Mathematical
Inequalities
& Applications
Volume 18, Number 4 (2015), 1443–1461 doi:10.7153/mia-18-112

WEIGHTED APPROXIMATION BY BASKAKOV OPERATORS

IVAN GADJEV

(*Communicated by I. Perić*)

Abstract. The weighted approximation errors of Baskakov operator is characterized for weights of the form $w(x) = x^{\gamma_0}(1+x)^{\gamma_\infty}$, where $\gamma_0 \in [-1,0]$, $\gamma_\infty \in \mathbb{R}$. Direct inequalities and strong converse inequalities of type A are proved in terms of the weighted K-functional.

1. Introduction

The weghted approximation by linear positive operators has been a widely discussed topic. In [5] Z. Ditzian proved for the Bernstein operator [19]

$$B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k}, \quad x \in [0,1],$$

the estimate

$$|B_n(f,x) - f(x)| \leqslant w^2_{\varphi^\lambda}\left(f, n^{-1/2}\varphi(x)^{1-\lambda}\right)$$

where $\lambda \in [0,1]$, $\varphi^2 = x(1-x)$ and

$$\omega^2_\phi(f,\delta) = \sup_{|h|\leqslant\delta} \sup_{x+h\phi(x)\in[0,1]} |f(x-\phi(x)h) - 2f(x) + f(x+\phi(x)h)|$$

is the second order modulus of smoothness of Ditzian-Totik [8]. This unifies the classical local ($\lambda = 0$) and the global (norm) ($\lambda = 1$) estimations for the Bernstein operator. The converse result is also true (see [6], [23]), which means the equivalence

$$|B_n(f,x) - f(x)| = O\left(\left(n^{-1/2}\varphi(x)^{1-\lambda}\right)^\alpha\right) \Leftrightarrow \omega^2_{\varphi^\lambda}(f,\delta) = O(\delta^\alpha)$$

holds for all $\alpha \in (0,2)$ and $\lambda \in [0,1]$.

Later, Felten [10], [11] extended this result for the more general weights (including non-simetrical weights) by replacing $\varphi(x)^{1-\lambda}$ with $\frac{\varphi(x)}{\phi(x)}$, where $\phi(x)$ is an admissible step-weight function and for the operators of exponential type, in particular for Szász-Mirakjan operator. In all of the results, mentioned above, the inverse approximation

Mathematics subject classification (2010): 41A36, 41A25, 41A27, 41A17.
Keywords and phrases: Baskakov operator, K-functional, direct theorem, strong converse theorem.

statement is formulated as the equivalence of the rates of convergence. A different type of converse inequality is proved by E. Van Wickeren in [21].

In [7] Ditzian and Ivanov suggested a classification and defined four types of strong converse inequalities. The strongest are the inequalities of type A. In unweighted case, they proved in the same paper a strong converse inequality of type B for the Bernstein operator. In [20] Totik proved the inequalities of type A for the Bernstein, Szász-Mirakjan and Baskakov operators. Another proofs are given: for the Bernstein operator by Knoop and Zhou in [18] and for the Baskakov operator by Gadjev in [14].

Regarding weighted approximation, we reference: by Bernstein operator – [4], [6], [10], [17], [23], and by Szász-Mirakjan – [1], [3], [9], [13], [16].

In this paper we characterize the weighted approximation by Baskakov operators. The weights under consideration are defined as

$$w(x) = x^{\gamma_0}(1+x)^{\gamma_\infty}, \quad \text{where} \quad \gamma_0 \in [-1,0], \quad \gamma_\infty \in \mathbb{R}. \tag{1}$$

For functions $f \in C[0;\infty)$ the Baskakov operator is given by (see [2])

$$V_n f(x) = (V_n f, x) = V_n(f,x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) V_{n,k}(x) \quad \text{for} \quad 0 \leqslant x < \infty, \tag{2}$$

where

$$V_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}. \tag{3}$$

In [3] Becker studied the global weighted (for weights $w(x) = 1 + x^N$, $n \in \mathbb{N}$) approximation by Baskakov operators and proved the direct inequality. In [16] Holhoş improved the result for weights (1), $\gamma_0 = 0$, $\gamma_\infty \leqslant 0$. Guo and Qi gave in [15] a strong converse inequality of type B for simmetrical weights $\gamma_0 = \gamma_\infty \in (0,1]$. In [13] Finta generalized the method of [15], which allows him to extend the result to some non-simmetrical weights.

In this paper we prove direct and strong converse inequalities of type A for the widdest reasonable class of weights (1), where $\gamma_0 \in [-1,0]$, $\gamma_\infty \in \mathbb{R}$. All previously known results are, at best, for weaker inequalities of type B. In the process of proving them, we establish two inequalities, which are important of their own: Voronovskaya-type inequality (Theorem 2) and Bernstein-type inequality (Theorem 3), which takes into account the growth of the constant with respect to the number of iterates of the Baskakov operator.

Before stating our main result, let us introduce some notations. We denote the first derivative operator by $D = \frac{d}{dx}$. Thus, $Dg(x) = g'(x)$ and $D^2 g(x) = g''(x)$. By $\psi(x) = x(1+x)$ we denote the weight which is naturally connected with the second derivative of Baskakov operator (2) (see Lemmas 6 and 2 below).

As usual, $C[0,\infty)$ denotes the space of all continuous on $[0,\infty)$ functions (without the requirement for boundedness at ∞) and $L_\infty[0,\infty)$ is the space of all Lebesgue measurable and essentially bounded in $[0,\infty)$ functions equipped with the uniform norm $\|.\|$.

For the weight function w, defined by (1) we set

$$C(w)=\{g\in C[0,\infty);\quad wg\in L_\infty[0,\infty)\},$$
$$W^2(w\psi)=\left\{g,Dg\in AC_{loc}(0,\infty)\quad\text{and}\quad w\psi D^2g\in L_\infty[0,\infty)\right\},$$
$$W^3(w\psi^{3/2})=\left\{g,Dg,D^2g\in AC_{loc}(0,\infty)\quad\text{and}\quad w\psi^{3/2}D^3g\in L_\infty[0,\infty)\right\}.$$

The weighted approximation error of V_n will be compared with the K-functional between the weighted spaces $C(w)$ and $W^2(w\psi)$, which is defined by

$$K_w(f,t)=\inf\left\{\|w(f-g)\|+t\left\|w\psi D^2g\right\|:g\in W^2(w\psi),f-g\in C(w)\right\}$$

for every function $f\in C(w)+W^2(w\psi)$ and every $t>0$.

Our main result is the following theorem, establishing a full equivalence between the K-functional $K_w\left(f,\frac{1}{n}\right)$ and $\|w(V_nf-f)\|$. It consists of a direct inequality (the first inequality in (4)) and a strong converse inequality of type A in the terminology in [7] (the second inequality in (4)).

THEOREM 1. *For w defined by* (1) *there exist positive constants C_1, C_2 and L such that for every natural $n\geqslant L$ and for all $f\in C(w)+W^2(w\psi)$ there holds*

$$C_1\|w(V_nf-f)\|\leqslant K_w\left(f,\frac{1}{n}\right)\leqslant C_2\|w(V_nf-f)\|. \tag{4}$$

Several comments about the range of γ_0 follow. The range of γ_0 cannot be reasonably extended, because for $\gamma_0<-1$ we must assume that $f(x)=0$ in a neighbourhood of 0, otherwise V_nf will not be bounded.

On the other hand, if $\gamma_0>0$, then $f(x)$ is not generally defined at $x=0$ and hence V_nf is not defined. Even if we restrict f in such a way that $wf\in C[0,\infty)$, we cannot settle this case because then V_n would not be a bounded operator in the weighted uniform norm (as implied by an equivalence like (4)).

We remark that for smaller classes of functions equivalence theorems in modified norms are known for different ranges of γ_0. See e.g. [12], [22].

Although Theorem 1 is formulated and proved for integer n it also holds true if n is assumed to be a continuous positive parameter. In this case

$$V_{n,k}(x)=\frac{\Gamma(n+k)}{k!\Gamma(n)}x^k(1+x)^{-n-k},$$

where Γ stands for the Gamma function, V_n is defined again by (2).

The paper is organized as follows. Some auxiliary results are proved in section 2. The main result is proved in section 3.

For the rest of this paper the constant C will always be an absolute constant, which means it does not depend on f and n although it may depend on γ_0 and γ_∞. It may be different on each occurrence. And $w^{-1}(x)$ will always denote $(w(x))^{-1}=\frac{1}{w(x)}$.

2. Auxiliary results

We will mention some properties of Baskakov operator, which can be found in [2]

$$V_n \quad \text{is a linear, positive operator with} \quad \|V_n f\| \leqslant \|f\| \tag{5}$$

$$V_n(1,x)=1, \qquad V_n(t-x,x)=0, \qquad V_n\left((t-x)^2,x\right)=\frac{\psi(x)}{n}. \tag{6}$$

For $k \geqslant 0$ we have the next easily verified identities [8]

$$(DV_{n,k})(x)=DV_{n,k}(x)=n\Big(V_{n+1,k-1}(x)-V_{n+1,k}(x)\Big), \tag{7}$$

$$DV_{n,k}(x)=\frac{n}{\psi(x)}\left(\frac{k}{n}-x\right)V_{n,k}(x) \tag{8}$$

where $V_{n,-1}(x)=0$.

Differentiating (6) we have

$$\sum_{k=0}^{\infty}\frac{k}{n}\left(\frac{k}{n}-x\right)^2 V_{n,k}(x)=\frac{(1+2x)\psi(x)}{n^2}+\frac{x\psi(x)}{n}. \tag{9}$$

The next three inequalities are valid for all integers m and can be found in [8, (9.6.4, page 142), (9.6.3, page 141), (9.4.14, page 128)]. The constant C depends only on m.

$$\sum_{k=1}^{\infty}\left(\frac{n}{k}\right)^m V_{n,k}(x)\leqslant Cx^{-m}, \tag{10}$$

$$\sum_{k=0}^{\infty}\left(1+\frac{k}{n}\right)^m V_{n,k}(x)\leqslant C(1+x)^m, \tag{11}$$

$$V_n\left((t-x)^{2m},x\right)\leqslant C\left(\frac{\psi(x)}{n}\right)^m \quad \text{for} \quad x\geqslant\frac{1}{n}. \tag{12}$$

LEMMA 1. *For $\beta\in\mathbb{R}$ there exists a constant C such that for every natural $n\geqslant|\beta|$ and every $x\in[0,\infty)$ and $V_{n,k}$ defined by (3)*

$$\sum_{k=0}^{\infty}\left(1+\frac{k}{n}\right)^{\beta}V_{n,k}(x)\leqslant\left(1+\frac{C}{n}\right)(1+x)^{\beta}. \tag{13}$$

Proof. Applying Hölder's inequality for the smallest $m\in\mathbb{N}$ and $m\geqslant|\beta|$ we get

$$\sum_{k=0}^{\infty}\left(1+\frac{k}{n}\right)^{\beta}V_{n,k}(x)\leqslant\left[\sum_{k=0}^{\infty}\left(1+\frac{k}{n}\right)^{sign(\beta)m}V_{n,k}(x)\right]^{\frac{|\beta|}{m}}.$$

Let us estimate the last sum. For every integer s we have

$$\sum_{k=0}^{\infty}\left(1+\frac{k}{n}\right)^{s}V_{n,k}(x)=(1+x)^{s}\sum_{k=0}^{\infty}\frac{(n+k)^{s}n(n+1)\dots(n+s-1)}{n^{s}(n+k)\dots(n+k+s-1)}V_{n+s,k}(x).$$

But

$$\frac{(n+k)^{s}n(n+1)\dots(n+s-1)}{n^{s}(n+k)\dots(n+k+s-1)}\leqslant 1+\frac{C(s)}{n}$$

and consequently

$$\sum_{k=0}^{\infty}\left(1+\frac{k}{n}\right)^{s}V_{n,k}(x)\leqslant\left(1+\frac{C(s)}{n}\right)(1+x)^{s},$$

i.e.

$$\left[\sum_{k=0}^{\infty}\left(1+\frac{k}{n}\right)^{sign(\beta)m}V_{n,k}(x)\right]^{\frac{|\beta|}{m}}\leqslant\left(1+\frac{C(m)}{n}\right)^{\frac{\beta}{m}}(1+x)^{\beta}\leqslant\left(1+\frac{C(\beta)}{n}\right)(1+x)^{\beta}.\quad\square$$

LEMMA 2. *For $\alpha\in[0,1]$, $\beta\in\mathbb{R}$ there exists a constant C such that for every natural $n\geqslant|\beta|$ and every $x\in[0,\infty)$ and $V_{n,k}$ defined by (3)*

$$\sum_{k=0}^{\infty}\left(\frac{k}{n}\right)^{\alpha}\left(1+\frac{k}{n}\right)^{\beta}V_{n,k}(x)\leqslant\left(1+\frac{C}{n}\right)x^{\alpha}(1+x)^{\beta}. \tag{14}$$

Proof. We consider two cases.

1. $\alpha=1$.

By Cauchy's inequality and (13) we have

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{k}{n}\left(1+\frac{k}{n}\right)^{\beta}V_{n,k}(x)&=x\sum_{k=1}^{\infty}\left(1+\frac{k}{n}\right)^{\beta}V_{n+1,k-1}(x)\\&=x\sum_{k=0}^{\infty}\left(1+\frac{k+1}{n}\right)^{\beta}V_{n+1,k}(x)\\&=\left(1+\frac{1}{n}\right)^{\beta}x\sum_{k=0}^{\infty}\left(1+\frac{k}{n+1}\right)^{\beta}V_{n+1,k}(x)\\&\leqslant\left(1+\frac{C(\beta)}{n}\right)x(1+x)^{\beta}.\end{aligned}$$

2. $0\leqslant\alpha<1$.

By Hölder's inequality, (6) and (13) we have

$$\begin{aligned}\sum_{k=0}^{\infty}\left(\frac{k}{n}\right)^{\alpha}\left(1+\frac{k}{n}\right)^{\beta}V_{n,k}(x)&\leqslant\left[\sum_{k=0}^{\infty}\frac{k}{n}V_{n,k}(x)\right]^{\alpha}\left[\sum_{k=0}^{\infty}\left(1+\frac{k}{n}\right)^{\frac{\beta}{1-\alpha}}V_{n,k}(x)\right]^{1-\alpha}\\&\leqslant\left(1+\frac{C(\alpha,\beta)}{n}\right)x^{\alpha}(1+x)^{\beta}.\quad\square\end{aligned}$$

Of course, if we use notation (1) (with $\alpha = -\gamma_0$ and $\beta = -\gamma_\infty$) we can write (14) in the way

$$V_n\left(w^{-1}(x)\right) = \sum_{k=0}^{\infty} w^{-1}\left(\frac{k}{n}\right) V_{n,k}(x) \leqslant \left(1+\frac{C}{n}\right) w^{-1}(x). \tag{15}$$

LEMMA 3. *For* $\alpha \in [-2,0]$, $m+\alpha \geqslant 0$, $\beta \in \mathbb{R}$ *and* $x,t \in [0,\infty)$

$$\left|\int_x^t (t-u)^m u^\alpha (1+u)^\beta du\right| \leqslant x^\alpha |t-x|^{m+1}\left[(1+x)^\beta + (1+t)^\beta\right]. \tag{16}$$

Proof. It is obvious that for u between x and t

$$(1+u)^\beta \leqslant (1+x)^\beta + (1+t)^\beta$$

because $(1+u)^\beta$ is a monotonic function. We consider two cases.

Case 1. $t \geqslant x$.

Then $u^\alpha \leqslant x^\alpha$ and consequently

$$\begin{aligned}\left|\int_x^t (t-u)^m u^\alpha (1+u)^\beta du\right| &\leqslant x^\alpha\left[(1+x)^\beta + (1+t)^\beta\right]\int_x^t (t-u)^m du \\ &= \frac{x^\alpha}{m+1}(t-x)^{m+1}\left[(1+x)^\beta + (1+t)^\beta\right] \\ &\leqslant x^\alpha |t-x|^{m+1}\left[(1+x)^\beta + (1+t)^\beta\right].\end{aligned}$$

Case 2. $t < x$.

Then

$$\begin{aligned}\left|\int_x^t (t-u)^m u^\alpha (1+u)^\beta du\right| &\leqslant \left[(1+x)^\beta + (1+t)^\beta\right]\int_t^x (u-t)^m u^\alpha du \\ &= \left[(1+x)^\beta + (1+t)^\beta\right]\int_t^x \left(1-\frac{t}{u}\right)^{-\alpha}(u-t)^{m+\alpha} du \\ &\leqslant \left[(1+x)^\beta + (1+t)^\beta\right]\int_t^x \left(1-\frac{t}{x}\right)^{-\alpha}(u-t)^{m+\alpha} du \\ &= \frac{x^\alpha (x-t)^{m+1}}{m+\alpha+1}\left[(1+x)^\beta + (1+t)^\beta\right] \\ &\leqslant x^\alpha |t-x|^{m+1}\left[(1+x)^\beta + (1+t)^\beta\right]. \quad \square\end{aligned}$$

LEMMA 4. *For* $\alpha \in [-2,0]$, $\delta \in (-1,0]$, $\alpha \leqslant \delta$, $m+\alpha \geqslant 0$, $\beta \in \mathbb{R}$ *there exists a constant* C *such that for every natural* $n \geqslant |\beta|$ *and every* $x \in [0,\infty)$

$$\sum_{k=0}^{\infty}\left|\int_x^{k/n}\left(\frac{k}{n}-u\right)^m u^\alpha (1+u)^\beta du\right| V_{n,k}(x) \leqslant \begin{cases} Cx^\alpha (1+x)^\beta \left[\frac{\psi(x)}{n}\right]^{\frac{m+1}{2}}, & x \geqslant \frac{1}{n}, \\ Cx^\alpha (1+x)^\beta \frac{[\psi(x)]^{1-\delta}}{n^{m+\delta}}, & x < \frac{1}{n}. \end{cases} \tag{17}$$

Proof.

Case 1. $x \geqslant \frac{1}{n}$.

By (16) we get

$$\sum_{k=0}^{\infty}\left|\int_x^{k/n}\left(\frac{k}{n}-u\right)^m u^{\alpha}(1+u)^{\beta}du\right|V_{n,k}(x)$$
$$\leqslant x^{\alpha}\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|^{m+1}\left[(1+x)^{\beta}+\left(1+\frac{k}{n}\right)^{\beta}\right]V_{n,k}(x).$$

Using Cauchy's inequality and (12) we have for the first sum on the right

$$x^{\alpha}\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|^{m+1}(1+x)^{\beta}V_{n,k}(x)\leqslant x^{\alpha}(1+x)^{\beta}\left[\sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^{2(m+1)}V_{n,k}(x)\right]^{\frac{1}{2}}$$
$$\leqslant C(m)x^{\alpha}(1+x)^{\beta}\left[\frac{\psi(x)}{n}\right]^{\frac{m+1}{2}}.$$

For the second one, again by using Cauchy's inequality, (12) and (13) we get

$$x^{\alpha}\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|^{m+1}\left(1+\frac{k}{n}\right)^{\beta}V_{n,k}(x)$$
$$\leqslant x^{\alpha}\left[\sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^{2(m+1)}V_{n,k}(x)\right]^{\frac{1}{2}}\left[\sum_{k=0}^{\infty}\left(1+\frac{k}{n}\right)^{2\beta}V_{n,k}(x)\right]^{\frac{1}{2}}$$
$$\leqslant C(m,\beta)x^{\alpha}(1+x)^{\beta}\left[\frac{\psi(x)}{n}\right]^{\frac{m+1}{2}}.$$

Case 2. $x < \frac{1}{n}$.

We estimate the terms in the sum separately for $k=0,\ k=1,...,k=2m+1$ and for $k \geqslant 2(m+1)$.

$$I_0=(1+x)^{-n}\int_0^x u^{m+\alpha}(1+u)^{\beta}du\leqslant\left[(1+x)^{\beta}+1\right]\int_0^x u^{m+\alpha}du$$
$$\leqslant C(\beta)x^{m+\alpha+1}\leqslant C(\beta)\frac{x^{\alpha+1}}{n^m}.$$

For $1 \leqslant k \leqslant 2m+1$ we have

$$\begin{aligned} I_k &= \int_x^{k/n} \left(\frac{k}{n} - u\right)^m u^\alpha (1+u)^\beta V_{n,k}(x) \\ &\leqslant \left[(1+x)^\beta + \left(1+\frac{k}{n}\right)^\beta\right] \left(\frac{k}{n}\right)^m V_{n,k}(x) \int_x^{k/n} u^\alpha du \\ &\leqslant \frac{C(m,k,\beta)}{n^m} V_{n,k}(x) x^{\alpha-\delta} \int_x^{k/n} u^\delta du \leqslant \frac{C(m,k,\beta,\delta)}{n^m} V_{n,k}(x) x^{\alpha-\delta} \left(\frac{k}{n}\right)^{\delta+1} \\ &\leqslant \frac{C(m,k,\beta,\delta)}{n^{m+\delta+1}} V_{n,k}(x) x^{\alpha-\delta} \leqslant \frac{Cx^{\alpha-\delta+1}}{n^{m+\delta}} \end{aligned}$$

because for $x \leqslant \frac{1}{n}$ and $k \geqslant 1$

$$V_{n,k}(x) \leqslant C(k)nx \quad \text{and} \quad (1+x)^\beta + \left(1+\frac{k}{n}\right)^\beta \leqslant C(m,\beta).$$

Now we estimate

$$\sum_{k=2(m+1)}^{\infty} I_k = \sum_{k=2(m+1)}^{\infty} \left| \int_x^{k/n} \left(\frac{k}{n} - u\right)^m u^\alpha (1+u)^\beta du \right| V_{n,k}(x).$$

We have

$$\sum_{k=2(m+1)}^{\infty} I_k \leqslant \sum_{k=2(m+1)}^{\infty} \left[(1+x)^\beta + \left(1+\frac{k}{n}\right)^\beta\right] \left(\frac{k}{n}\right)^{m+1} x^\alpha V_{n,k}(x) = \frac{x^\alpha}{n^{m+1}} (J_1 + J_2).$$

$$\begin{aligned} J_1 &= \sum_{k=2(m+1)}^{\infty} k^{m+1} (1+x)^\beta V_{n,k}(x) \\ &= (1+x)^\beta x^{m+1} \sum_{k=2(m+1)}^{\infty} \frac{n(n+1)\dots(n+m)k^m}{(k-m)(k-m+1)\dots(k-1)} V_{n+m+1,k-m-1}(x) \\ &\leqslant C(m)(1+x)^\beta x^{m+1} n^{m+1} \sum_{k=2(m+1)}^{\infty} V_{n+m+1,k-m-1}(x) \leqslant C(m)(xn)^{m+1}. \end{aligned}$$

For the second term by Cauchy's inequality and using (10) and (13) we have

$$\begin{aligned} J_2 &= \sum_{k=2(m+1)}^{\infty} k^{m+1} \left(1+\frac{k}{n}\right)^\beta V_{n,k}(x) \\ &\leqslant \left[\sum_{k=2(m+1)}^{\infty} k^{2(m+1)} V_{n,k}(x)\right]^{\frac{1}{2}} \left[\sum_{k=2(m+1)}^{\infty} \left(1+\frac{k}{n}\right)^{2\beta} V_{n,k}(x)\right]^{\frac{1}{2}} \\ &\leqslant C(m,\alpha,\beta)(xn)^{m+1}. \end{aligned}$$

Then,

$$\sum_{k=2(m+1)}^{\infty} I_k \leqslant \frac{x^{\alpha}}{n^{m+1}} C(m,\alpha,\beta)(xn)^{m+1} = C(m,\alpha,\beta)x^{m+\alpha+1} \leqslant \frac{C(m,\alpha,\beta)x^{\alpha-\delta+1}}{n^{m+\delta}},$$

and consequently

$$\sum_{k=0}^{\infty} I_k \leqslant \frac{C(m,\alpha,\beta)x^{\alpha-\delta+1}}{n^{m+\delta}} \leqslant Cx^{\alpha}(1+x)^{\beta}\frac{[\psi(x)]^{1-\delta}}{n^{m+\delta}}. \quad \square$$

LEMMA 5. *For w, defined by (1) there exists a constant C such that for every natural $n \geqslant |\gamma_\infty|$ we have*

$$\|wV_n f\| \leqslant C\|wf\| \tag{18}$$

for every function $f \in C(w)$.

Proof. Using (5) and (15) we get

$$\begin{aligned} |V_n f(x)| &= \left|V_n\left((wf)w^{-1}\right)(x)\right| \leqslant V_n\left(\|wf\|w^{-1}\right)(x) \\ &= \|wf\|V_n\left(w^{-1}\right)(x) \leqslant C\|wf\|w^{-1}(x). \quad \square \end{aligned}$$

LEMMA 6. *For w, defined by (1) there exists a constant C such that for every natural $n \geqslant |\gamma_\infty|$ we have*

$$\|w(V_n g - g)\| \leqslant \frac{C}{n}\left\|w\psi D^2 g\right\| \tag{19}$$

for every function $g \in W^2(w\psi)$.

Proof. We have

$$g(t) = g(x) + (t-x)Dg(x) + \int_x^t (t-v)D^2 g(v)dv.$$

Multiplying both sides by $V_{n,k}(x)$, summing with respect to k and using (6) we obtain

$$V_n g(x) = g(x) + V_n\left(\int_x^{(.)} (.-v)D^2 g(v)dv\right).$$

Using the positivity of V_n we get

$$\begin{aligned} |V_n g(x) - g(x)| &= \left|V_n\left(\int_x^{(.)} (.-v)D^2 g(v)dv\right)\right| \\ &\leqslant \left|V_n\left(\int_x^{(.)} (.-v)(w\psi)^{-1}(v)dv\right)\right| \|w\psi D^2 g\| \\ &= \|w\psi D^2 g\| \left|\sum_{k=0}^{\infty}\int_x^{k/n}\left(\frac{k}{n}-v\right)(w\psi)^{-1}(v)dvV_{n,k}(x)\right| \\ &\leqslant \|w\psi D^2 g\| \sum_{k=0}^{\infty}\left|\int_x^{k/n}\left(\frac{k}{n}-v\right)v^{-1-\gamma_0}(1+v)^{-1-\gamma_\infty}dv\right| V_{n,k}(x). \end{aligned}$$

The use of (17) of Lemma 4 with $\alpha=-1-\gamma_0$, $\beta=-1-\gamma_\infty$, $m=1$, $\delta=0$ completes the proof. $\square$

As an elementary consequence of this lemma we have that if a function $g\in W^2(w\psi)$ then $V_ng-g\in C(w)$.

THEOREM 2. *For w, defined by (1) there exists a constant C such that for every natural $n\geqslant|\gamma_\infty|$ we have*

$$\left\|w\left(V_ng-g-\frac{1}{2n}\psi D^2g\right)\right\|\leqslant\frac{C}{n^{3/2}}\left\|w\psi^{3/2}D^3g\right\| \tag{20}$$

for every function $g\in W^3(w\psi^{3/2})$.

Proof. By Taylor's formula we have

$$g(t)=g(x)+(t-x)Dg(x)+\frac{(t-x)^2}{2}D^2g(x)+\frac{1}{2}\int_x^t(t-v)^2D^3g(v)dv.$$

Multiplying both sides by $V_{n,k}(x)$, summing with respect to k and using the identities (6) we get

$$\begin{aligned}&\left|V_ng(x)-g(x)-\frac{1}{2n}\psi D^2g(x)\right|\\&\qquad\leqslant\frac{1}{2}\sum_{k=0}^{\infty}\left|\int_x^{k/n}\left(\frac{k}{n}-v\right)^2D^3g(v)dv\right|V_{n,k}(x)\\&\qquad\leqslant\frac{1}{2}\left\|w\psi^{3/2}D^3g\right\|\sum_{k=0}^{\infty}\left|\int_x^{k/n}\left(\frac{k}{n}-v\right)^2w^{-1}(v)\psi^{-3/2}(v)dv\right|V_{n,k}(x).\end{aligned}$$

Now we can use (17) of Lemma 4 with $\alpha=-\gamma_0-\frac{3}{2}$, $\beta=-\gamma_\infty-\frac{3}{2}$, $m=2$, $\delta=-\frac{1}{2}$ to complete the proof. $\square$

LEMMA 7. *For w, defined by (1) there exists a constant C such that for every natural $n\geqslant|\gamma_\infty|$ we have*

$$\left\|w\psi D^2V_nf\right\|\leqslant Cn\|wf\| \tag{21}$$

for every function $f\in C(w)$.

Proof. We consider two cases.
Case 1. $x\leqslant\frac{1}{n}$. We have [8]

$$D^2V_nf(x)=n(n+1)\sum_{k=0}^{\infty}\Delta^2_{\frac{1}{n}}f\left(\frac{k}{n}\right)V_{n+2,k}(x),$$

where, as usual,

$$\Delta_h^rf\left(\frac{k}{n}\right)=\sum_{k=0}^{r}(-1)^k\binom{r}{k}f\big(x+(r-k)h\big).$$

Then, by (15) we get

$$\begin{aligned}
&\left|w(x)\psi(x)D^2V_nf(x)\right| \\
&= n(n+1)w(x)\psi(x)\left|\sum_{k=0}^{\infty}\left[f\left(\frac{k+2}{n}\right)-2f\left(\frac{k+1}{n}\right)+f\left(\frac{k}{n}\right)\right]V_{n+2,k}(x)\right| \\
&\leqslant n(n+1)w(x)\psi(x)\|wf\|\sum_{k=0}^{\infty}\left[w^{-1}\left(\frac{k+2}{n}\right)+2w^{-1}\left(\frac{k+1}{n}\right)+w^{-1}\left(\frac{k}{n}\right)\right]V_{n+2,k}(x) \\
&\leqslant C(\gamma_\infty)n(n+1)w(x)\psi(x)\|wf\|\sum_{k=0}^{\infty}w^{-1}\left(\frac{k}{n+2}\right)V_{n+2,k}(x) \\
&\leqslant Cn\|wf\|.
\end{aligned}$$

Case 2. $x > \frac{1}{n}$. We have

$$\begin{aligned}
&\left|\psi(x)D^2V_nf(x)\right| \\
&= \frac{n^2}{\psi(x)}\sum_{k=0}^{\infty}\left[\left(\frac{k}{n}-x\right)^2-\frac{1+2x}{n}\left(\frac{k}{n}-x\right)-\frac{\psi(x)}{n}\right]f\left(\frac{k}{n}\right)V_{n,k}(x) \\
&\leqslant \|wf\|\frac{n^2}{\psi(x)}\sum_{k=0}^{\infty}\left[\left(\frac{k}{n}-x\right)^2+\frac{1+2x}{n}\left|\frac{k}{n}-x\right|+\frac{\psi(x)}{n}\right]w^{-1}\left(\frac{k}{n}\right)V_{n,k}(x) \\
&= \|wf\|\left(I_1+I_2+I_3\right).
\end{aligned}$$

Now we estimate I_i separately.

For I_1 and I_2 we consider two cases.

1. $\gamma_0 = -1$

Applying two times Cauchy's inequality for I_1 and I_2 and using (10), (12) and (13) we obtain

$$\begin{aligned}
I_1 &= \frac{n^2}{\psi(x)}\sum_{k=0}^{\infty}\frac{k}{n}\left(\frac{k}{n}-x\right)^2\left(1+\frac{k}{n}\right)^{-\gamma_\infty}V_{n,k}(x) \\
&\leqslant \frac{n^2}{\psi(x)}\left[\sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^4V_{n,k}(x)\right]^{1/2}\left[\sum_{k=0}^{\infty}\left(\frac{k}{n}\right)^4V_{n,k}(x)\right]^{1/4}\left[\sum_{k=0}^{\infty}\left(1+\frac{k}{n}\right)^{-4\gamma_\infty}V_{n,k}(x)\right]^{1/4} \\
&\leqslant C(\gamma_\infty)nw^{-1}(x) \\
&= Cnw^{-1}(x).
\end{aligned}$$

$$I_2 = \frac{n(1+2x)}{\psi(x)} \sum_{k=0}^{\infty} \frac{k}{n} \left| \frac{k}{n} - x \right| \left(1 + \frac{k}{n}\right)^{-\gamma_\infty} V_{n,k}(x)$$

$$\leqslant \frac{n(1+2x)}{\psi(x)} \left[\sum_{k=0}^{\infty} \left(\frac{k}{n} - x \right)^2 V_{n,k}(x) \right]^{1/2}$$

$$\times \left[\sum_{k=0}^{\infty} \left(\frac{k}{n} \right)^4 V_{n,k}(x) \right]^{1/4} \left[\sum_{k=0}^{\infty} \left(1 + \frac{k}{n}\right)^{-4\gamma_\infty} V_{n,k}(x) \right]^{1/4}$$

$$\leqslant \frac{C(\gamma_\infty) n(1+2x)}{\psi(x)} \left[\frac{\psi(x)}{n} \right]^{\frac{1}{2}} w^{-1}(x) \leqslant 2Cn^{\frac{1}{2}} w^{-1}(x) \sqrt{1 + \frac{1}{x}} \leqslant Cnw^{-1}(x).$$

2. $\gamma_0 > -1$

By Hölder's inequality we get for I_1

$$I_1 = \frac{n^2}{\psi(x)} \sum_{k=0}^{\infty} \left(\frac{k}{n} - x \right)^2 w^{-1} \left(\frac{k}{n} \right) V_{n,k}(x)$$

$$\leqslant \frac{n^2}{\psi(x)} \left[\sum_{k=0}^{\infty} \frac{k}{n} \left(\frac{k}{n} - x \right)^2 V_{n,k}(x) \right]^{-\gamma_0} \left[\sum_{k=0}^{\infty} \left(\frac{k}{n} - x \right)^2 \left(1 + \frac{k}{n}\right)^{-\frac{\gamma_\infty}{1+\gamma_0}} V_{n,k}(x) \right]^{1+\gamma_0}.$$

From (9) it follows

$$\sum_{k=0}^{\infty} \frac{k}{n} \left(\frac{k}{n} - x \right)^2 V_{n,k}(x) \leqslant \frac{4x\psi(x)}{n}.$$

Now using Cauchy's inequality, (12) and (14) we have

$$\sum_{k=0}^{\infty} \left(\frac{k}{n} - x \right)^2 \left(1 + \frac{k}{n}\right)^{-\frac{\gamma_\infty}{1+\gamma_0}} V_{n,k}(x)$$

$$\leqslant \left[\sum_{k=0}^{\infty} \left(\frac{k}{n} - x \right)^4 V_{n,k}(x) \right]^{\frac{1}{2}} \left[\sum_{k=0}^{\infty} \left(1 + \frac{k}{n}\right)^{-\frac{2\gamma_\infty}{1+\gamma_0}} V_{n,k}(x) \right]^{\frac{1}{2}}$$

$$\leqslant C(\gamma_0, \gamma_\infty) \frac{\psi(x)}{n} (1+x)^{-\frac{\gamma_\infty}{1+\gamma_0}}$$

and consequently $I_1 \leqslant Cnw^{-1}(x)$.

Again, by Hölder's inequality and (6) we get for I_2:

$$I_2 = \frac{n(1+2x)}{\psi(x)} \sum_{k=0}^{\infty} \left| \frac{k}{n} - x \right| w^{-1} \left(\frac{k}{n} \right) V_{n,k}(x)$$

$$\leqslant \frac{n(1+2x)}{\psi(x)} \left[\sum_{k=0}^{\infty} \frac{k}{n} V_{n,k}(x) \right]^{-\gamma_0} \left[\sum_{k=0}^{\infty} \left(1 + \frac{k}{n}\right)^{-\frac{\gamma_\infty}{1+\gamma_0}} \left| \frac{k}{n} - x \right|^{\frac{1}{1+\gamma_0}} V_{n,k}(x) \right]^{1+\gamma_0}$$

$$= \frac{n(1+2x)x^{-\gamma_0}}{\psi(x)} \left[\sum_{k=0}^{\infty} \left(1 + \frac{k}{n}\right)^{-\frac{\gamma_\infty}{1+\gamma_0}} \left| \frac{k}{n} - x \right|^{\frac{1}{1+\gamma_0}} V_{n,k}(x) \right]^{1+\gamma_0}.$$

Let m is the smallest natural number such that $m > \frac{1}{2(1+\gamma_0)}$. Then by Hölder's inequality, (12) and (14) we get

$$\begin{aligned}
&\sum_{k=0}^{\infty}\left(1+\frac{k}{n}\right)^{-\frac{\gamma_\infty}{1+\gamma_0}}\left|\frac{k}{n}-x\right|^{\frac{1}{1+\gamma_0}}V_{n,k}(x)\\
&\quad\leqslant\left[\sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^{2m}V_{n,k}(x)\right]^{\frac{1}{2m(1+\gamma_0)}}\left[\sum_{k=0}^{\infty}\left(1+\frac{k}{n}\right)^{\frac{-2m\gamma_\infty}{2m(1+\gamma_0)-1}}V_{n,k}(x)\right]^{\frac{2m(1+\gamma_0)-1}{2m(1+\gamma_0)}}\\
&\quad\leqslant C(\gamma_0,\gamma_\infty)\left[\frac{\psi(x)}{n}\right]^{\frac{1}{2(1+\gamma_0)}}(1+x)^{-\frac{\gamma_\infty}{1+\gamma_0}}
\end{aligned}$$

and

$$\begin{aligned}
I_2&\leqslant\frac{C(\gamma_0,\gamma_\infty)n(1+2x)}{\psi(x)}\left[\frac{\psi(x)}{n}\right]^{\frac{1}{2}}w^{-1}(x)\\
&\leqslant 2C(\gamma_0,\gamma_\infty)n^{\frac{1}{2}}w^{-1}(x)\sqrt{1+\frac{1}{x}}\leqslant Cnw^{-1}(x).
\end{aligned}$$

For I_3 by (14) we have

$$I_3=\sum_{k=0}^{\infty}nw^{-1}\left(\frac{k}{n}\right)V_{n,k}(x)\leqslant Cnw^{-1}(x).$$

The proof of the lemma is complete. □

3. Proof of Theorem 1

The proof is based on

THEOREM 3. *For w, defined by (1) there exists an absolute constant L such that for $n\geqslant L$*

$$\left\|w\psi^{\frac{3}{2}}D^3V_n^Ng\right\|\leqslant K(N)\sqrt{n}\left\|w\psi D^2g\right\|\quad \textit{where}\quad \lim_{N\to\infty}K(N)=0 \tag{22}$$

holds for all $g\in W^2(w\psi)$.

Proof. Inequality (4.12) of [14] gives the following estimate of the third derivatives of the N-th power of Baskakov operator

$$\begin{aligned}
\left|D^3V_n^Ng(x)\right|&\leqslant\frac{n(n+1)}{N-1}\sqrt{\sum_N V_{n+3,k_N}(x)P(k_1,..k_N;n)\,Q^2}\\
&\quad\times\sqrt{\sum_N\left[\int_0^{1/n}\int_0^{1/n}D^2g\left(\frac{k_1}{n}+u_1+v_1\right)du_1dv_1\right]^2V_{n+3,k_N}(x)P(k_1,..k_N;n)}
\end{aligned}$$

where

$$\sum_N = \sum_{k_1=0}^{\infty} \cdots \sum_{k_N=0}^{\infty}, \quad P(k_1,..k_N;n) = \prod_{j=1}^{N-1} T_{2,k_j}\left(\frac{k_{j+1}}{n}\right)$$

and

$$T_{2,k}(x) = n(n+1)\int_0^{1/n}\int_0^{1/n} V_{n+2,k}(x+t_1+t_2)\,dt_1dt_2,$$

Instead of giving the definition of Q we use the estimate from Lemma 4.2 of [14] which for $2 \leqslant N \leqslant n$ and $n \geqslant 10$ is

$$\sum_N V_{n+3,k_N}(x)P(k_1,..k_N;n)\,Q^2 \leqslant CnN\psi^{-1}(x).$$

For the second factor, using (23) of Lemma 8 and (24) of Lemma 9 (below) we have

$$\begin{aligned}
&\sum_N \left[\int_0^{1/n}\int_0^{1/n} D^2g\left(\frac{k_1}{n}+u_1+v_1\right) du_1dv_1\right]^2 V_{n+3,k_N}(x)P(k_1,..k_N;n)\\
&\leqslant \left\|w\psi D^2g\right\|^2 \sum_N \left[\int_0^{1/n}\int_0^{1/n}(w\psi)^{-1}\left(\frac{k_1}{n}+u_1+v_1\right) du_1dv_1\right]^2 V_{n+3,k_N}(x)P(k_1,..k_N;n)\\
&\leqslant Cn^{-4}\left\|w\psi D^2g\right\|^2 \sum_N V_{n+3,k_N}(x)P(k_1,..k_N;n)\,(w\psi)^{-2}\left(\frac{k_1+1}{n}\right)\\
&\leqslant Cn^{-4}K_1(N)\left\|w\psi D^2g\right\|^2 (w\psi)^{-2}(x)
\end{aligned}$$

where

$$\lim_{N\to\infty}\frac{K_1(N)}{N} = 0.$$

Then,

$$\begin{aligned}
\left|D^3V_n^N g(x)\right| &\leqslant \frac{Cn(n+1)}{N-1}\sqrt{CnN\psi^{-1}(x)}\sqrt{Cn^{-4}K_1(N)\,(w\psi)^{-2}(x)}\,\left\|w\psi D^2g\right\|\\
&= K(N)\sqrt{n}w^{-1}(x)\psi^{-3/2}(x)\left\|w\psi D^2g\right\|
\end{aligned}$$

with

$$K(N) = C\sqrt{\frac{K_1(N)}{N}}. \quad \square$$

LEMMA 8. *For $\alpha \in [-1,0]$, $\beta \in \mathbb{R}$ there exists an absolute constant C such that*

$$\int_0^{1/n}\int_0^{1/n}\left(\frac{k}{n}+u+v\right)^{\alpha}\left(1+\frac{k}{n}+u+v\right)^{\beta} dudv \leqslant \frac{C}{n^2}\left(\frac{k+1}{n}\right)^{\alpha}\left(1+\frac{k+1}{n}\right)^{\beta}. \tag{23}$$

Proof. Because of

$$\left(1+\frac{k}{n}+u+v\right)^{\beta} \leqslant C(\beta)\left(1+\frac{k+1}{n}\right)^{\beta} \quad \text{for} \quad 0 \leqslant u,v \leqslant \frac{1}{n}$$

we need to prove only

$$\int_0^{1/n}\int_0^{1/n}\left(\frac{k}{n}+u+v\right)^{\alpha}dudv \leqslant \frac{C}{n^2}\left(\frac{k+1}{n}\right)^{\alpha}.$$

By Hölder's inequality we have

$$\int_0^{1/n}\int_0^{1/n}\left(\frac{k}{n}+u+v\right)^{\alpha}dudv \leqslant n^{-2(1+\alpha)}\left[\int_0^{1/n}\int_0^{1/n}\left(\frac{k}{n}+u+v\right)^{-1}dudv\right]^{-\alpha}.$$

Here,

$$\int_0^{1/n}\int_0^{1/n}\left(\frac{k}{n}+u+v\right)^{-1}dudv \leqslant \left[\int_0^{1/n}\left(\frac{k}{n}+u\right)^{-\frac{1}{2}}du\right]^2$$
$$= 4\left[\left(\frac{k+1}{n}\right)^{\frac{1}{2}}-\left(\frac{k}{n}\right)^{\frac{1}{2}}\right]^2 \leqslant \frac{4}{(k+1)n}$$

and consequently

$$\int_0^{1/n}\int_0^{1/n}\left(\frac{k}{n}+u+v\right)^{\alpha}dudv \leqslant n^{-2(1+\alpha)}\left[\frac{4}{(k+1)n}\right]^{-\alpha} = \frac{4^{-\alpha}}{n^2}\left(\frac{k+1}{n}\right)^{\alpha}. \quad \square$$

LEMMA 9. *For w defined by (1), $n,N\in\mathbb{N}$ such that $N\leqslant\frac{n-2}{2}$ and $n\geqslant|\gamma_\infty|$ and $x\in(0,\infty)$ we have*

$$\sum_N V_{n+3,k_N}(x)P(k_1,..k_N;n)\,(w\psi)^{-2}\left(\frac{k_1+1}{n}\right) \leqslant K_1(N)\,(w\psi)^{-2}(x), \quad \lim_{N\to\infty}\frac{K_1(N)}{N}=0. \tag{24}$$

Proof. For $\gamma_0\in[-1,0)$ by Hölder's inequality we get

$$\sum_N V_{n+3,k_N}(x)P(k_1,..k_N;n)\,(w\psi)^{-2}\left(\frac{k_1+1}{n}\right)$$
$$\leqslant \left[\sum_N V_{n+3,k_N}(x)P(k_1,..k_N;n)\left(\frac{k_1+1}{n}\right)^{-2}\left(1+\frac{k_1+1}{n}\right)^{-2}\right]^{1+\gamma_0}$$
$$\times\left[\sum_N V_{n+3,k_N}(x)P(k_1,..k_N;n)\left(1+\frac{k_1+1}{n}\right)^{\frac{-2(\gamma_0-\gamma_\infty)}{\gamma_0}}\right]^{-\gamma_0}.$$

For the first factor on the right we have from lemma 4.3 of [14]

$$\sum_N V_{n+3,k_N}(x)P(k_1,..k_N;n)\left(\frac{k_1+1}{n}\right)^{-2}\left(1+\frac{k_1+1}{n}\right)^{-2} \leqslant CN^{3/4}\ln N\psi^{-2}(x). \tag{25}$$

For the second one we have

$$\sum_N V_{n+3,k_N}(x)P(k_1,..k_N;n)\left(1+\frac{k_1+1}{n}\right)^{\beta}$$
$$=\sum_{k_N=0}^{\infty}\cdots\sum_{k_2=0}^{\infty}V_{n+3,k_N}(x)\prod_{j=2}^{N-1}T_{2,k_j}\left(\frac{k_{j+1}}{n}\right)\sum_{k_1=0}^{\infty}\left(1+\frac{k_1+1}{n}\right)^{\beta}T_{2,k_1}\left(\frac{k_2}{n}\right).$$

Now, by (13)

$$\sum_{k_1=0}^{\infty}\left(1+\frac{k_1+1}{n}\right)^{\beta}T_{2,k_1}\left(\frac{k_2}{n}\right)\leqslant\left(1+\frac{C}{n}\right)\left(1+\frac{k_2+1}{n}\right)^{\beta},$$

and inductively we obtain for $N\leqslant\frac{n-2}{2}$

$$\sum_N V_{n+3,k_N}(x)P(k_1,..k_N;n)\left(1+\frac{k_1+1}{n}\right)^{\beta}\leqslant\left(1+\frac{C}{n}\right)^{N}(1+x)^{\beta}\leqslant C(1+x)^{\beta}. \quad (26)$$

Consequently

$$\sum_N V_{n+3,k_N}(x)P(k_1,..k_N;n)\,(w\psi)^{-2}\left(\frac{k_1+1}{n}\right)$$
$$\leqslant C\left[N^{3/4}\ln N\right]^{1+\gamma_0}\psi^{-2(1+\gamma_0)}(x)(1+x)^{2(\gamma_0-\gamma_\infty)}=K_1(N)\,(w\psi)^{-2}(x)$$

with

$$K_1(N)=C\left[N^{3/4}\ln N\right]^{1+\gamma_0}.$$

For $\gamma_0=0$ it is enough to prove it for $\gamma_\infty\in(-1,0)$. Indeed, if $\gamma_\infty\notin(-1,0)$ we can choose an integer s such that $-2(1+\gamma_\infty)\leqslant s\leqslant-2\gamma_\infty$. Then $-2(1+\gamma_\infty)=s-2(1+\gamma_\infty^*)$ where $\gamma_\infty^*\in(-1,0)$ and because of

$$\left(1+\frac{k_1+1}{n}\right)^{-2(1+\gamma_\infty)}V_{n,k}(x)=\left(1+\frac{k_1+1}{n}\right)^{s}\left(1+\frac{k_1+1}{n}\right)^{-2(1+\gamma_\infty^*)}V_{n,k}(x)$$
$$\leqslant\left(1+\frac{C(s)}{n}\right)(1+x)^{s}\left(1+\frac{k_1+1}{n}\right)^{-2(1+\gamma_\infty^*)}V_{n+s,k}(x)$$

inductively we get

$$\sum_N V_{n+3,k_N}(x)P(k_1,..k_N;n)\left(\frac{k_1+1}{n}\right)^{-2}\left(1+\frac{k_1+1}{n}\right)^{-2(1+\gamma_\infty)}$$
$$\leqslant\left(1+\frac{C}{n}\right)^{N}(1+x)^{2(\gamma_\infty^*-\gamma_\infty)}$$
$$\times\sum_N V_{n+3,k_N}(x)P(k_1,..k_N;n)\left(\frac{k_1+1}{n}\right)^{-2}\left(1+\frac{k_1+1}{n}\right)^{-2(1+\gamma_\infty^*)}.$$

For $\gamma_\infty \in (-1,0)$ by Hölder's inequality we have

$$\sum_N V_{n+3,k_N}(x)P(k_1,..k_N;n)\left(\frac{k_1+1}{n}\right)^{-2}\left(1+\frac{k_1+1}{n}\right)^{-2(1+\gamma_\infty)}$$
$$\leqslant \left[\sum_N V_{n+3,k_N}(x)P(k_1,..k_N;n)\left(\frac{k_1+1}{n}\right)^{-2}\left(1+\frac{k_1+1}{n}\right)^{-2}\right]^{1+\gamma_\infty}$$
$$\times\left[\sum_N V_{n+3,k_N}(x)P(k_1,..k_N;n)\left(\frac{k_1+1}{n}\right)^{-2}\right]^{-\gamma_\infty}.$$

For the first factor, again, we use the estimation (25). For the second one, using the simple inequality

$$\frac{1}{x^2}\leqslant 2\left[\frac{1}{x^2(1+x)^2}+\frac{1}{(1+x)^2}\right]$$

and the estimations (25) and (13) we get

$$\sum_N V_{n+3,k_N}(x)P(k_1,..k_N;n)\left(\frac{k_1+1}{n}\right)^{-2}\leqslant CN^{3/4}\ln N\psi^{-2}(x)$$

and the Lemma 9 follows. □

Proof of Theorem 1. For every $g\in W^2(w\psi)$ such that $f-g\in C(w)$ we get from (18) and (19)

$$\|w(f-V_nf)\|\leqslant\|w(f-g)\|+\|w(g-V_ng)\|+\|wV_n(f-g)\|$$
$$\leqslant 2\|w(f-g)\|+\frac{C}{n}\left\|w\psi D^2g\right\|\leqslant C\left\{\|w(f-g)\|+\frac{1}{n}\left\|w\psi D^2g\right\|\right\}.$$

Taking infimum on $g\in W^2(w\psi)$ such that $f-g\in C(w)$ in the above inequality we get the first inequality of Theorem 1.

For the second one we have

$$K_w\left(f,\frac{1}{n}\right)=\inf\left\{\|w(f-g)\|+\frac{1}{n}\left\|w\psi D^2g\right\|\right\}\leqslant\|w(f-V_nf)\|+\frac{1}{n}\left\|w\psi D^2V_nf\right\|$$

which means that it is sufficient to show that for some constant C

$$\frac{1}{n}\left\|w\psi D^2V_nf\right\|\leqslant C\|w(f-V_nf)\|. \tag{27}$$

We have

$$\frac{1}{n}\left\|w\psi D^2V_nf\right\|=\frac{1}{n}\left\|w\psi D^2\left(V_nf-V_n^{N+1}f+V_n^{N+1}f\right)\right\|$$
$$\leqslant\frac{1}{n}\left\|w\psi D^2V_n\left(f-V_n^Nf\right)\right\|+\frac{1}{n}\left\|w\psi D^2V_n^{N+1}f\right\|.$$

Applying (21) and (18) we get

$$\frac{1}{n}\left\|w\psi D^2V_n\left(f-V_n^Nf\right)\right\| \leqslant C_1\left\|w(f-V_n^Nf)\right\| \leqslant C_1\sum_{i=0}^{N-1}\left\|w\left(V_n^if-V_n^{i+1}f\right)\right\| \leqslant C_1N\left\|w(f-V_nf)\right\|.$$

For the second term, after using (20), (22) and (18) we obtain

$$\begin{aligned}\frac{1}{2n}\left\|w\psi D^2V_n^{N+1}f\right\| &\leqslant \left\|w\left(V_n^{N+2}f-V_n^{N+1}f-\frac{\psi}{2n}D^2V_n^{N+1}f\right)\right\| \\ &\quad+\left\|w\left(V_n^{N+2}f-V_n^{N+1}f\right)\right\| \\ &\leqslant C_2n^{-3/2}\left\|w\psi^{3/2}D^3V_n^{N+1}f\right\|+C_3\left\|w(f-V_nf)\right\| \\ &\leqslant C_2n^{-1}K(N)\left\|w\psi D^2V_nf\right\|+C_3\left\|w(f-V_nf)\right\|,\end{aligned}$$

i.e.

$$\frac{1}{n}\left\|w\psi D^2V_nf\right\| \leqslant (C_1N+2C_3)\left\|w(f-V_nf)\right\|+2C_2n^{-1}K(N)\left\|w\psi D^2V_nf\right\|.$$

Because of $\lim_{N\to\infty}K(N)=0$ we can choose N such that $2C_2K(N)\leqslant\frac{1}{2}$ and (27) follows.

This completes the proof of Theorem 1. □

Acknowledgements. The author would like to thank Professor K. G. Ivanov for his valuable comments and suggestions which greatly help us to improve the presentation of this paper.

REFERENCES

[1] N. T. AMANOV, *On the weighted approximation by Szász-Mirakjan operators*, Anal. Math, **18**, (1992), 167–184.
[2] V. A. BASKAKOV, *An instance of a sequence of the linear positive operators in the space of continuous functions*, Docl. Akad. Nauk SSSR, **113**, (1957), 249–251.
[3] M. BECKER, *Global approximation theorems for Szász-Mirakjan and Baskakov operators in polynomial weght spaces*, Indiana Univ. Math. J., 1 **27**, (1978), 127–142.
[4] Z. DITZIAN, *A global inverse theorem for combinations of Bernstein polynomials*, J. Approx. Theory, **26**, (1979), 277–292.
[5] Z. DITZIAN, *Direct estimate for Bernstein polynomials*, J. Approx. Theory, **79**, (1994), 165–166.
[6] Z. DITZIAN, *Rate convergence for Bernstein polynomials revisited*, J. Approx. Theory, **50**, (1987), 40–48.
[7] Z. DITZIAN AND K. G. IVANOV, *Strong converse inequalities*, J. Anal. Math., **61**, (1993), 61–111.
[8] Z. DITZIAN AND V. TOTIK, *Moduli of Smoothness*, Springer, Berlin, New York, 1987.
[9] B. R. DRAGANOV AND K. G. IVANOV, *Natural weights for uniform approximation by the Szász-Mirakjan operator*, Constructive Theory of Functions, Sozopol 2013, (2014), 1–27.
[10] M. FELTEN, *Direct and inverse estimates for Bernstein polynomials*, Constr. Approx., **14**, (1998), 459–468.
[11] M. FELTEN, *Local and global approximation theorems for positive linear operators*, J. Approx. Theory, **94**, (1998), 396–419.

[12] GUO FENG, *Direct and inverse approximation theorems for Baskakov operators with the Jacobi-Type weights*, Abstract and Applied Analysis, Volume **2011**, (2011).
[13] Z. FINTA, *On converse approximation theorems*, J. Math. Anal. Appl., **312**, (2005), 159–180.
[14] I. GADJEV, *Strong converse result for Baskakov operator*, Serdica Math. Journal, **40**, (2014), 273–318.
[15] SH. GUO AND Q. QI, *Strong converse inequalities for Baskakov operators*, J. Approx. Theory, **124**, (2003), 219–231.
[16] A. HOLHOŞ, *Uniform weighted approximation by positive linear operators*, Stud. Univ. Babes-Bolyai Math, **56** No. 3, (2011), 135–146.
[17] K. G. IVANOV, *Natural weights for uniform approximation by Bernstein polynomials*, manuscript, (2011).
[18] HANS-BERND KNOOP AND XIN-LONG ZHOU, *The lower estimate for linear positive operators (II)*, Results in Mathematics, **25**, (1994), 316–330.
[19] G. LORENTZ, *Bernstein Polynomials*, Mathematical Expositions, Univ. of Toronto Press 8, 1953.
[20] V. TOTIK, *Approximation by Bernstein polynomials*, Amer. J. Math., **116**, (1994), 995–1018.
[21] E. V. WICKEREN, *Stechkin-Marchaud-type inequalities in connection with Bernstein polynomials*, Constr. Approx., **2**, (1986), 331–337.
[22] P. C. XUN AND D. X. ZHOU, *Rate of convergence for Baskakov operators with Jacobi-weights*, Acta Mathematics Applicatae Sinica, **18**, (1995), 127–139.
[23] DING-XUAN ZHOU, *On a conjecture of Z. Ditzian*, J. Approx. Theory, **69**, (1992), 167–172.

(Received October 19, 2014)

Ivan Gadjev
Department of Mathematics and Informatics
University of Transport
158 Geo Milev str., 1574 Sofia, Bulgaria
e-mail: gadjevivan@hotmail.com