$(-1)^{k-1} D^{(k)}_{x,y}(X,Y) = a(X+Y)^k - bX^k - cY^k,$$

with

$$a := (-1)^{k-1} G^{(k-2)}(x+y), \qquad b := (-1)^{k-1} G^{(k-2)}(x), \qquad c := (-1)^{k-1} G^{(k-2)}(y).$$

Constants $a$, $b$, $c$ are negative by (7), and $a \geqslant \max\{b,c\}$ because $x,y$ are nonnegative and $(-1)^{k-1}G^{(k-2)}(x)$ is a strictly increasing function (because its derivative is $(-1)^{k-1}G^{(k-1)}(x)$, which is positive). In inhomogeneous coordinates, the positivity of the differential means that the function

$$\ell(\lambda) := a(1+\lambda)^k - b - c\lambda^k$$

is positive in $(0,+\infty)$, and the positive definiteness (for even $k$) that it is positive in $\mathbb{R}$. The solutions of $\ell'(\lambda) = 0$ satisfy $(1+1/\lambda)^{k-1} = c/a$, thus there are two roots (one negative and one positive) when $k$ is odd, and only one (positive) when $k$ is even. Moreover, $\ell$ is evidently positive at $0$ and $+\infty$ (and at $-\infty$ if $k$ is even), therefore the function is positive in $(0,+\infty)$ (and in $\mathbb{R}$ for an even $k$) if and only if its value at the positive stationary point is positive. After some algebra this condition becomes

$$(-a)^{-1/(k-1)} > (-b)^{-1/(k-1)} + (-c)^{-1/(k-1)}$$

so that in order to prove the claim for a $k \geqslant 2$ we need to prove that

$$((-1)^\kappa G^{(\kappa)}(x+y))^{-1/(\kappa+1)} > ((-1)^\kappa G^{(\kappa)}(x))^{-1/(\kappa+1)} + ((-1)^\kappa G^{(\kappa)}(y))^{-1/(\kappa+1)} \tag{8}$$

for $x,y > 0$, where $\kappa := k-2$. In other words, we need to prove that the function $((-1)^\kappa G^{(\kappa)}(x))^{-1/(\kappa+1)}$ is strictly super-additive in $(0,+\infty)$. This function admits a regular continuation in $x=0$ with value $0$, hence (8) can be showed by proving that it is a strictly convex function.

After some computations the condition $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\big[((-1)^\kappa G^{(\kappa)}(x))^{-1/(\kappa+1)}\big] > 0$ becomes

$$(\kappa+2)(G^{(\kappa+1)}(x))^2 > (\kappa+1)G^{(\kappa)}(x)G^{(\kappa+2)}(x) \quad x>0. \tag{9}$$

Hence, summarizing, we can prove (∗) for a given $k \geqslant 2$ by proving (9) for $\kappa = k-2$.

REMARK. In terms of $f_\kappa(x) := \frac{(-1)^\kappa}{(\kappa+1)!}G^{(\kappa)}(x) = \int_{0^+}^{+\infty} \frac{d\{t\}}{(t+x)^{\kappa+2}}$, Inequality (9) states that

$$(f_\kappa'(x))^2 > \frac{\kappa+1}{\kappa+2} f_\kappa(x)\, f_\kappa''(x) \qquad x>0. \tag{10}$$

However, we also have

$$(f_\kappa'(x))^2 \leqslant f_\kappa(x)\, f_\kappa''(x) \qquad x>0$$

because $f_\kappa$ is completely monotone (see [22, Th. 1] and [41, Ch. IV, Th. 16]). Thus, up to a constant, this inequality is the opposite of (10). Together, they show that for a very high index $\kappa$ the values of functions $(f_\kappa')^2$ and $f_\kappa f_\kappa''$ are very close.

LEMMA 1. (9) *holds for* $k \leqslant 25$, *i.e.* $\kappa \leqslant 23$.

*Proof.* By (6) we have $G(x) = \frac{1}{x} - \frac{1}{1+x} - \frac{1}{(1+x)^2} + G(1+x)$. When iterated $m$ times this produces the equality

$$G(x) = \frac{1}{x} - \frac{1}{m+x} - \sum_{j=1}^{m} \frac{1}{(j+x)^2} + G(m+x).$$

Deriving this equality $\kappa$ times we get

$$\frac{(-1)^\kappa G^{(\kappa)}(x)}{\kappa!} = \frac{1}{x^{\kappa+1}} - \frac{1}{(m+x)^{\kappa+1}} - \sum_{j=1}^{m} \frac{\kappa+1}{(j+x)^{\kappa+2}} + \frac{(-1)^\kappa G^{(\kappa)}(m+x)}{\kappa!}. \tag{11}$$

Equality (7) for $\kappa$ may be also written as

$$\frac{(-1)^\kappa G^{(\kappa)}(x)}{(\kappa+1)!} = (\kappa+2)\int_{0^+}^{+\infty} \frac{\{t\}\,\mathrm{d}t}{(t+x)^{\kappa+3}} = \frac{1}{2x^{\kappa+2}} + (\kappa+2)\int_{0^+}^{+\infty} \frac{B_1(\{t\})\,\mathrm{d}t}{(t+x)^{\kappa+3}}.$$

The last integral is the typical integral appearing in Euler-McLaurin's formula. Since the map $t \mapsto \frac{1}{(t+x)^{\kappa+3}}$ decreases, from the usual bounds for the remainder term in this formula we get

$$\frac{1}{2x^{\kappa+2}} - \frac{\kappa+2}{12x^{\kappa+3}} \leqslant \frac{(-1)^{\kappa} G^{(\kappa)}(x)}{(\kappa+1)!} \leqslant \frac{1}{2x^{\kappa+2}} \tag{12}$$

(see [18, Ch. VI, Ex. to Par. 3, p. 296] or [26, Eq. 9.80, p. 475]). We already know that $(-1)^{\kappa} G^{(\kappa)}(x)$ is positive, thus the lower bound is nontrivial only for $x \geqslant (\kappa+2)/6$.

Combining (11) and (12) with the shift $x \mapsto x+m$ we get for $m \geqslant (\kappa+3)/6$

$$\frac{(-1)^{\kappa+1} G^{(\kappa+1)}(x)}{(\kappa+1)!} \geqslant \frac{1}{x^{\kappa+2}} - \frac{1}{(m+x)^{\kappa+2}} - \sum_{j=1}^{m} \frac{\kappa+2}{(j+x)^{\kappa+3}} + \frac{\kappa+2}{2(m+x)^{\kappa+3}} - \frac{(\kappa+3)(\kappa+2)}{12(m+x)^{\kappa+4}} \geqslant 0$$

and

$$0 \leqslant \frac{(-1)^{\kappa} G^{(\kappa)}(x)}{(\kappa)!} \leqslant \frac{1}{x^{\kappa+1}} - \frac{1}{(m+x)^{\kappa+1}} - \sum_{j=1}^{m} \frac{\kappa+1}{(j+x)^{\kappa+2}} + \frac{\kappa+1}{2(m+x)^{\kappa+2}}.$$

Thus we can prove (9) by showing that there exists $m = m(\kappa) \geqslant (\kappa+3)/6$ such that

$$\begin{aligned}
&\left(\frac{1}{x^{\kappa+2}} - \frac{1}{(m+x)^{\kappa+2}} - \sum_{j=1}^{m} \frac{\kappa+2}{(j+x)^{\kappa+3}} + \frac{\kappa+2}{2(m+x)^{\kappa+3}} - \frac{(\kappa+3)(\kappa+2)}{12(m+x)^{\kappa+4}}\right)^2 \\
&\quad > \left(\frac{1}{x^{\kappa+1}} - \frac{1}{(m+x)^{\kappa+1}} - \sum_{j=1}^{m} \frac{\kappa+1}{(j+x)^{\kappa+2}} + \frac{\kappa+1}{2(m+x)^{\kappa+2}}\right) \\
&\quad \times \left(\frac{1}{x^{\kappa+3}} - \frac{1}{(m+x)^{\kappa+3}} - \sum_{j=1}^{m} \frac{\kappa+3}{(j+x)^{\kappa+4}} + \frac{\kappa+3}{2(m+x)^{\kappa+4}}\right) \qquad x > 0.
\end{aligned}$$

For $\kappa \leqslant 23$ we have found a value for $m = m(\kappa)$ for which the difference of the functions appearing on the left and right sides is actually a rational function (in $x$) with denominator $6\prod_{j=1}^{m}(j+x)^{\kappa+4}$ and whose numerator has only nonnegative integer coefficients: this obviously proves the claim in a strong form. These values are collected in Table 1.

Table 1: $m(\kappa)$ *for* $\kappa \leqslant 23$.

| $\kappa$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 10 | 13 | 15 | 18 | 21 |
| $\kappa$ | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
| $m$ | 24 | 28 | 32 | 35 | 40 | 44 | 49 | 54 | 59 | 65 | 70 | 76 |

□

The nonnegativity of the coefficients in the difference seems to be a 'stable' property in some sense, since if it holds for $m$, then it seems to hold also for $m+1$; the values appearing in the previous table are only the smallest ones ensuring that property. It is not clear how we can select for a given $\kappa$ a good candidate for $m$: the empirical data show that $m(\kappa)$ grows approximately as $m(\kappa) \approx \kappa^2$.

Moreover, in order to prove that the difference has positive coefficients we have not found a better argument than their explicit computation. This computation becomes quickly very complicated, due to the large size of the coefficients involved (for example, the greatest coefficient appearing in the numerator of the difference for $\kappa = 10$ is $\approx 10^{444}$ and for $\kappa = 23$ is $\approx 10^{5880}$). Table 1 has been computed with PARI/GP [37]. Lacking both a good comprehension of the dependence of $m(\kappa)$ on $\kappa$ and a good method to check the positivity of those coefficients, the validity of $(*)$ in the other regions is proved via a different argument.

We notice that (9) has the form of a log concavity for the map $\kappa \mapsto (-1)^\kappa G^{(\kappa)}(x)/\kappa!$ having $x$ as nonnegative parameter, and that if this map is log-concave in $[\kappa, \kappa+2]$ then (9) holds for $\kappa$. This remark can be made effective since, according to (7), that map has a natural continuation to $\mathbb{R}_+$ (with a shift in the argument to simplify the next computations):

$$\mathscr{G}(s) := s\int_{0^+}^{+\infty} \frac{\mathrm{d}\{t\}}{(t+x)^{s+1}}.$$

Thus, if we are able to prove that $\log\mathscr{G}$ is strictly concave, i.e. that

$$(\mathscr{G}'(s))^2 > \mathscr{G}(s)\mathscr{G}''(s) \tag{13}$$

in $[k-1, k+1]$, then (9) is proved for $\kappa = k-2$ and $(*)$ for $k$. Next lemmas will obtain $(*)$ by proving (13).

LEMMA 2. $(*)$ *holds for* $k \leqslant 1.8x - 1$ *and* $y > 0$.

*Proof.* By Lemma 1 we can assume that $k \geqslant 26$. We prove (13) for $25 \leqslant s \leqslant 1.8x$. The integral representation of $\mathscr{G}$ gives

$$\mathscr{G}'(s) = -\int_{0^+}^{+\infty} \frac{(s\log(t+x)-1)\,\mathrm{d}\{t\}}{(t+x)^{s+1}},$$

$$\mathscr{G}''(s) = \int_{0^+}^{+\infty} \frac{(s\log^2(t+x)-2\log(t+x))\,\mathrm{d}\{t\}}{(t+x)^{s+1}}.$$

Inequality (13) therefore may be written as

$$\Big(\int_{0^+}^{+\infty} \frac{s\log(t+x)-1}{(t+x)^{s+1}}\,\mathrm{d}\{t\}\Big)^2 > s\int_{0^+}^{+\infty} \frac{\mathrm{d}\{t\}}{(t+x)^{s+1}} \int_{0^+}^{+\infty} \frac{s\log^2(t+x)-2\log(t+x)}{(t+x)^{s+1}}\,\mathrm{d}\{t\}.$$

Since $x > 0$ we can normalize the measure $t \mapsto (t+x)^{-s-1}\mathrm{d}\{t\}$ by introducing $\mathrm{d}\mu_{s,x}(t) := c^{-1}(t+x)^{-s-1}\mathrm{d}\{t\}$ with $c := \int_{0^+}^{+\infty}(t+x)^{-s-1}\mathrm{d}\{t\}$. In terms of $\mu_{s,x}$ Inequality (13) is written as

$$\Big(\int_{0^+}^{+\infty} (s\log(t+x)-1)\,\mathrm{d}\mu_{s,x}(t)\Big)^2 > \int_{0^+}^{+\infty} ((s\log(t+x)-1)^2-1)\,\mathrm{d}\mu_{s,x}(t),$$

or, using a probabilistic language, as

$$\mathrm{Var}[s\log(t+x)-1]_{\mu_{s,x}} < 1, \tag{14}$$

where the variance is computed with respect to the measure $\mu_{s,x}$. It is now evident that (14) is true if and only if the analog where we substitute $-1$ with any function of $s$ and $x$ holds. We prove now that $\mathrm{Var}[s\log(t+x)]_{\mu_{s,x}} < 1$ for $25 \leqslant s \leqslant 1.8x$. We need some computations. From (12) and the definitions of $c$ and $\mathscr{G}$ we get

$$c = \frac{1}{s}\mathscr{G}(s) \geqslant \frac{1}{2x^{s+1}} - \frac{s+1}{12x^{s+2}},$$

i.e.

$$2cx^{s+1} \geqslant 1 - \frac{s+1}{6x}. \tag{15}$$

Let $\mathfrak{m}$ be the mean value of $s\log(t+x)$ with respect to $\mu_{s,x}$. Then

$$c(s\log x - \mathfrak{m}) = -s\int_{0^+}^{+\infty} \frac{\log(1+t/x)}{(t+x)^{s+1}}\,\mathrm{d}B_1(\{t\});$$

integrating by parts one time it becomes

$$= s\int_{0^+}^{+\infty} B_1(\{t\})\Big[\frac{\log(1+t/x)}{(t+x)^{s+1}}\Big]'\,\mathrm{d}t$$

and two further integrations by parts give

$$= \frac{-s}{12x^{s+2}} + \frac{s}{6}\int_{0^+}^{+\infty} B_3(\{t\})\Big[\frac{\log(1+t/x)}{(t+x)^{s+1}}\Big]'''\,\mathrm{d}t.$$

Let $M_3 := \max_{[0,1]} B_3(t) = \sqrt{3}/36$, then

$$= \frac{-s}{12x^{s+2}} + \theta\,\frac{s}{6}M_3\int_{0^+}^{+\infty} \Big|\Big[\frac{\log(1+t/x)}{(t+x)^{s+1}}\Big]'''\Big|\,\mathrm{d}t$$

$$= \frac{-s}{12x^{s+2}} + \theta\,\frac{s}{6}M_3\int_{0^+}^{+\infty} \Big[\frac{3s^2+12s+11}{(t+x)^{s+4}} + (s+1)(s+2)(s+3)\frac{\log(1+t/x)}{(t+x)^{s+4}}\Big]\,\mathrm{d}t.$$

The integral may be explicitly evaluated, giving

$$= \frac{-s}{12x^{s+2}} + \theta\,\frac{M_3}{6}\,\frac{s}{s+3}\,\frac{4s^2+15s+13}{x^{s+3}}.$$

Let $\Delta_t := s\log(t+x) - \mathfrak{m}$, then the previous computation shows that

$$\Delta_0 = \frac{1}{2cx^{s+1}}\Big(\frac{-s}{6x} + \theta\,\frac{M_3}{3}\,\frac{s}{s+3}\,\frac{4s^2+15s+13}{x^2}\Big). \tag{16}$$

We observe that $\Delta_0$ is negative when $25 \leqslant s \leqslant 1.8x$. In the same range we also have $2s + (6x - s - 1)\Delta_0 \geqslant 0$. This can be seen using the upper bound for $|\Delta_0|$ coming from (16), the lower bound for $c$ coming from (15) and elementary arguments.

We also have

$$c\,\mathrm{Var}[s\log(t+x)-1]_{\mu_{s,x}} = \int_{0^+}^{+\infty} \frac{\Delta_t^2}{(t+x)^{s+1}}\,\mathrm{d}B_1(\{t\})$$

that after three integrations by parts becomes

$$= \frac{\Delta_0^2}{2x^{s+1}} + \frac{2s\Delta_0 - (s+1)\Delta_0^2}{12x^{s+2}} + \frac{1}{6}\int_{0^+}^{+\infty} B_3(\{t\})\Big[\frac{-\Delta_t^2}{(t+x)^{s+1}}\Big]'''\,\mathrm{d}t$$

which is, as noticed above,

$$\leqslant \frac{1}{6}\int_{0^+}^{+\infty} B_3(\{t\})\Big[\frac{-\Delta_t^2}{(t+x)^{s+1}}\Big]'''\,\mathrm{d}t.$$

One has

$$\Big[\frac{-\Delta_t^2}{(t+x)^{s+1}}\Big]''' = \frac{A - B\Delta_t + C\Delta_t^2}{(t+x)^{s+4}}$$

with

$$A := 6s^2(s+2), \qquad B := 2s(3s^2+12s+11), \qquad C := (s+1)(s+2)(s+3).$$

Since $\Delta_t = \Delta_0 + s\log(1+t/x)$, we also have

$$\Big[\frac{-\Delta_t^2}{(t+x)^{s+1}}\Big]''' = \frac{A - B\Delta_0 + C\Delta_0^2}{(t+x)^{s+4}} + \frac{(-B+2C\Delta_0)s\log(1+t/x) + Cs^2\log^2(1+t/x)}{(t+x)^{s+4}}.$$

Moreover,

$$0 \leqslant \int_{0^+}^{+\infty} \frac{B_3(\{t\})}{(t+x)^{s+4}}\,\mathrm{d}t \leqslant \frac{1}{120x^{s+4}}.$$

In addition

$$\int_{0^+}^{+\infty} B_3(\{t\})\frac{s\log(1+t/x)}{(t+x)^{s+4}}\,\mathrm{d}t = \theta M_3 \int_{0^+}^{+\infty} \frac{s\log(1+t/x)}{(t+x)^{s+4}}\,\mathrm{d}t = \theta\frac{M_3}{x^{s+3}}\frac{s}{(s+3)^2}$$

and

$$\int_{0^+}^{+\infty} B_3(\{t\})\frac{s^2\log^2(1+t/x)}{(t+x)^{s+4}}\,\mathrm{d}t = \theta\frac{M_3}{x^{s+3}}\frac{2s^2}{(s+3)^3}.$$

Hence, using the lower bound for $c$ given by (15), we get

$$\begin{aligned}\mathrm{Var}&[s\log(t+x)-1]_{\mu_{s,x}}\\ &\leqslant \frac{6s^2(s+2) + 2s(3s^2+12s+11)|\Delta_0| + (s+1)(s+2)(s+3)\Delta_0^2}{60x^2(6x-s-1)}\\ &\qquad + 4M_3\frac{s^2(4s^2+15s+13) + s(s+1)(s+2)(s+3)|\Delta_0|}{(s+3)^2x(6x-s-1)},\end{aligned}$$

where $\Delta_0$ is given in (16) and thus bounded by

$$|\Delta_0| \leqslant \frac{6x}{6x-s-1}\Big(\frac{s}{6x}+\frac{M_3}{3}\frac{s}{s+3}\frac{4s^2+15s+13}{x^2}\Big).$$

In these formulas the functions appearing on the right hand sides depend essentially on $s$ and $x$ via the quotient $s/x$ and therefore (14) will be true for $s/x$ small enough. We determine how small $s/x$ has to be as follows. The upper bound for the variance increases in $s$ and in $|\Delta_0|$; the upper bound for $|\Delta_0|$ also increases in $s$, thus substituting the bound for $|\Delta_0|$ into the upper bound for the variance we get a new function increasing again in $s$, whose upper bound is therefore reached when $s = 1.8x$. The resulting function is a rational function decreasing in $x$. Its value for $x = 25/1.8$ is smaller than $1$, thus it is so for all $25 \leqslant s \leqslant 1.8x$. □

LEMMA 3. (∗) *holds in* $x \in (0,1]$, *for every* $k \geqslant 1$ *and* $y > 0$.

*Proof.* By Lemma 1 we can assume that $k \geqslant 26$ and $s \geqslant 25$. The decomposition of the measure $\mathrm{d}\{t\}$ as $\mathrm{d}t - \mathrm{d}\lfloor t \rfloor$ produces the alternative representation $\mathscr{G}(s) = x^{-s} - s\zeta(s+1,x+1)$, so that (13) becomes

$$(-x^{-s}\log x - \zeta - s\zeta')^2 > (x^{-s} - s\zeta)(x^{-s}\log^2 x - 2\zeta' - s\zeta'')$$

which we write as

$$(\zeta + s\zeta')^2 + 2x^{-s}\log x(\zeta + s\zeta') + sx^{-s}\zeta\log^2 x + (x^{-s} - s\zeta)(2\zeta' + s\zeta'') > 0, \qquad (17)$$

and where we have used the notation $\zeta$ for $\zeta(s+1,x+1)$, and $\zeta'$ and $\zeta''$ for $\partial_s\zeta(s+1,x+1)$ and $\partial_s^2\zeta(s+1,x+1)$. The term $x^{-s} - s\zeta$ is positive for every $x > 0$ and for every $s$. Moreover,

$$\zeta + s\zeta' = \frac{1 - s\log(1+x)}{(1+x)^{s+1}} + \sum_{j=2}^{+\infty}\frac{1 - s\log(j+x)}{(j+x)^{s+1}}$$

is negative for every $x \geqslant e^{1/s} - 1$ and

$$2\zeta' + s\zeta'' = \frac{s\log^2(1+x) - 2\log(1+x)}{(1+x)^{s+1}} + \sum_{j=2}^{+\infty}\frac{s\log^2(j+x) - 2\log(j+x)}{(j+x)^{s+1}} \qquad (18)$$

is positive for $x \geqslant e^{2/s} - 1$. This fact and the representation (17) already suffice to prove the claim for $x \in [e^{2/s} - 1, 1]$.

The assumption $s \geqslant 25$ ensures that $e^{2/s} - 1 < 1/4$, therefore in order to complete the proof it is sufficient to prove it in $(0, 1/4]$. For $s \geqslant 25$ each term of the series in (18) is positive, hence (17) may be also written as

$$\frac{2\log x}{x^s}(\zeta + s\zeta') + \frac{s\zeta\log^2 x}{x^s} + (x^{-s} - s\zeta)\frac{(s\log(1+x) - 2)\log(1+x)}{(1+x)^{s+1}} + \mathrm{Pos.} > 0$$

therefore it is sufficient to prove that

$$2\log x(\zeta+s\zeta')+s\zeta\log^2 x+(1-sx^s\zeta)\frac{(s\log(1+x)-2)\log(1+x)}{(1+x)^{s+1}}>0.$$

Under the assumption $x\leqslant 1/4$ it is sufficient to prove that

$$2\zeta\log x+s\zeta\log^2 x-2(1-sx^s\zeta)\frac{\log(1+x)}{(1+x)^{s+1}}>0$$

(because $1-sx^s\zeta>0$ for every $x$ and $\zeta'\log x>0$ for $x<1$), or even the stronger

$$2\log x+s\log^2 x-\frac{2\log 2}{\zeta}>0.$$

Since $\zeta>1$, it is sufficient to prove that

$$2\log x+s\log^2 x-2\log 2>0$$

and this is true for $s\geqslant 25$, $x\leqslant 1/4$. □

LEMMA 4. *(*) holds for $k\geqslant 1.6x+2$, $x>1$ and $y>0$.*

*Proof.* By Lemma 1 we can assume that $k\geqslant 26$. We prove the claim by showing (13) for $s\geqslant\max\{25,1.6x+1\}$. Substituting

$$s\zeta=\sum_{j=1}^{\infty}\frac{s}{(j+x)^{s+1}},\qquad -\zeta-s\zeta'=\sum_{j=1}^{\infty}\frac{s\log(j+x)-1}{(j+x)^{s+1}},$$
$$2\zeta'+s\zeta''=\sum_{j=1}^{\infty}\frac{s\log^2(j+x)-2\log(j+x)}{(j+x)^{s+1}}$$

in (17), after some algebra the inequality becomes:

$$\Big(\sum_{j=1}^{\infty}\frac{s\log(j+x)}{(j+x)^{s+1}}\Big)^2+\Big(\sum_{j=1}^{\infty}\frac{1}{(j+x)^{s+1}}\Big)^2+\frac{s}{x^s}\sum_{j=1}^{\infty}\frac{(\log(j+x)-\log x)^2}{(j+x)^{s+1}}$$
$$-\frac{2}{x^s}\sum_{j=1}^{\infty}\frac{\log(j+x)-\log x}{(j+x)^{s+1}}-\sum_{j=1}^{\infty}\frac{s}{(j+x)^{s+1}}\sum_{j=1}^{\infty}\frac{s\log^2(j+x)}{(j+x)^{s+1}}>0,$$

which is equivalent to

$$\frac{s}{x^s}\sum_{j=1}^{\infty}\frac{\log^2(1+j/x)}{(j+x)^{s+1}}-\frac{2}{x^s}\sum_{j=1}^{\infty}\frac{\log(1+j/x)}{(j+x)^{s+1}}+\Big(\sum_{j=1}^{\infty}\frac{1}{(j+x)^{s+1}}\Big)^2$$
$$-s^2\sum_{1\leqslant j<k}\frac{\log^2((k+x)/(j+x))}{(j+x)^{s+1}(k+x)^{s+1}}>0. \quad (19)$$

We have

$$\sum_{k=j+1}^{\infty} \frac{\log^2((k+x)/(j+x))}{(k+x)^{s+1}} \leqslant \frac{4e^{-2}}{s^2(j+x)^{s+1}} + \frac{2}{s^3(j+x)^s}. \tag{20}$$

In fact, the function $z^{-s-1}\log^2 z$ has a unique maximum at $z_0 = e^{2/(s+1)}$, therefore

$$\begin{aligned}\sum_{k=j+1}^{\infty} \frac{\log^2((k+x)/(j+x))}{(k+x)^{s+1}} &\leqslant \int_j^{\infty} \frac{\log^2((k+x)/(j+x))}{(k+x)^{s+1}}\,\mathrm{d}k \\ &\quad + \left(\frac{2}{s+1}\right)^2 \frac{1}{(e^{2/(s+1)})^{s+1}} \frac{1}{(j+x)^{s+1}} \\ &= \frac{4e^{-2}}{(s+1)^2} \frac{1}{(j+x)^{s+1}} + \frac{1}{(j+x)^s} \int_1^{\infty} \frac{\log^2 z}{z^{s+1}}\,\mathrm{d}z \\ &= \frac{4e^{-2}}{(s+1)^2} \frac{1}{(j+x)^{s+1}} + \frac{2}{s^3(j+x)^s}.\end{aligned}$$

And since the map $j \mapsto \frac{1}{(j+x)^{s+1}}$ is completely monotone, thus convex, we further have

$$\sum_{j=1}^{\infty} \frac{1}{(j+x)^{s+1}} \geqslant \frac{1}{s(1+x)^s} + \frac{1/2}{(1+x)^{s+1}}. \tag{21}$$

As a consequence, by (20) and (21), the function appearing on the left-hand side in (19) is greater than

$$S := \sum_{j=1}^{\infty} \frac{r(j,x,s)}{(jx+x^2)^{s+1}}$$

with

$$\begin{aligned} r(j,x,s) := \frac{s}{x}\left(x\log\left(1+\frac{j}{x}\right) - \frac{x}{s}\right)^2 - \frac{x}{s} \\ - \frac{x^s}{(j+x)^s}\left(\frac{2x}{s} + 4e^{-2}\frac{x}{j+x}\right) + \frac{x^s}{(1+x)^s}\left(\frac{x}{s} + \frac{x/2}{1+x}\right), \end{aligned}$$

and we obtain (19) showing that $S > 0$. We simplify this expression using $\frac{1}{(1+x)^s} \geqslant \frac{1}{(j+x)^s}$ and $4e^{-2}\frac{x}{j+x} - \frac{x/2}{1+x} \leqslant 4e^{-2} - \frac{1}{2} \leqslant \frac{1}{20}$, thus

$$r(j,x,s) \geqslant \tilde{r}(j,x,s)$$

with

$$\tilde{r}(j,x,s) := sx\log^2\left(1+\frac{j}{x}\right) - 2x\log\left(1+\frac{j}{x}\right) - \frac{1}{(1+j/x)^s}\left(\frac{x}{s} + \frac{1}{20}\right).$$

The derivatives $\partial_s\tilde{r}$ and $\partial_j\tilde{r}$ are

$$\partial_s\tilde{r}(j,x,s) = x\log^2\left(1+\frac{j}{x}\right) + \frac{1}{(1+j/x)^s}\left(\log\left(1+\frac{j}{x}\right)\left(\frac{x}{s}+\frac{1}{20}\right) + \frac{x}{s^2}\right)$$

$$\partial_j\tilde{r}(j,x,s) = \frac{2x}{j+x}\Big(s\log\Big(1+\frac{j}{x}\Big)-1\Big)+\frac{s/x}{(1+j/x)^{s+1}}\Big(\frac{x}{s}+\frac{1}{20}\Big)$$

and hence are positive for $s \geqslant 1.6x+1$, $x \geqslant 1$, $j \geqslant 1$ (for $\partial_j$, notice that $s\log(1+j/x) \geqslant (1.6x+1)\log(1+1/x) \geqslant 1.6$ for every $x \geqslant 1$). As a consequence, for $x \geqslant 1$, $s \geqslant 1.6x+1$ and $j \geqslant 2$ one has $\tilde{r}(j,x,s) > 0$ since $\tilde{r}(2,x,1.6x+1) > 0$ here, as one can check easily.

Unfortunately $\tilde{r}(1,x,s)$ is negative for every $x \geqslant 1$ when $s = 1.6x+1$, thus the positivity of $S$ cannot be proved in this simple way. However, $\tilde{r}(1,x,2.1x+1)$ is positive for every $x \geqslant 1$. This suffices to prove that $S$ is positive for $s \geqslant 2.1x+1$, thus it remains to prove that $S$ is positive also in the intermediate range $1.6x+1 \leqslant s \leqslant 2.1x+1$. Moreover, $\tilde{r}(1,x,\max(25,1.6x+1)) > 0$ for $x \leqslant 10$, thus we can further take $x \geqslant 10$.

Lastly, the positivity of each $\tilde{r}(j,\cdot,\cdot)$ in the given range for $j \geqslant 2$ allows us to get the claim showing that a truncated sum is already positive. In fact, we prove now that

$$x^{s+1}S > \sum_{j=1}^{4}\frac{\tilde{r}(j,x,s)}{(j+x)^{s+1}} \geqslant 0$$

for $x \geqslant 10$ and $1.6x+1 \leqslant s \leqslant 2.1x+1$.

Let $U_j := s\log(1+j/x)$. Then,

$$\frac{s}{x}\tilde{r}(j,x,s) = U_j^2 - 2U_j - \Big(1+\frac{s}{20x}\Big)e^{-U_j}.$$

Since we are assuming $s \leqslant 2.1x+1$ and $x \geqslant 10$, it is

$$\geqslant U_j^2 - 2U_j - \frac{10}{9}e^{-U_j},$$

so that it is enough to prove that

$$sx^{s-1}\sum_{j=1}^{4}\frac{\tilde{r}(j,x,s)}{(j+x)^{s+1}} \geqslant \sum_{j=1}^{4}\frac{(U_j^2-2U_j)e^{-U_j}-\frac{10}{9}e^{-2U_j}}{j+x} \tag{22}$$

is positive.

We compute lower bounds $m_j$ for the numerators in (22), so that

$$(22) \geqslant \sum_{j=1}^{4}\frac{m_j}{j+x}, \tag{23}$$

and in order to reduce to an algebraic problem the check of its positivity we look for absolute constants. We compute the range of $U_j$ for $j \leqslant 4$: this is easy because $U_j$, as a function of $x$ and $s$, does not have stationary points, so that its extremal values are along the border of the regions. We then compute the lower bounds $m_j$ for $(u^2-2u)e^{-u}-\frac{10}{9}e^{-2u}$ with $u$ in each range. Ranges and lower bounds are in Table 2.

Table 2: $m_j$ *for* $j \leqslant 4$.

| range of $U_1$ | range of $U_j$, $j=2,3,4$ | $m_1$ | $m_2$ | $m_3$ | $m_4$ |
|---|---|---|---|---|---|
| $(1.6,2.1)$ | $[17\log(1+j/10),2.1j)$ | -0.175 | 0.138 | 0.049 | 0.012 |

Elementary arguments prove that (23) is positive for $x \geqslant 10$. □

Now we can finally prove the theorem. Lemma 3 proves (∗) for every $k \geqslant 1$ when $x \in (0,1]$. Assuming $x > 1$, Lemma 2 proves (∗) when $1 \leqslant k \leqslant 1.8x-1$ and Lemma 4 when $k \geqslant 1.6x+2$. Thus, all values for $k$ are covered by Lemmas 2 and 4 when $x \geqslant 15$. For $1 \leqslant x \leqslant 15$ all $k$ are covered by Lemma 1 and Lemma 4.

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*Loïc Grenié*
*Dipartimento di Ingegneria Gestionale*
*dell'Informazione e della Produzione*
*Università di Bergamo*
*viale Marconi 5, 24044 Dalmine (BG), Italy*
*e-mail:* `loic.grenie@gmail.com`

*Giuseppe Molteni*
*Dipartimento di Matematica*
*Università di Milano*
*via Saldini 50, 20133 Milano, Italy*
*e-mail:* `giuseppe.molteni1@unimi.it`