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EMBEDDING OF CLASSES OF FUNCTIONS WITH Λ_φ–BOUNDED VARIATION INTO GENERALIZED LIPSCHITZ CLASSES

HEPING WANG

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Abstract. In this note, we obtain the sufficient and necessary condition for the embedding of the classes $\Lambda_\varphi \mathrm{BV}$ of functions with Λ_φ-bounded variation into the generalized Lipschitz classes H_q^ω, $1 \leqslant q < \infty$.

1. Introduction and main results

Let φ be a strictly increasing convex continuous function on $[0,\infty)$ with $\varphi(0)=0$, and let $\Lambda =: \{\lambda_k\}$ be an increasing sequence of positive numbers such that $\sum\limits_{k=1}^{\infty}\frac{1}{\lambda_k} = +\infty$. We say that a real valued function f on $[a,b]$ is of $\Lambda\varphi$-bounded variation and denoted by $f \in \Lambda\varphi\mathrm{BV}$ if

$$V_{\Lambda_\varphi}(f) := \sup_{\mathscr{I}} \sum_{k=1}^{\infty} \frac{\varphi(|f(I_k)|)}{\lambda_k} := \sup_{\mathscr{I}} \sum_{k=1}^{\infty} \frac{\varphi(|f(b_k)-f(a_k)|)}{\lambda_k} < \infty,$$

where the supremum is taken over all sequences $\mathscr{I} = \{I_k\} = \{[a_k,b_k]\}$ of non-overlapping intervals in $[a,b]$, $f(I_k) = f(b_k) - f(a_k)$. The class $\Lambda\varphi\mathrm{BV}$ is introduced in Schramm and Waterman's paper [11] (see also [7]). In the paper, we suppose that $[a,b] = [0,1]$, and functions in $\Lambda\varphi\mathrm{BV}$ are 1-periodic.

In the case $\varphi(x) = x^p$ $(p \geqslant 1)$, f is said to be of bounded p-Λ-variation variation. The corresponding class $\Lambda\mathrm{BV}^{(p)}$ was introduced in 1980 by Shiba in [12] and called by the Waterman-Shiba class. If $p=1$, f is said to be of Λ-bounded variation, and we denote $f \in \Lambda\mathrm{BV}$. The corresponding class is the well-known Waterman class $\Lambda\mathrm{BV}$.

In the case $\Lambda = \{1\}$, we get the class BV_φ of φ-bounded variation, which was introduced by Young (see [21]). More specifically, when $\varphi(x) = x^p$ $(p \geqslant 1)$, we get the class BV_p which is called the Wiener class. The class BV_1 is the well known class of bounded variation BV.

It is easily seen from the definition that $\Lambda\varphi\mathrm{BV}$ functions are bounded, and the discontinuities of a $\Lambda\varphi\mathrm{BV}$ function are simple and, therefore, at most denumerable.

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The class $\Lambda\varphi BV$ had been studied mainly because of their applicability to the theory of Fourier series and some good approximative properties (see [20, 21, 17, 18, 19, 10, 13]).

Let $\omega(t)$ be a modulus of continuity, i.e., a continuous, subadditive, and increasing function on $[0,+\infty)$ satisfying $\omega(0)=0$. For $1\leqslant q\leqslant\infty$, denote by $H_q^{\omega}\equiv H_q^{\omega(t)}$ the class of 1-periodic functions for which $\|f\|_{H_q^{\omega}} := \|f\|_q + \sup\limits_{t>0}\frac{\omega(f;t)_q}{\omega(t)} < \infty$, where

$$\omega(f;t)_q := \begin{cases} \sup_{0\leqslant h\leqslant t}\left\{\int_0^1 |f(x+h)-f(x)|^q dx\right\}^{\frac{1}{q}}, & 1\leqslant q<\infty, \\ \sup_{0\leqslant h\leqslant t}\sup_{x\in[0,1]}|f(x+h)-f(x)|, & q=\infty \end{cases}$$

is the L_q modulus of continuity of f. If $\omega(t)=t^{\alpha}$, $\alpha\in(0,1]$, then $H_q^{\omega}\equiv H_q^{\alpha}$ coincides with the Lipschitz class $\mathrm{Lip}(\alpha,q)$.

In recent years, much attention is drawn on the relationship of the class $\Lambda_{\varphi}BV$ and the Lipschitz class H_q^{ω}. For $q=\infty$, Medvedeva and Leindler gave sufficient and necessary conditions for the embeddings $H_{\infty}^{\omega}\subset\Lambda BV$ and $H_{\infty}^{\omega}\subset\Lambda_{\varphi}BV$ in [9] and [7]. For $1<q<\infty$ and $\alpha\in(0,1)$, the present author and Lind obtained sufficient and necessary conditions for the embeddings $H_q^{\alpha}\subset\{n^{\beta}\}BV$ $(0<\beta\leqslant 1)$ and $H_q^{\alpha}\subset\Lambda BV$ in [13] and [8]. Finally, the present author obtained a sufficient and necessary condition for the embeddings $H_q^{\omega}\subset\Lambda BV$ in [14] under some weak restriction on ω.

On the other hand, the reverse embedding is also investigated. Sharp estimates of the L_q-modulus of continuity $(1\leqslant q<\infty)$ of a function in terms of its Λ_{φ}-variation were obtained in [16], [5], [6], and [15]. Furthermore, Goginava, Hormozi, etc. gave the necessary and sufficient conditions in [1], [4], and [3] for the inclusion of the class $\Lambda BV^{(p)}$ in the class H_q^{ω} $(1\leqslant p,q<\infty)$.

This note is devoted to investigating the embedding $\Lambda_{\varphi}BV\subset H_q^{\omega}$, $1\leqslant q<\infty$. Our main result can be formulated as follows.

THEOREM 1. *Suppose that $1\leqslant q<\infty$, and ψ is the inverse function of the function φ. Then the embedding $\Lambda_{\varphi}BV\subset H_q^{\omega}$ holds if and only if*

$$\overline{\lim_{n\to\infty}}\frac{1}{n^{1/q}\omega(1/n)}\max_{1\leqslant k\leqslant n} k^{1/q}\psi\Big(\big(\sum_{i=1}^{k}1/\lambda_i\big)^{-1}\Big)<\infty. \tag{1.1}$$

REMARK 1. When $\varphi(t)=t^p$, $1\leqslant p<\infty$, Theorem 1 recedes to Theorem 2.2 in [3]. For the proof of necessity we use the method in [3].

2. Upper estimates for L_q modulus of continuity of functions in $\Lambda_{\varphi}BV$

This section is devoted to investigating the upper estimates for the L_q modulus of continuity of functions in $\Lambda_{\varphi}BV$ in terms of its Λ_{φ}-variation. Our main result of this section can be formulated as follows.

THEOREM 2. *Let ψ be the inverse function of the function φ and $f\in\Lambda_{\varphi}BV$. Then*

(1) for $q=1$, we have

$$\omega(f;1/n)_1 \leqslant \psi\Big(\Big(\sum_{k=1}^{n}\frac{1}{\lambda_k}\Big)^{-1}V_{\Lambda_\varphi}(f)\Big) = \max_{1\leqslant m\leqslant n}\frac{m}{n}\,\psi\Big(\Big(\sum_{k=1}^{m}\frac{1}{\lambda_k}\Big)^{-1}V_{\Lambda_\varphi}(f)\Big); \quad (2.1)$$

(2) for $1<q<\infty$, we have

$$\omega(f;1/n)_q \leqslant 16n^{-1/q}\max_{1\leqslant k\leqslant n} k^{1/q}\psi\Big(\Big(\sum_{i=1}^{k}1/\lambda_i\Big)^{-1}V_{\Lambda_\varphi}(f)\Big). \quad (2.2)$$

REMARK 2. In the case $q=1$, Theorem 2 is essentially given in [15, Theorem 2] in a slightly different form. The authors also showed in [15] that the upper estimate for L_1 modulus of continuity of functions in $\Lambda_\varphi\mathrm{BV}$ is sharp in the sense of order.

Let $f\in\Lambda_\varphi\mathrm{BV}$ and $1\leqslant q<\infty$. We have

$$\begin{aligned}\omega(f;1/n)_q^q &= \sup_{0<h\leqslant 1/n}\int_0^1 |f(x+h)-f(x)|^q\,dx\\ &= \sup_{0<h\leqslant 1/n}\int_0^{1/n}\sum_{k=1}^{n}\Big|f\Big(x+\frac{k-1}{n}+h\Big)-f\Big(x+\frac{k-1}{n}\Big)\Big|^q\,dx.\end{aligned}$$

For $h\leqslant 1/n$ and fixed $x\in[0,1/n]$, the intervals $I_k:=[x+\frac{k-1}{n},x+\frac{k-1}{n}+h]$, $k=1,\ldots,n,$ are non-overlapping intervals. We set $x_k=|f(I_k)|$. We reorder x_k such that

$$x_1\geqslant x_2\geqslant\ldots\geqslant x_n\geqslant 0.$$

Since $f\in\Lambda_\varphi\mathrm{BV}$, we get

$$\sum_{k=1}^{n}\frac{\varphi(x_k)}{\lambda_k}\leqslant V_{\Lambda_\varphi}(f).$$

We put

$$I_{n,q}(f)=\sup\Big\{\Big(\sum_{k=1}^{n}|x_k|^q\Big)^{1/q} : \sum_{k=1}^{n}\frac{\varphi(x_k)}{\lambda_k}\leqslant V_{\Lambda_\varphi}(f)\text{ and }x_1\geqslant x_2\geqslant\ldots\geqslant x_n\geqslant 0\Big\}.$$

It follows that

$$\omega(f;1/n)_q\leqslant\sup_{0<h\leqslant 1/n}\Big(\int_0^{1/n}I_{n,q}(f)^q\,dx\Big)^{1/q}\leqslant n^{-1/q}I_{n,q}(f). \quad (2.3)$$

Let ψ be the inverse function of φ and let $y_k=\varphi(x_k)$. Then ψ is a strictly increasing concave function on $[0,\infty)$ with $\psi(0)=0$,

$$y_1\geqslant y_2\geqslant\ldots\geqslant y_n\geqslant 0,\;\; x_k=\psi(y_k),\; 1\leqslant k\leqslant n,$$

and

$$I_{n,q}(f)=\sup\Big\{\Big(\sum_{k=1}^{n}|\psi(y_k)|^q\Big)^{1/q} : \sum_{k=1}^{n}\frac{y_k}{\lambda_k}\leqslant V_{\Lambda_\varphi}(f)\text{ and }y_1\geqslant y_2\geqslant\ldots\geqslant y_n\geqslant 0\Big\}.$$

Now we estimate $I_{n,q}(f)$ for $1\leqslant q<\infty$. We have

LEMMA 1. *Let ψ be a strictly increasing concave function on $[0,\infty)$ with $\psi(0)=0$, and $q=1$. Then*

$$I_{n,1}(f)=n\,\psi\Big(\Big(\sum_{k=1}^{n}\frac{1}{\lambda_k}\Big)^{-1}V_{\Lambda_\varphi}(f)\Big)=\max_{1\leqslant m\leqslant n} m\,\psi\Big(\Big(\sum_{k=1}^{m}\frac{1}{\lambda_k}\Big)^{-1}V_{\Lambda_\varphi}(f)\Big). \tag{2.4}$$

Proof. Let $\{y_k\}_{k=1}^{n}$ satisfy $y_1\geqslant y_2\geqslant\ldots\geqslant y_n\geqslant 0$ and $\sum_{k=1}^{n}\frac{y_k}{\lambda_k}\leqslant V_{\Lambda_\varphi}(f)$. Using Tchebychef's inequality (see [2, (2.17.1)]), we get

$$\Big(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\lambda_k}\Big)\Big(\frac{1}{n}\sum_{k=1}^{n}|\psi(y_k)|\Big)\leqslant\frac{1}{n}\sum_{k=1}^{n}\frac{|\psi(y_k)|}{\lambda_k}.$$

It follows from the above inequality and the concavity of ψ that

$$\begin{aligned}\sum_{k=1}^{n}|\psi(y_k)|&\leqslant n\Big(\sum_{k=1}^{n}\frac{1}{\lambda_k}\Big)^{-1}\sum_{k=1}^{n}\frac{|\psi(y_k)|}{\lambda_k}\\&\leqslant n\psi\Big(\Big(\sum_{k=1}^{n}\frac{1}{\lambda_k}\Big)^{-1}\sum_{k=1}^{n}\frac{y_k}{\lambda_k}\Big)\\&\leqslant n\psi\Big(\Big(\sum_{k=1}^{n}\frac{1}{\lambda_k}\Big)^{-1}V_{\Lambda_\varphi}(f)\Big),\end{aligned}$$

which implies

$$I_{n,1}(f)\leqslant n\psi\Big(\Big(\sum_{k=1}^{n}\frac{1}{\lambda_k}\Big)^{-1}V_{\Lambda_\varphi}(f)\Big).$$

On the other hand, for $1\leqslant m\leqslant n$ we let $y_1=\ldots=y_m=\big(\sum_{k=1}^{m}\frac{1}{\lambda_k}\big)^{-1}V_{\Lambda_\varphi}(f)$ and $y_{m+1}=\ldots=y_n=0$. Then $\sum_{k=1}^{n}\frac{y_k}{\lambda_k}\leqslant V_{\Lambda_\varphi}(f)$ and

$$I_{n,1}(f)\geqslant\max_{1\leqslant m\leqslant n} m\,\psi\Big(\Big(\sum_{k=1}^{m}\frac{1}{\lambda_k}\Big)^{-1}V_{\Lambda_\varphi}(f)\Big)\geqslant n\,\psi\Big(\Big(\sum_{k=1}^{n}\frac{1}{\lambda_k}\Big)^{-1}V_{\Lambda_\varphi}(f)\Big).$$

This completes the proof of Lemma 1. □

LEMMA 2. *Let ψ be a strictly increasing concave function on $[0,\infty)$ with $\psi(0)=0$, and $1<q<\infty$. Then*

$$I_{n,q}(f)\leqslant 16\max_{1\leqslant k\leqslant n}k^{1/q}\psi\Big(\Big(\sum_{i=1}^{k}1/\lambda_i\Big)^{-1}V_{\Lambda_\varphi}(f)\Big). \tag{2.5}$$

REMARK 3. We conjecture that

$$I_{n,q}(f)=\max_{1\leqslant k\leqslant n}k^{1/q}\psi\Big(\Big(\sum_{i=1}^{k}1/\lambda_i\Big)^{-1}V_{\Lambda_\varphi}(f)\Big).$$

However, we cannot prove it. This conjecture is true in the case $\psi(t)=t^{1/p}$ for $1<p<\infty$ (see [3, Corollary 2.4]).

Proof. Without loss of generality we may assume that $\lambda_1 = 1$. We set

$$V = V_{\Lambda_\varphi}(f) \text{ and } M = \max_{1\leqslant k\leqslant n} k^{1/q}\psi((\sum_{i=1}^{k} 1/\lambda_i)^{-1}V_{\Lambda_\varphi}(f)).$$

Let $\{y_k\}_{k=1}^n$ satisfy $y_1 \geqslant y_2 \geqslant \ldots \geqslant y_n \geqslant 0$ and $\sum_{k=1}^n \frac{y_k}{\lambda_k} \leqslant V$. We set

$$\sigma_s = \{1 \leqslant j \leqslant n \,:\, y_j \geqslant 2^{-s}V\},\ v_s = \#(\sigma_s),\ \text{and } u_s = \#(\sigma_s \setminus \sigma_{s-1}),$$

where $\#A$ denotes the number of elements of the finite set A. Then $\sigma_s \setminus \sigma_{s-1} = \{1 \leqslant j \leqslant n \,:\, 2^{-s}V \leqslant y_j < 2^{-s+1}V\}$. We have

$$\begin{aligned} V \geqslant \sum_{k=1}^{n} \frac{y_k}{\lambda_k} &= \sum_{s=0}^{\infty} \sum_{j\in\sigma_s\setminus\sigma_{s-1}} \frac{y_j}{\lambda_j} \\ &\geqslant \sum_{s=0}^{\infty} 2^{-s}V \sum_{j\in\sigma_s\setminus\sigma_{s-1}} \frac{1}{\lambda_j} \\ &= \sum_{s=0}^{\infty} 2^{-s-1}V \sum_{j\in\sigma_s} \frac{1}{\lambda_j}, \end{aligned}$$

where in the last equality we used the Abel transform $\sum_{k=1}^m a_k b_k = \sum_{k=1}^m (a_k - a_{k+1})(b_1 + \ldots + b_k)$ with $b_{m+1} = 0$ and $m \in \mathbb{N}$. Let s_0 be such that $v_{s_0} > 0$ and $v_{s_0-1} = 0$. It follows that

$$\sum_{s=s_0}^{\infty} 2^{-s-1} \sum_{j=1}^{v_s} \frac{1}{\lambda_j} \leqslant 1.$$

Let s_1 satisfy $2^{s_1+1} < \sum_{k=1}^n \frac{1}{\lambda_k} \leqslant 2^{s_1+2}$. For $s_0 \leqslant s \leqslant s_1$, there exists a m_s, $1 \leqslant m_s \leqslant n$ such that

$$2^{s+1} < \sum_{k=1}^{m_s} \frac{1}{\lambda_k} \leqslant 2^{s+2}.$$

This means that $2^{-s-2} \leqslant (\sum_{k=1}^{m_s} \frac{1}{\lambda_k})^{-1} \leqslant 2^{-s-1}$. Since ψ is an increasing concave function and $\psi(0) = 0$, we get that $\psi(8t) \leqslant 8\psi(t)$ for any $t > 0$. Then we have

$$\begin{aligned} \sum_{k=1}^{n} (\psi(y_k))^q &= \sum_{s=s_0}^{\infty} \sum_{j\in\sigma_s\setminus\sigma_{s-1}} (\psi(y_j))^q \leqslant \sum_{s=s_0}^{\infty} \sum_{j\in\sigma_s\setminus\sigma_{s-1}} (\psi(2^{-s+1}V))^q \\ &\leqslant 8^q \sum_{s=s_0}^{\infty} (\psi(2^{-s-2}V))^q u_s \\ &\leqslant 8^q \Big(\sum_{s=s_0}^{s_1} (\psi((\sum_{k=1}^{m_s} \frac{1}{\lambda_k})^{-1}V))^q v_s + (\psi((\sum_{k=1}^{n} \frac{1}{\lambda_k})^{-1}V))^q \sum_{s=s_1+1}^{\infty} u_s \Big) \\ &\leqslant 8^q \Big(\sum_{s=s_0}^{s_1} (\psi((\sum_{k=1}^{m_s} \frac{1}{\lambda_k})^{-1}V))^q v_s + n(\psi((\sum_{k=1}^{n} \frac{1}{\lambda_k})^{-1}V))^q \Big) \\ &\leqslant 8^q M^q \Big(1 + \sum_{s=s_0}^{s_1} \frac{v_s}{m_s} \Big), \end{aligned}$$

where $u_s = \#(\sigma_s \setminus \sigma_{s-1})$, $v_s = \#(\sigma_s)$ and $M = \max_{1\leqslant k\leqslant n} k^{1/q}\psi((\sum_{i=1}^k 1/\lambda_i)^{-1}V)$. We note that

$$1 \geqslant \sum_{s=s_0}^{\infty} 2^{-s-1} \sum_{j=1}^{v_s} \frac{1}{\lambda_j} \geqslant \sum_{s=s_0}^{s_1} \Big(\sum_{i=1}^{m_s} 1/\lambda_i\Big)^{-1} \Big(\sum_{j=1}^{v_s} \frac{1}{\lambda_j}\Big),$$

and for $k \leqslant m$,

$$\frac{k}{m} \leqslant \Big(\sum_{i=1}^{m} 1/\lambda_i\Big)^{-1} \Big(\sum_{j=1}^{k} \frac{1}{\lambda_j}\Big).$$

Clearly, $v_s \leqslant m_s$. We have

$$\sum_{s=s_0}^{s_1} \frac{v_s}{m_s} \leqslant \sum_{s=s_0}^{s_1} \Big(\sum_{i=1}^{m_s} 1/\lambda_i\Big)^{-1} \Big(\sum_{j=1}^{v_s} \frac{1}{\lambda_j}\Big) \leqslant 1.$$

Hence, we obtain that

$$\Big(\sum_{k=1}^{n} (\psi(y_k))^q\Big)^{1/q} \leqslant 2^{1/q} \cdot (8M) \leqslant 16M.$$

Lemma 2 is proved. □

Proof of Theorem 2. Theorem 2 follows from (2.3), (2.4), and (2.5) immediately. □

3. Proof of Theorem 1

Proof of Theorem 1.

Sufficiency. Suppose that (1.1) holds. Then there exists a constant $C > 0$ such that for any n, we have

$$\max_{1\leqslant k\leqslant n} k^{1/q}\psi((\sum_{i=1}^{k} 1/\lambda_i)^{-1}) \leqslant Cn^{1/q}\omega(1/n).$$

We shall show that $\Lambda_\varphi\mathrm{BV} \subset H_q^\omega$. Since ψ is a strictly increasing concave function with $\psi(0) = 0$, we get $\psi(at) \leqslant (1+a)\psi(t)$ for any $a, t > 0$. For any $f \in \Lambda_\varphi\mathrm{BV}$, by (2.1) and (2.2) we have

$$\begin{aligned}\omega(f;1/n)_q &\leqslant 16n^{-1/q} \max_{1\leqslant k\leqslant n} k^{1/q}\psi((\sum_{i=1}^{k} 1/\lambda_i)^{-1} V_{\Lambda_\varphi}(f)) \\ &\leqslant 16(1+V_{\Lambda_\varphi}(f))n^{-1/q} \max_{1\leqslant k\leqslant n} k^{1/q}\psi((\sum_{i=1}^{k} 1/\lambda_i)^{-1}) \\ &\leqslant 16C(1+V_{\Lambda_\varphi}(f))\omega(1/n),\end{aligned}$$

which implies that $f \in H_q^\omega$. Hence, we have $\Lambda_\varphi\mathrm{BV} \subset H_q^\omega$.

Necessity. It suffices to prove that the embedding $\Lambda_\varphi \mathrm{BV} \subset H_q^\omega$ does not hold if

$$\overline{\lim_{n\to\infty}} \frac{1}{n^{1/q}\omega(1/n)} \max_{1\leqslant k\leqslant n} k^{1/q}\psi((\sum_{i=1}^{k} 1/\lambda_i)^{-1}) = \infty. \tag{3.1}$$

Suppose that (3.1) holds. Then there exist sequences n_k and m_k such that

$$n_k \geqslant 2^{k+2}, \tag{3.2}$$

$$n_k \geqslant m_k \geqslant 1, \tag{3.3}$$

$$\frac{1}{\omega(\frac{1}{n_k})}\left(\frac{m_k}{n_k}\right)^{1/q}\psi((\sum_{i=1}^{m_k} 1/\lambda_i)^{-1}) \geqslant 2^{4k}, \tag{3.4}$$

where

$$\max_{1\leqslant j\leqslant n_k} j^{1/q}\psi((\sum_{i=1}^{j} 1/\lambda_i)^{-1}) = m_k^{1/q}\psi((\sum_{i=1}^{m_k} 1/\lambda_i)^{-1}).$$

We shall construct a 1-periodic function g such that $g \in \Lambda_\varphi \mathrm{BV}$ and $g \notin H_q^\omega$, which contradicts the embedding $\Lambda_\varphi \mathrm{BV} \subset H_q^\omega$. We set

$$a_k = \psi((\sum_{i=1}^{m_k} 1/\lambda_i)^{-1})2^{-k}, \quad k = 1,2,\ldots. \tag{3.5}$$

Consider

$$g_k(y) = \begin{cases} a_k, & y \in [\frac{1}{2^k} + \frac{2j-2}{n_k}, \frac{1}{2^k} + \frac{2j-1}{n_k}];\ 1 \leqslant j \leqslant N_k, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$s_k = \max\left\{ j \in \mathbb{N} \mid 2j \leqslant \frac{n_k}{2^k} + 1 \right\}, \tag{3.6}$$

and

$$N_k = \min\{m_k, s_k\}. \tag{3.7}$$

Hence, applying (3.6), the fact that $2(s_k+1) > \frac{n_k}{2^k} + 1$, and (3.2), we have

$$2^{-k-1} < \frac{2s_k - 1}{n_k} \leqslant 2^{-k}, \tag{3.8}$$

which means that the support of the function g_k is in $[2^{-k}, 2^{-k+1})$. Hence, the functions g_k have disjoint support, and correspondingly,

$$V_{\Lambda_\varphi}(g) \leqslant \sum_{k=1}^{\infty} V_{\Lambda_\varphi}(g_k). \tag{3.9}$$

Since φ is convex and $\varphi(0) = 0$, we get that $\varphi(tx) \leqslant t\varphi(x)$ for $t \in (0,1]$ and $x > 0$, and hence,

$$\varphi(a_k) \leqslant 2^{-k}\varphi(\psi((\sum_{i=1}^{m_k} 1/\lambda_i)^{-1})) = 2^{-k}(\sum_{i=1}^{m_k} 1/\lambda_i)^{-1}.$$

It follows from (3.7) that

$$V_{\Lambda_\varphi}(g_k) \leqslant \sum_{j=1}^{2N_k} \frac{\varphi(a_k)}{\lambda_j} \leqslant 2\varphi(a_k)\sum_{j=1}^{N_k}\frac{1}{\lambda_j} \leqslant 2\varphi(a_k)\sum_{j=1}^{m_k}\frac{1}{\lambda_j} \leqslant 2^{-k+1},$$

which combining with (3.9), deduces that

$$V_{\Lambda_\varphi}(g) \leqslant \sum_{k=1}^{\infty} 2^{-k+1} \leqslant 2.$$

This implies that $f \in \Lambda_\varphi BV$.

Next we show that $g \notin H_q^\omega$. We observe that $|g(x+\frac{1}{n_k})-g(x)| = a_k$ for $x \in [\frac{1}{2^k}, \frac{1}{2^k}+\frac{2N_k-1}{n_k}]$. Then

$$\begin{aligned}
\omega(g;\frac{1}{n_k})_q^q &= \sup_{0<\gamma\leqslant\frac{1}{n_k}} \int_0^1 |g(x+\gamma)-g(x)|^q\,dx \\
&\geqslant \int_0^1 \left|g\left(x+\frac{1}{n_k}\right)-g(x)\right|^q dx \\
&\geqslant \int_{\frac{1}{2^k}}^{\frac{1}{2^k}+\frac{2N_k-1}{n_k}} \left|g\left(x+\frac{1}{n_k}\right)-g(x)\right|^q dx \\
&= \frac{2N_k-1}{n_k}a_k^q.
\end{aligned} \tag{3.10}$$

If $N_k = s_k$, then by (3.8), (3.3), and (3.5) we have

$$\left(\frac{2N_k-1}{n_k}\right)^{1/q} a_k \geqslant 2^{-(k+1)/q}2^{-k}\psi((\sum_{i=1}^{m_k}1/\lambda_i)^{-1}),$$

and if $N_k = m_k$, then by (3.5) we get

$$\left(\frac{2N_k-1}{n_k}\right)^{1/q} a_k \geqslant 2^{-k}\left(\frac{m_k}{n_k}\right)^{1/q}\psi((\sum_{i=1}^{m_k}1/\lambda_i)^{-1}).$$

In both cases, we have

$$\left(\frac{2N_k-1}{n_k}\right)^{1/q} a_k \geqslant 2^{-(k+1)/q}2^{-k}\left(\frac{m_k}{n_k}\right)^{1/q}\psi((\sum_{i=1}^{m_k}1/\lambda_i)^{-1}). \tag{3.11}$$

It follows from (3.10), (3.11), and (3.4) that

$$\begin{aligned}
\frac{\omega(g;\frac{1}{n_k})_q}{\omega(\frac{1}{n_k})} &\geqslant 2^{-(k+1)/q}2^{-k}\left(\frac{m_k}{n_k}\right)^{1/q}\psi((\sum_{i=1}^{m_k}1/\lambda_i)^{-1})(\omega(\frac{1}{n_k}))^{-1} \\
&\geqslant 2^{-(k+1)/q}2^{-k}2^{4k} \geqslant 2^k \to +\infty, \quad \text{as } k\to\infty,
\end{aligned}$$

which shows that $g \notin H_q^\omega$.

The proof of Theorem 1 is finished. $\square$

REFERENCES

[1] U. GOGINAVA, *On the embedding of Waterman class in the class* H_p^ω, Ukrainian Math. J., **57**, 11 (2005), 1818–1824.

[2] G. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities* (Second Edition), Cambridge University Press, Cambridge, 1952.

[3] M. HORMOZI, *Inclusion of* $\Lambda BV^{(p)}$ *spaces in the classes* H_q^ω, J. Math. Anal. Appl., **404**, 2 (2013), 195–200.

[4] M. HORMOZI, A. A. LEDARI AND F. PRUS-WIŚNIOWSKI, *On* p-Λ*-bounded variation*, Bull. Iranian. Math. Soc., **37**, 4 (2011), 35–49.

[5] YU. E. KUPRIKOV, *On moduli of continuity of functions in Waterman classes*, Moscow Univ. Math. Bull., **52**, 5 (1997), 46–49.

[6] ZHONGKAI LI AND HEPING WANG, *Estimates of* L^p *modulus of continuity of functions of* Λ*-bounded variation and applications in Fourier series*, Applicable Analysis, **90**, 3–4 (2011), 475–482.

[7] L. LEINDLER, *A note on embedding of classes* H^ω, Anal. Math., **27**, (2001), 71–76.

[8] M. LIND, *On functions of bounded* Λ*-variation and integral smoothness*, to appear in Forum Math.. Also see arXiv.org/math. CA/1212.5402v1 (2012).

[9] M. V. MEDVEDEVA, *On embedding of the classes Hôz*, Math. Notes, **64**, (1998), 616–621.

[10] S. PERLMAN AND D. WATERMAN, *Some remarks on functions of* Λ*-bounded variation*, Proc. Amer. Math. Soc., **74**, 1 (1979), 113–123.

[11] M. SCHRAMM AND D. WATERMAN, *Absolute convergence of Fourier series of functions of* $\Lambda BV^{(p)}$ *and* $\phi\Lambda BV$, Acta Math. Acad. Sci. Hungar., **40**, 3–4 (1982), 273–276.

[12] M. SHIBA, *On the absolute convergence of Fourier series of functions of class* $\Lambda BV^{(p)}$, Sci. Rep. Fukushima Univ., **30**, (1980), 7–10.

[13] HEPING WANG, *Embedding of Lipschitz classes into classes of functions of* Λ*-bounded variation*, J. Math. Anal. Appl., **354**, (2009), 698–703.

[14] HEPING WANG, *Embedding of generalized Lipschitz classes into classes of functions with* Λ*-bounded variation*, preprint.

[15] HEPING WANG AND ZHAOYANG WU, *Estimates of* L_p *modulus of continuity of generalized bounded variation classes*, J. Funct. Spaces, (2014), Article ID 201801, 13 pages.

[16] SILEI WANG, *Some properties of the functions of* Λ*-bounded variation*, Sci. Sinica Ser. A, **25**, 2 (1982), 149–160.

[17] D. WATERMAN, *On convergence of Fourier series of functions of generalized bounded variation*, Studia Math., **44**, (1972), 107–117.

[18] D. WATERMAN, *On the summability of Fourier series of functions of* Λ*-bounded variation*, Studia Math., **55**, (1976), 87–95.

[19] D. WATERMAN, *On* Λ*-bounded variation*, Studia Math., **57**, (1976), 33–45.

[20] N. WIENER, *The quadratic variation of a function and its Fourier coefficients*, Massachusetts J. Math., **3**, (1924), 72–94.

[21] L. C. YOUNG, *Sur une généralization de la notion de variation de Wiener et sur la convergence des séries de Fourier*, C. R. Acad. Sci. Paris, **204**, (1937), 470–472.

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Heping Wang
School of Mathematical Sciences
Capital Normal University
Beijing 100048, China
e-mail: wanghp@cnu.edu.cn.