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THE STECHKIN INEQUALITY FOR FOURIER MULTIPLIERS ON VARIABLE LEBESGUE SPACES

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Abstract. We prove the Stechkin inequality for Fourier multipliers on variable Lebesgue spaces under some natural assumptions on variable exponents.

1. Introduction

Let $p\colon \mathbb{R}\to[1,\infty]$ be a measurable a.e. finite function. By $L^{p(\cdot)}(\mathbb{R})$ we denote the set of all complex-valued functions f on $\mathbb{R}$ such that

$$I_{p(\cdot)}(f/\lambda) := \int_{\mathbb{R}} |f(x)/\lambda|^{p(x)} dx < \infty$$

for some $\lambda > 0$. This set becomes a Banach function space when equipped with the norm

$$\|f\|_{p(\cdot)} := \inf\{\lambda > 0 : I_{p(\cdot)}(f/\lambda) \leqslant 1\}.$$

It is easy to see that if p is constant, then $L^{p(\cdot)}(\mathbb{R})$ is nothing but the standard Lebesgue space $L^p(\mathbb{R})$. The space $L^{p(\cdot)}(\mathbb{R})$ is referred to as a variable Lebesgue space.

We will always suppose that

$$1 < p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}} p(x), \quad \operatorname*{ess\,sup}_{x\in\mathbb{R}} p(x) =: p_+ < \infty. \tag{1}$$

Under these conditions, the space $L^{p(\cdot)}(\mathbb{R})$ is separable and reflexive, and the set $L_0^\infty(\mathbb{R})$ of all bounded compactly supported functions is dense $L^{p(\cdot)}(\mathbb{R})$ (see, e.g., [3, Chap. 2] or [5, Chap. 3]). We will denote by $\mathscr{L}(L^{p(\cdot)})$ the Banach algebra of all bounded linear operators on $L^{p(\cdot)}(\mathbb{R})$.

Let $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the Fourier transform,

$$(Ff)(x) := \int_{\mathbb{R}} f(t)e^{itx}\,dt, \quad x \in \mathbb{R},$$

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and let $F^{-1}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the inverse of F,

$$(F^{-1}g)(t) = \frac{1}{2\pi} \int_{\mathbb{R}} g(x) e^{-itx}\, dx, \quad t \in \mathbb{R}.$$

A function $a \in L^\infty(\mathbb{R})$ is called a Fourier multiplier on $L^{p(\cdot)}(\mathbb{R})$ if the map

$$f \mapsto F^{-1} a F f$$

maps $L^2(\mathbb{R}) \cap L^{p(\cdot)}(\mathbb{R})$ into itself and extends to a bounded operator on $L^{p(\cdot)}(\mathbb{R})$ (notice that $L^2(\mathbb{R}) \cap L^{p(\cdot)}(\mathbb{R})$ is dense in $L^{p(\cdot)}(\mathbb{R})$). The latter operator is then denoted by $W^0(a)$.

Let a be a complex-valued function of bounded total variation $V(a)$ on $\mathbb{R}$ where

$$V(a) := \sup \left\{ \sum_{k=1}^{n} |a(x_k) - a(x_{k-1})| : -\infty < x_0 < x_1 < \ldots < x_n < +\infty, n \in \mathbb{N} \right\}.$$

Hence at every point $x \in \dot{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ the one-sided limits

$$a(x \pm 0) = \lim_{t \to x^{\pm}} a(t)$$

exist, where $a(\pm\infty) = a(\infty \mp 0)$, and the set of discontinuities of a is at most countable (see, e.g., [17, Chap. VIII, Sections 3 and 9]).

For $f \in L^1_{\text{loc}}(\mathbb{R})$, let S be the Cauchy singular integral operator given by

$$(Sf)(x) := \frac{1}{\pi i} \int_{\mathbb{R}} \frac{f(t)}{t - x}, \quad x \in \mathbb{R},$$

where the integral is understood in the principal value sense. By the Marcel Riesz theorem, it is bounded on every standard Lebesgue space $L^p(\mathbb{R})$, $1 < p < \infty$, moreover, its norm is known (see, e.g., [10, Chap. 13, Theorem 1.3]):

$$\|S\|_{\mathscr{L}(L^p)} = \cot\left(\frac{\pi}{2\max(p,q)}\right), \quad \text{where} \quad q := \frac{p}{p-1}.$$

The following theorem provides a simple sufficient condition for the boundedness of the operator $W^0(a)$ on standard Lebesgue spaces.

THEOREM 1. (Stechkin, 1950) *Let* $1 < p < \infty$. *If* a *has a finite total variation* $V(a)$, *then the convolution operator* $W^0(a)$ *is bounded on the standard Lebesgue space* $L^p(\mathbb{R})$ *and*

$$\|W^0(a)\|_{\mathscr{L}(L^p)} \leqslant \|S\|_{\mathscr{L}(L^p)} (\|a\|_\infty + V(a)). \tag{2}$$

The discrete version of this theorem was obtained by Stechkin [18]. Its proof is also contained in [6, Theorem 20.7]. Another proof was suggested by Matsaev and published in [8, Chap. XIV, Theorem 1.2] (see also [9, Chap. 5, Theorem 2.2] and [2, Theorem 1.2]). The proof of Theorem 1, in the form stated here, is contained in

Duduchava's book [7, Theorem 2.11]. For its generalization to the case of standard Lebesgue spaces with Muckenhoupt weights, see [1, Theorem 17.1].

Inequality (2) is usually called the Stechkin inequality. The aim of this note is to extend Theorem 1 to the setting of variable Lebesgue spaces. To formulate our result, we will need the class of variable exponents $\mathscr{B}_M(\mathbb{R})$ related to the Hardy-Littlewood maximal operator.

Given $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, the Hardy-Littlewood maximal operator is defined by

$$(Mf)(x) := \sup_{J \ni x} \frac{1}{|J|} \int_J |f(t)|\, dt, \quad x \in \mathbb{R},$$

where the supremum is taken over all finite intervals J containing x. Here $|J|$ denotes the length of the interval $J \subset \mathbb{R}$.

By $\mathscr{B}_M(\mathbb{R})$ denote the set of all measurable functions $p : \mathbb{R} \to [1, \infty]$ such that (1) holds and the Hardy-Littlewood maximal operator is bounded on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R})$. To provide a simple sufficient conditions guaranteeing that $p \in \mathscr{B}_M(\mathbb{R})$, we need the following definition. Given a function $r : \mathbb{R} \to \mathbb{R}$, one says that r is locally log-Hölder continuous if there exists a constant $C_0 > 0$ such that

$$|r(x) - r(y)| \leqslant \frac{C_0}{-\log|x-y|}$$

for all $x, y \in \mathbb{R}$ such that $|x - y| < 1/2$. One says that $r : \mathbb{R} \to \mathbb{R}$ is log-Hölder continuous at infinity if there exist constants C_∞ and r_∞ such that for all $x \in \mathbb{R}$,

$$|r(x) - r_\infty| \leqslant \frac{C_\infty}{\log(e + |x|)}.$$

The class of functions $r : \mathbb{R} \to \mathbb{R}$ that are simultaneously locally log-Hölder continuous and log-Hölder continuous at infinity is denoted by $LH(\mathbb{R})$. From [3, Proposition 2.3 and Theorem 3.16] we obtain that if $p \in LH(\mathbb{R})$ satisfies (1), then $p \in \mathscr{B}_M(\mathbb{R})$. Although the latter result provides a nice sufficient condition for the boundedness of the Hardy-Littlewood maximal operator on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R})$, it is not necessary. Notice that all functions in $LH(\mathbb{R})$ are continuous and have limits at infinity. Lerner [16] (see also [3, Example 4.68]) proved that if $p_0 > 1$ and $\mu \in \mathbb{R}$ is sufficiently close to zero, then the following variable exponent

$$p(x) = p_0 + \mu \sin(\log\log(1 + \max\{|x|, 1/|x|\})), \quad x \neq 0,$$

belongs to $\mathscr{B}_M(\mathbb{R})$. It is clear that the function p does not have limits at zero or infinity. We refer to the recent monographs [3, 5] for further discussions concerning the fascinating and still mysterious class $\mathscr{B}_M(\mathbb{R})$.

Since the Cauchy singular integral operator S is a Calderón-Zygmund operator, from [3, Theorem 5.39] or [5, Corollary 6.3.10] we obtain that if $p \in \mathscr{B}_M(\mathbb{R})$, then S is bounded on $L^{p(\cdot)}(\mathbb{R})$. Moreover, if S is bounded on $L^{p(\cdot)}(\mathbb{R})$, then (1) is fulfilled, as it is shown in [3, Theorem 5.42].

THEOREM 2. (Main result) *Let* $p \in \mathscr{B}_M(\mathbb{R})$. *If* a *has a finite total variation* $V(a)$, *then the convolution operator* $W^0(a)$ *is bounded on the variable Lebesgue space* $L^{p(\cdot)}(\mathbb{R})$ *and*

$$\|W^0(a)\|_{\mathscr{L}(L^{p(\cdot)})} \leqslant \|S\|_{\mathscr{L}(L^{p(\cdot)})}(\|a\|_\infty + V(a)). \tag{3}$$

Notice that if we do not require that the constant on the right hand side of (3) is equal to the norm of the operator S on $L^{p(\cdot)}(\mathbb{R})$, then a version of the above result (with a different constant) can be obtained from [11, Corollary 5.6]. However, the proofs in [11] involve deep results related to the celebrated Carleson-Hunt theorem, which can be avoided in the much simpler proof presented below.

We also refer to [4, Section 2.5] and [14, Theorems 5.1–5.3], [15, Theorems 4.5–4.11] for other results on Fourier multipliers on variable Lebesgue spaces.

The proof of Theorem 2 is based on three main ingredients collected in Section 2: the approximation of an arbitrary function of finite total variation a by a sequence $\{a_n\}_{n=1}^\infty$ of piecewise constant functions in the norm of $L^\infty(\mathbb{R})$; the interpolation theorem of Riesz-Thorin type for variable Lebesgue spaces; and a remarkable fact due to Diening saying that each exponent $p \in \mathscr{B}_M(\mathbb{R})$ can be written in the form

$$\frac{1}{p(x)} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1(x)}, \quad x \in \mathbb{R},$$

for some constants $p_0 \in (1,\infty)$, $\theta \in (0,1)$, and another variable exponent $p_1 \in \mathscr{B}_M(\mathbb{R})$.

Theorem 2 will be proved in Section 3 following the main lines of the proof of [1, Theorem 17.1]. For piecewise constant functions a_n, the proof of inequality (3) is straightforward. The proof in the general case is developed by passing to the limit. The nontrivial step consists in proving that $\{W^0(a_n)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathscr{L}(L^{p(\cdot)})$. This is done by applying the interpolation theorem two times: first with $L^{p(\cdot)}(\mathbb{R})$ as the interpolation space between $L^{p_0}(\mathbb{R})$ and $L^{p_1(\cdot)}(\mathbb{R})$ and then with $L^{p_0}(\mathbb{R})$ as the interpolation space between $L^2(\mathbb{R})$ and some standard Lebesgue space $L^q(\mathbb{R})$. This trick will reduce the problem to the proof of that $\{W^0(a_n)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathscr{L}(L^2)$, but the latter is granted because the norm of $W^0(a_n) - W^0(a_m)$ on $L^2(\mathbb{R})$ coincides with the norm of $a_n - a_m$ in $L^\infty(\mathbb{R})$.

2. Preliminaries

2.1. Approximation by piecewise constant functions

LEMMA 3. ([7, Lemma 2.10]) *If* a *has a finite total variation* $V(a)$, *then there exists a sequence of piecewise constant functions* a_n *of finite total variation* $V(a_n)$ *such that*

$$\lim_{n\to\infty} \|a_n - a\|_\infty = 0, \quad \sup_{n\in\mathbb{N}} V(a_n) \leqslant V(a). \tag{4}$$

2.2. Interpolation

We will need the following interpolation theorem for variable Lebesgue spaces.

THEOREM 4. ([5, Corollary 7.1.4]) *Let* $p_j : \mathbb{R} \to [1,\infty]$, $j = 0,1$, *be a.e. finite measurable functions, and let* $p_\theta : \mathbb{R} \to [1,\infty]$ *be defined for* $\theta \in [0,1]$ *by*

$$\frac{1}{p_\theta(x)} = \frac{\theta}{p_0(x)} + \frac{1-\theta}{p_1(x)}, \quad x \in \mathbb{R}.$$

Suppose A *is a linear operator defined on* $L^{p_0(\cdot)}(\mathbb{R}) + L^{p_1(\cdot)}(\mathbb{R})$. *If* $A \in \mathscr{L}(L^{p_j(\cdot)}(\mathbb{R}))$ *for* $j = 0,1$, *then* $A \in \mathscr{L}(L^{p_\theta(\cdot)}(\mathbb{R}))$ *for all* $\theta \in [0,1]$ *and*

$$\|A\|_{\mathscr{L}(L^{p_\theta(\cdot)})} \leqslant 4\|A\|^{\theta}_{\mathscr{L}(L^{p_0(\cdot)})}\|A\|^{1-\theta}_{\mathscr{L}(L^{p_1(\cdot)})}. \tag{5}$$

If p_j, $j = 1,2$, are constant, then the above result is the classical Riesz-Thorin interpolation theorem, and inequality (5) holds with the interpolation constant 1 in the place of 4.

2.3. A property of exponents in $\mathscr{B}_M(\mathbb{R})$

The following property of the class $\mathscr{B}_M(\mathbb{R})$ was communicated to the authors of [12] by Diening.

THEOREM 5. ([12, Theorem 4.1]) *If* $p \in \mathscr{B}_M(\mathbb{R})$, *then there exist two constants* $p_0 \in (1,\infty)$, $\theta \in (0,1)$, *and a variable exponent* $p_1 \in \mathscr{B}_M(\mathbb{R})$ *such that*

$$\frac{1}{p(x)} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1(x)}, \quad x \in \mathbb{R}. \tag{6}$$

3. Proof of the main result

3.1. Two lemmas on the operator S

As usual, we denote by I the identity operator. From [13, Theorem 3.8(a)] and [5, Theorem 5.7.2] we extract the following.

LEMMA 6. *If* $p \in \mathscr{B}_M(\mathbb{R})$, *then* $S^2 = I$ *on* $L^{p(\cdot)}(\mathbb{R})$.

The next statement explains why convolution operators and the operator S are closely related. Let χ_J denote the characteristic function of an interval $J \subset \mathbb{R}$. For $\lambda, x \in \mathbb{R}$, put $e_\lambda(x) := e^{i\lambda x}$.

LEMMA 7. *Let* $p \in \mathscr{B}_M(\mathbb{R})$. *If* $\lambda \in \mathbb{R}$, *then*

$$W^0(\chi_{(\lambda,\infty)}) = (I - e_{-\lambda} S e_\lambda I)/2 \tag{7}$$

on $L^{p(\cdot)}(\mathbb{R})$ *and*

$$\|W^0(\chi_{(\lambda,\infty)})\|_{\mathscr{L}(L^{p(\cdot)})} \leqslant \|S\|_{\mathscr{L}(L^{p(\cdot)})}. \tag{8}$$

Proof. Let $a(x) = \chi_{(\lambda,\infty)}(x) = (1+\operatorname{sgn}(x-\lambda))/2$ for $x \in \mathbb{R}$. By [1, Example 1.18], for every $f \in L^2(\mathbb{R})$,

$$W^0(a)f = (f - e_{-\lambda}Se_{\lambda}f)/2.$$

Since $L^2(\mathbb{R}) \cap L^{p(\cdot)}(\mathbb{R})$ is dense in $L^{p(\cdot)}(\mathbb{R})$ and the operators S and $e_{\pm\lambda}I$ of multiplication by $e_{\pm\lambda}$ are bounded on $L^{p(\cdot)}(\mathbb{R})$, from the above identity we get (7). Obviously, $\|e_{\pm\lambda}I\|_{\mathscr{L}(L^{p(\cdot)})} = 1$. From Lemma 6 we deduce that $1 \leqslant \|S\|_{\mathscr{L}(L^{p(\cdot)})}$. Thus, from (7) we obtain $\|W(a)\|_{\mathscr{L}(L^{p(\cdot)})} \leqslant (1+\|S\|_{\mathscr{L}(L^{p(\cdot)})})/2 \leqslant \|S\|_{\mathscr{L}(L^{p(\cdot)})}$, that is, (8) holds. □

3.2. The Stechkin inequality for piecewise constant functions

LEMMA 8. *Let $p \in \mathscr{B}_M(\mathbb{R})$. If a is a piecewise constant function, then* (3) *holds.*

Proof. Let a be the piecewise constant function. Then there exist a partition

$$-\infty = \lambda_0 < \lambda_1 < \ldots < \lambda_n < \lambda_{n+1} = +\infty$$

and constants $\alpha_0, \ldots, \alpha_n \in \mathbb{C}$ such that

$$a = \sum_{k=0}^{n} \alpha_k \chi_{(\lambda_k,\lambda_{k+1})}.$$

Since $\chi_{(\lambda_0,\lambda_1)} = 1 - \chi_{(\lambda_1,\infty)}$, $\chi_{(\lambda_n,\lambda_{n+1})} = \chi_{(\lambda_n,\infty)}$, and

$$\chi_{(\lambda_{k-1},\lambda_k)} = \chi_{(\lambda_{k-1},\infty)} - \chi_{(\lambda_k,\infty)}, \quad k \in \{1,\ldots,n-1\},$$

we can rewrite a as follows:

$$a = \alpha_0 + \sum_{k=1}^{n} (\alpha_k - \alpha_{k-1})\chi_{(\lambda_k,\infty)}.$$

By Lemma 7,

$$\begin{aligned} \|W^0(a)\|_{\mathscr{L}(L^{p(\cdot)})} &\leqslant |\alpha_0| + \sum_{k=1}^{n} |\alpha_k - \alpha_{k-1}| \, \|S\|_{\mathscr{L}(L^{p(\cdot)})} \\ &\leqslant \left(|\alpha_0| + \sum_{k=1}^{n} |\alpha_k - \alpha_{k-1}| \right) \|S\|_{\mathscr{L}(L^{p(\cdot)})} \\ &\leqslant (\|a\|_\infty + V(a))\|S\|_{\mathscr{L}(L^{p(\cdot)})} \end{aligned}$$

because $1 \leqslant \|S\|_{\mathscr{L}(L^{p(\cdot)})}$ in view of Lemma 6, $|\alpha_0| \leqslant \|a\|_\infty$, and

$$\sum_{k=1}^{n} |\alpha_k - \alpha_{k-1}| \leqslant V(a).$$

Thus, (3) holds. □

3.3. Proof of Theorem 2

Let $a \in L^\infty(\mathbb{R})$ be an arbitrary function such that $V(a) < \infty$. Then, by Lemma 3, there exists a sequence $\{a_n\}_{n=1}^\infty$ of piecewise constant functions such that (4) is fulfilled. Let us show that $\{W^0(a_n)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathscr{L}(L^{p(\cdot)})$.

From Theorem 5 it follows that there exist two constants $p_0 \in (1,\infty)$, $\theta \in (0,1)$, and a variable exponent $p_1 \in \mathscr{B}_M(\mathbb{R})$ such that (6) holds. If $2 \leqslant p_0 < \infty$, then take $q \in (p_0,\infty)$. If $1 < p_0 < 2$, then take $q \in (1,p_0)$. In both cases choose η such that

$$\frac{1}{p_0} = \frac{\eta}{2} + \frac{1-\eta}{q}. \tag{9}$$

Then, as it is easily seen, in both cases

$$\eta = \frac{2p_0 - 2q}{2p_0 - p_0 q} \in (0,1].$$

Since $p_1, q \in \mathscr{B}_M(\mathbb{R})$, by Lemma 8,

$$\|W^0(a_n - a_m)\|_{\mathscr{L}(L^{p_1(\cdot)})} \leqslant \|S\|_{\mathscr{L}(L^{p_1(\cdot)})}\big(\|a_n - a_m\|_\infty + V(a_n - a_m)\big), \tag{10}$$

$$\|W^0(a_n - a_m)\|_{\mathscr{L}(L^q)} \leqslant \|S\|_{\mathscr{L}(L^q)}\big(\|a_n - a_m\|_\infty + V(a_n - a_m)\big). \tag{11}$$

On the other hand, it is well known that

$$\|W^0(a_n - a_m)\|_{\mathscr{L}(L^2)} = \|a_n - a_m\|_\infty. \tag{12}$$

Thus, the operator $W^0(a_n - a_m)$ is bounded on the spaces $L^{p_1(\cdot)}(\mathbb{R})$, $L^q(\mathbb{R})$, and $L^2(\mathbb{R})$. By the Riesz-Thorin interpolation theorem, taking into account (9) and (12), we obtain

$$\|W^0(a_n - a_m)\|_{\mathscr{L}(L^{p_0})} \leqslant \|a_n - a_m\|_\infty^\eta \|W^0(a_n - a_m)\|_{\mathscr{L}(L^q)}^{1-\eta}. \tag{13}$$

On the other hand, from equality (6) and Theorem 4 it follows that

$$\|W^0(a_n - a_m)\|_{\mathscr{L}(L^{p(\cdot)})} \leqslant 4\|W^0(a_n - a_m)\|_{\mathscr{L}(L^{p_0})}^\theta \|W^0(a_n - a_m)\|_{\mathscr{L}(L^{p_1(\cdot)})}^{1-\theta}. \tag{14}$$

Combining (13)–(14) with (10)–(11), we arrive at

$$\begin{aligned}\|W^0(a_n - a_m)\|_{\mathscr{L}(L^{p(\cdot)})} \leqslant& 4\left(\|a_n - a_m\|_\infty^\eta \|W^0(a_n - a_m)\|_{\mathscr{L}(L^q)}^{1-\eta}\right)^\theta \\ &\times \|W^0(a_n - a_m)\|_{\mathscr{L}(L^{p_1(\cdot)})}^{1-\theta} \\ \leqslant& C\|a_n - a_m\|_\infty^\varphi \big(\|a_n - a_m\|_\infty + V(a_n - a_m)\big)^\psi \end{aligned} \tag{15}$$

with

$$C := 4\|S\|_{\mathscr{L}(L^q)}^{\theta(1-\eta)} \|S\|_{\mathscr{L}(L^{p_1(\cdot)})}^{1-\theta}, \quad \varphi := \theta\eta \in (0,1), \quad \psi := \theta(1-\eta) + (1-\theta) \in (0,1).$$

From (4) it follows that

$$V(a_n - a_m) \leqslant V(a_n) + V(a_m) \leqslant 2V(a) \tag{16}$$

and that $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^{\infty}(\mathbb{R})$. From (15)–(16) we obtain

$$\|W^0(a_n) - W^0(a_m)\|_{\mathscr{L}(L^{p(\cdot)})} \leqslant C\|a_n - a_m\|_{\infty}^{\varphi}\big(\|a_n - a_m\|_{\infty} + 2V(a)\big)^{\psi},$$

whence $\{W^0(a_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathscr{L}(L^{p(\cdot)})$.

Thus, the sequence $\{W^0(a_n)\}_{n=1}^{\infty}$ has a limit A in $\mathscr{L}(L^{p(\cdot)})$. Since

$$\|W^0(a_n) - W^0(a)\|_{\mathscr{L}(L^2)} = \|a_n - a\|_{\infty} = o(1) \quad \text{as} \quad n \to \infty,$$

we see that $W^0(a)f = Af$ for all $f \in L^2(\mathbb{R}) \cap L^{p(\cdot)}(\mathbb{R})$. The set $L^2(\mathbb{R}) \cap L^{p(\cdot)}(\mathbb{R})$ is dense in $L^{p(\cdot)}(\mathbb{R})$, whence $W^0(a) = A$ on the space $L^{p(\cdot)}(\mathbb{R})$ and

$$\lim_{n\to\infty} \|W^0(a) - W^0(a_n)\|_{\mathscr{L}(L^{p(\cdot)})} = 0. \tag{17}$$

From (4) and Lemma 8 it follows that

$$\lim_{n\to\infty} \|a_n\|_{\infty} = \|a\|_{\infty}. \tag{18}$$

and

$$\|W^0(a_n)\|_{\mathscr{L}(L^{p(\cdot)})} \leqslant \|S\|_{\mathscr{L}(L^{p(\cdot)})}(\|a_n\|_{\infty} + V(a_n)) \leqslant \|S\|_{\mathscr{L}(L^{p(\cdot)})}(\|a_n\|_{\infty} + V(a)),$$

whence

$$\begin{aligned}\|W^0(a)\|_{\mathscr{L}(L^{p(\cdot)})} &\leqslant \|W^0(a_n)\|_{\mathscr{L}(L^{p(\cdot)})} + \|W^0(a) - W^0(a_n)\|_{\mathscr{L}(L^{p(\cdot)})} \\ &\leqslant \|S\|_{\mathscr{L}(L^{p(\cdot)})}(\|a_n\|_{\infty} + V(a)) + \|W^0(a) - W^0(a_n)\|_{\mathscr{L}(L^{p(\cdot)})}.\end{aligned}$$

Passing to the limit in the above inequality as $n \to \infty$ and taking into account (17)–(18), we arrive at (3). □

REFERENCES

[1] A. BÖTTCHER, YU. I. KARLOVICH, AND I. M. SPITKOVSKY, *Convolution Operators and Factorization of Almost Periodic Matrix Functions*, Birkhäuser, Basel, 2002.

[2] A. BÖTTCHER AND M. SEYBOLD, *Wackelsatz and Stechkin's inequality for discrete Muckenhoupt weights*, Fakultät für Mathematik, TU Chemnitz, **99–7**, Unpublished preprint is available on A. Böttcher's web page, 1999.

[3] D. CRUZ-URIBE AND A. FIORENZA, *Variable Lebesgue Spaces*, Birkhäuser, Basel, 2013.

[4] D. CRUZ-URIBE, A. FIORENZA, J. M. MARTELL, AND C. PÉREZ, *The boundedness of classical operators on variable L^p spaces*, Ann. Acad. Sci. Fenn., Math. **31** (1), 2006, 239–264.

[5] L. DIENING, P. HARJULEHTO, P. HÄSTÖ, AND M. RŮŽIČKA, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer, Berlin, 2011.

[6] H. R. DOWSON, *Spectral Theory of Linear Operators*, Academic Press, London, New York, San Francisco, 1978.

[7] R. DUDUCHAVA, *Integral Equations with Fixed Singularities*, Teubner Verlagsgesellschaft, Leipzig, 1979.
[8] I. GOHBERG AND N. KRUPNIK, *Introduction to the Theory of One-Dimensional Singular Integral Equations*, Shtiintsa, Kishinev, 1973 (in Russian).
[9] I. GOHBERG AND N. KRUPNIK, *One-Dimensional Linear Singular Integral Equations. I. Introduction*, Birkhäuser, Basel, 1992.
[10] I. GOHBERG AND N. KRUPNIK, *One-Dimensional Linear Singular Integral Equations. II. General Theory and Applications*, Birkhäuser, Basel, 1992.
[11] A. YU. KARLOVICH, *Maximally modulated singular integral operators and their applications to pseudodifferential operators on Banach function spaces*, Contemporary Mathematics **645**, in "Function Spaces in Analysis", 2015, 165–178.
[12] A. YU. KARLOVICH AND I. M. SPITKOVSKY, *Pseudodifferential operators on variable Lebesgue spaces*, Operator Theory: Advances and Applications **228**, in: "Operator Theory, Pseudo-Differential Equations, and Mathematical Physics. The Vladimir Rabinovich Anniversary Volume", 2013, 173–183.
[13] A. YU. KARLOVICH A. YU. AND I. M. SPITKOVSKY, *The Cauchy singular integral operator on weighted variable Lebesgue spaces*, Operator Theory: Advances and Applications **236**, in: "Concrete Operators, Spectral Theory, Operators in Harmonic Analysis and Approximation", 2014, 275–291.
[14] V. M. KOKILASHVILI AND S. G. SAMKO, *Singular operators and Fourier multipliers in weighted Lebesgue spaces with variable index*, Vestn. St. Petersbg. Univ., Math. **41** (2), 2008, 134–144.
[15] V. M. KOKILASHVILI AND S. G. SAMKO, *Operators of harmonic analysis in weighted spaces with non-standard growth*, J. Math. Anal. Appl. **352** (1), 2009, 15–34.
[16] A. K. LERNER, *Some remarks on the Hardy-Littlewood maximal function on variable L^p spaces*, Math. Z. **251** (3), 2005, 509–521.
[17] I. P. NATANSON, *Theory of Functions of a Real Variable*, Frederick Ungar Publishing Co., New York, 1955.
[18] S. B. STECHKIN, *On bilinear forms*, Dokl. Akad. Nauk SSSR **71** (2), 1950, 237–240 (in Russian).

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