Mathematical
Inequalities
& Applications
Volume 18, Number 4 (2015), 1483–1500 doi:10.7153/mia-18-115

JACKSON KERNELS: A TOOL FOR ANALYSING THE DECAY OF EIGENVALUE SEQUENCES OF INTEGRAL OPERATORS ON THE SPHERE

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(*Communicated by I. Perić*)

Abstract. Decay rates for the sequence of eigenvalues of positive and compact integral operators have been largely investigated for a long time in the literature. In this paper, the focus will be on positive integral operators acting on square integrable functions on the unit sphere and generated by a kernel satisfying a Hölder type assumption defined by average operators. In the approach to be presented here, the decay rate will be reached from convenient estimations on the eigenvalues of the operator themselves, with the help of specific properties of a generic approximation operator defined through the so-called generalized Jackson kernels. The decay rate has the same structure of those known to hold in the cases in which the Hölder condition is the classical one. Therefore, within the spherical setting, the abstract approach to be introduced here extends some classical results on the topic.

1. Introduction

The present paper provides a concise approach to obtain decay rates for the eigenvalue sequence of positive integral operators acting on square integrable functions on the sphere, in the case when the generating kernel of the operator satisfies an abstract Hölder condition. Recent results on this same topic can be found in [1, 3, 8] and other references mentioned there. However, the scope here is different, since the differentiability of the generating kernel of the operators will not be an explicit requirement. This section begins with a description of some known results, mainly those which pertain to the scope of this paper.

A brief feedback about this subject cannot omit results obtained in the late 80's by several authors. We start with a function K in $L^2([0,1]\times[0,1])$ and consider the compact operator $\mathscr{L}_K : L^2([0,1]) \to L^2([0,1])$ generated by it

$$\mathscr{L}_K(f)(x) = \int_0^1 K(x,y)f(y)\,dy, \quad f \in L^2([0,1]), \quad x \in [0,1].$$

Mathematics subject classification (2010): 41A36, 45P05, 47A75, 47B34.

Keywords and phrases: Eigenvalue inequalities, eigenvalue decay, convolutions, integral operators, singular values, positive definiteness, Hölder inequality.

Both authors partially supported by FAPESP, grants # 2014/06209-1 and # 2014/00277-5 respectively.

In this case, and also in others in which the interval is replaced with a more general space, we simply call K the kernel and $\mathscr{L}_K$ the operator. The introduction of the symmetry assumption

$$K(x,y)=\overline{K(y,x)}, \quad (x,y)\in[0,1]\times[0,1],$$

makes the operator $\mathscr{L}_K$ self-adjoint and, therefore, its eigenvalue sequence $\{\lambda_n\}$ can be ordered in a decreasing manner

$$|\lambda_1|\geqslant|\lambda_2|\geqslant|\lambda_3|\geqslant\cdots,.$$

taking into account multiplicities. In particular, $\{\lambda_n\}$ approaches zero as $n\to\infty$. By the way, the equality above and some others in the paper need to be interpreted as equalities a. e..

A classical result of Weyl states that

$$\lambda_n=o(n^{-k-1/2}), \quad (n\to\infty),$$

whenever $K\in C^k([0,1]\times[0,1])$. After the introduction of a positivity assumption, Reade ([22]) established the faster decay rate

$$\lambda_n=o(n^{-k-1}), \quad (n\to\infty).$$

The positivity mentioned above refers to the property

$$\int_0^1\int_0^1 K(x,y)f(x)\overline{f(y)}\,dxdy\geqslant 0, \quad f\in L^2([0,1]),$$

which, in the most important cases, corresponds to the usual positive definiteness of the kernel K.

Later on, still keeping the positiveness in the setting, the same author deduced the decay rate ([23])

$$\lambda_n=O(n^{-1-r}), \quad (n\to\infty),$$

under the following Hölder assumption on the generating kernel K:

$$|K(x,y)-K(z,w)|\leqslant C(|x-z|^r+|y-w|^r), \quad x,y,z,w\in[0,1], \quad r\in(0,1).$$

A few years later, an outstanding generalization of the results above appeared in [16] with the replacement of $[0,1]$ with a compact C^∞ manifold. If the manifold is the usual m-dimensional unit sphere S^m endowed with its surface measure σ_m, the compact operator $\mathscr{L}_K$ now acts on $L^2(S^m):=L^2(S^m,\sigma_m)$, that is,

$$\mathscr{L}_K(f)(x)=\int_{S^m}K(x,y)f(y)\,d\sigma_m(y), \quad f\in L^2(S^m), \quad x\in S^m,$$

in which $K\in L^2(S^m\times S^m):=L^2(S^m\times S^m,\sigma_m\times\sigma_m)$. If K is continuous and satisfies the standard Hölder condition

$$|K(x,w)-K(y,w)|\leqslant B(w)d_m(x,y)^\beta, \quad x,y,w\in S^m,$$

for some $B \in L^1(S^m)$ and $\beta \in [0,1)$, the main result in [16] produced the decay

$$\lambda_n = O(n^{-1-\beta/m}), \quad (n \to \infty),$$

for the eigenvalue sequence $\{\lambda_n\}$ of $\mathscr{L}_K$. As for integral operators generated by kernels on metric spaces satisfying a similar condition, it is also logical and fair to mention the important references [4, 12, 15] where the role of the dimension is replaced by metric entropy.

The papers mentioned above included examples of integral operators for which the decay of the eigenvalue sequence matches exactly the decay obtained. In other words, in all cases above, the decay rates are best possible within each setting.

The setting in the present paper will be the spherical one and the focus will be on integral operators generated by (not necessarily continuous) kernels satisfying a Hölder type assumption defined by average operators. Below we stress a few points the reader should consider before and during the reading of this paper. They provide a reason why we have written the paper in the format it is:

(i) The spherical setting allows different approaches to the problem and that permits variations in the assumptions;

(ii) The approach adopted here is comparable to others found in the literature, however, it has its own characteristics;

(iii) The approach permits the consideration of slightly weaker general assumptions still reaching the same decay rates found in the literature;

(iv) All the results to be proved can be considered in other settings, as long as they have a background structure similar to that available in the spherical setting (two-point homogeneous spaces for example);

(v) The spherical setting has practical relevancy in other areas, for instance, in Geo-mathematics and meteorological sciences in general ([10]).

An outline of the paper is as follows. In Section 2, we introduce the abstract setting along with an abstract Hölder condition defined by the spherical convolution operator on which the main results of the paper will be based upon. We also include two motivational examples that may justify why we consider a general and abstract setting. In Section 3, we define the so called approximation operators, here manufactured with the help of the generalized Jackson kernels. At the end of the section, we show the approximation operator has finite rank when the setting is either one in the two motivational examples. Section 4 begins with the notion of positive integral operator that pertains to this work. It is followed by a technical inequality for integrals involving the generalized Jackson kernels and inequalities involved in the estimation of the approximations numbers of the square root of the positive integral operator. An estimation for the approximation numbers of the integral operator itself, under the assumption that the rank of the attached approximation operator is finite, comes after that. Finally, the section is closed with the main result in the paper. Section 5 is reserved for relevant remarks and the pointing of some open questions.

2. A Hölder condition based on spherical convolutions

Let us begin with the basic structure to be used in the paper. In addition to the spaces $L^2(S^m)$, we will stick to the usual spaces $L^p(S^m)$, $p = 1,\infty$. The norm in all of them will be written $\|\cdot\|_p$, $p = 1,2,\infty$. Finally, we want to emphasize from the outset that, throughout the whole paper, the dimension m will be fixed.

We will consider a Hölder assumption based upon a fixed family of nonnegative functions $\{\mathscr{Z}_t^m : t \in (0,\pi)\}$ belonging to $L^1([-1,1],d\omega_m)$, in which

$$d\omega_m(u) = (1-u^2)^{(m-2)/2}du, \quad u \in [-1,1].$$

If τ_m is the surface measure of S^m, then the norm in this space is

$$\|\phi\|_{1,m} := \frac{\tau_{m-1}}{\tau_m}\int_{-1}^{1}|\phi(u)|d\omega_m(u), \quad \phi \in L^1([-1,1],d\omega_m),$$

and the formula

$$Z_t^m(x,y) := \mathscr{Z}_t^m(x\cdot y), \quad x,y \in S^m,$$

in which $\cdot$ is the usual inner product of $\mathbb{R}^{m+1}$, defines an associated family $\{Z_t^m : t \in (0,\pi)\}$ of isotropic kernels on S^m. As usual, *isotropy* of a kernel refers to its invariance with respect to orthogonal transformations of the space where S^m sits. The setting to be undertaken here demands two assumptions:

A1 – The family $\{\mathscr{Z}_t^m : t \in (0,\pi)\}$ is uniformly bounded in $L^1([-1,1],d\omega_m)$.

A2 – If $V_m(t)$ is the surface area of the support of $Z_t^m(x,\cdot)$ in S^m, then there exists a positive integer $\alpha(m)$ and positive constants c_m and C_m so that

$$V_m(t) \leqslant C_m t^{\alpha(m)}, \quad t \in (0,\pi),$$

and

$$c_m t^{\alpha(m)} \leqslant V_m(t), \quad t \in (0,\pi/2).$$

The surface area of the support of $Z_t^m(x,\cdot)$ in S^m mentioned in $A2$ does not depend upon $x \in S^m$. Indeed, fix $x_1,x_2 \in S^m$ and for each $i \in \{1,2\}$, write $S_i^m(t)$ to denote the support of $Z_t^m(x_i,\cdot)$, and put

$$V_m(t,x_i) := \int_{S_i^m(t)} d\sigma_m(z).$$

Since each Z_t^m is not necessarily continuous, we have that

$$S_i^m(t) = S^m - \cup\{A : A \text{ is open in } S^m \text{ and } \sigma_m(\{x \in A : Z_t^m(x_i,x) \neq 0\}) = 0\}, \quad i = 1,2.$$

Using the isotropy of Z_t^m and some straightforward computations, it is easily seen that $\mathscr{O}(S_1^m(t)) = S_2^m(t)$, whenever $\mathscr{O}$ is an orthogonal transformation of $\mathbb{R}^{m+1}$ satisfying $\mathscr{O}(x_1) = x_2$. It is now clear that

$$V_m(t,x_2) = \int_{\mathscr{O}(S_1^m(t))} d\sigma_m(z) = \int_{S_1^m(t)} d\sigma_m(z) = V_m(t,x_1).$$

Under the setting introduced above, we define

$$T_t(f) := Z_t^m * f, \quad f \in L^p(S^m), \quad t \in (0,\pi),$$

in which

$$(Z_t^m * f)(x) = \frac{1}{\tau_m} \int_{S^m} \mathscr{Z}_t^m(x \cdot y) f(y) d\sigma_m(y), \quad x \in S^m, \quad f \in L^p(S^m),$$

is the *spherical convolution* of Z_t^m and f in S^m. Every T_t is a well-defined bounded linear operator from $L^p(S^m)$ into itself with $\|T_t\| \leqslant \|\mathscr{Z}_t^m\|_{1,m}$. If $\beta \in (0,2]$ and B is a nonnegative function from $L^\infty(S^m)$, then a kernel K on S^m is (T_t, B, β)*-Hölder* if

$$|T_t(K(y,\cdot))(x) - K(y,x)| \leqslant B(y) t^\beta, \quad x,y \in S^m, \quad t \in (0,\pi).$$

Below, we discuss two particular cases which served as motivation for the abstract setting introduced above and also for the consideration of the Hölder condition just defined.

EXAMPLE 1. For $t \in [0,\pi]$, let $C_m(t)$ be the total volume of the cap

$$C_t^x = \{y \in S^m : x \cdot y \geqslant \cos t\}$$

of S^m defined by t and "the pole" x. Clearly,

$$C_m(t) = \tau_{m-1} \int_0^t (\sin h)^{m-1} dh, \quad t \in (0,\pi),$$

a quantity that does not depend upon x. The formula

$$\mathscr{Z}_{n,t}^m(x \cdot y) = \begin{cases} \tau_m C_m(t)^{-1} (x \cdot y - \cos t)^{n-1} (1 - \cos t)^{-(n-1)}, & \text{if } \cos t \leqslant x \cdot y \leqslant 1 \\ 0, & \text{otherwise} \end{cases}$$

defines families $\{\mathscr{Z}_{n,t}^m : t \in (0,\pi)\}$ of locally supported kernels on S^m. A construction of such kernels by iteration with spherical convolutions can be found in [17, 25]. Since

$$\|\mathscr{Z}_{n,t}^m\|_{1,m} = \frac{\tau_{m-1}}{\tau_m} \int_{-1}^1 \mathscr{Z}_{n,t}^m(u) d\omega_m(u) \leqslant \frac{\tau_{m-1}}{C_m(t)} \int_{\cos t}^1 (1-u^2)^{(m-2)/2} du = 1,$$

each family $\{\mathscr{Z}_{n,t}^m : t \in (0,\pi)\}$ is uniformly bounded in $L^1([-1,1], d\omega_m)$. On the other hand, it is easily seen that the surface area $V_m(t)$ of the support of $Z_{n,t}^m(x,\cdot)$ in S^m is precisely $C_m(t)$ (there is no dependence on n) while direct computation yields

$$\frac{\tau_{m-1}}{m} \left(\frac{2}{\pi}\right)^{m-1} t^m \leqslant V_m(t) \leqslant \tau_{m-1} t^m, \quad t \in (0,\pi).$$

Thus, both assumptions A1 and A2 hold in this case. The particular case $n = 1$ recovers the usual *average operators* M_t on S^m ([2]), that is,

$$M_t(f)(x) = (\mathscr{Z}_{1,t}^m * f)(x) = \frac{1}{C_m(t)} \int_{C_t^x} f(w) dr(w), \quad x \in S^m, \quad t \in (0,\pi),$$

and the abstract (T_t, B, β)-Hölder condition turns itself into the *averaged Hölder condition*

$$|M_t(K(y,\cdot))(x) - K(y,x)| \leqslant B(y)t^{\beta}, \quad x,y \in S^m, \quad t \in (0,\pi).$$

EXAMPLE 2. Here we will consider the *Stekelov-type mean* operator introduced and discussed in [7]. If $R_m(t) := \tau_{m-1}(\sin t)^{m-1}$, $t \in (0,\pi)$, then it has the form

$$E_t(f)(x) = \frac{1}{D_m(t)} \int_0^t \frac{C_m(s)}{R_m(s)} M_s(f)(x)ds, \quad x \in S^m, \quad t \in (0,\pi),$$

the normalizing constant $D_m(t)$ being chosen so that $E_t(1) = 1$. In order to see that the operators E_t fit into the convolution structure we are using, it suffices to consider the family of locally supported kernels

$$W_t^m(x,y) := \mathscr{W}_t^m(x\cdot y) := \begin{cases} \displaystyle\int_0^t \frac{1}{R_m(s)} \mathscr{Z}_{1,s}^m(x\cdot y)ds, & \text{if } \cos t \leqslant x\cdot y \leqslant 1 \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathscr{Z}_{1,s}^m$ are the kernels described in the previous example. Clearly,

$$\begin{aligned}
\|\mathscr{W}_t^m\|_{1,m} &= \frac{\tau_{m-1}}{\tau_m} \int_{-1}^1 \mathscr{W}_t^m(u)d\omega_m(u) \\
&\leqslant \frac{\tau_{m-1}}{\tau_m D_m(t)} \int_{\cos t}^1 \left[\int_0^t \frac{C_m(s)}{R_m(s)} |\mathscr{Z}_{1,s}^m(u)|ds\right](1-u^2)^{(m-2)/2}du \\
&= \frac{1}{D_m(t)} \int_0^t \frac{C_m(s)}{R_m(s)} \left[\frac{\tau_{m-1}}{\tau_m} \int_{\cos t}^1 |\mathscr{Z}_{1,s}^m(u)(1-u^2)^{(m-2)/2}du\right] ds \\
&= \frac{1}{D_m(t)} \int_0^t \frac{C_m(s)}{R_m(s)} ds.
\end{aligned}$$

The normalization we have chosen for $D_m(t)$ provides the uniform boundedness for the family $\{\mathscr{W}_t^m : t \in (0,\pi)\}$ in $L^1([-1,1], d\omega_m)$. The support of $W_t^m(x,\cdot)$ is $C_m(t)$ and $A2$ holds as in the first example. Since the kernel $\mathscr{W}_t^m$ is isotropic, we have

$$E_t(f)(x) = (W_t^m * f)(x), \quad f \in L^2(S^m), \quad t \in (0,\pi).$$

In this case, the abstract (T_t, B, β)-Hölder condition turns itself into the *Stekelov-mean Hölder condition*

$$|E_t(K(y,\cdot))(x) - K(y,x)| \leqslant B(y)t^{\beta}, \quad x,y \in S^m, \quad t \in (0,\pi).$$

Before closing the section, let us return to the standard Hölder condition introduced in Section 1, now considering $B \in L^{\infty}(S^m)$. It is straightforward to verify that a kernel K satisfying the usual Hölder condition also satisfies an averaged Hölder condition (with the same index β but not necessarily the same B). Likewise, a kernel satisfying an averaged Hölder condition also satisfies an Stekelov-mean Hölder condition (with the same index β but not necessarily the same B). Thus, we have a chain of conditions from the stronger usual Hölder condition to the weaker Stekelov-mean Hölder condition.

3. The approximating operators

In this section, we introduce the approximation operators we intend to use in some critical arguments in the paper where we need to estimate the approximation numbers of our operators. They will depend on the setting introduced in Section 2 and on the generalized Jackson kernels. The use of these kernels were influenced by the paper [6] wherein standard Jackson kernels were used to obtain decay rates for the sequence of eigenvalues of the integral operator on $L^2([0,1])$ in the case K is differentiable in $[0,1]^2$ up to a certain order. On the other hand, it is well known that the generalized Jackson kernels imply optimal results in many problems in analysis and approximation theory.

We will assume that the setting at the beginning of Section 2 has been fixed here. For positive integers l and $\mu \geqslant 2$, tied to each other via the formula $\nu = l(\mu-1)$, the *generalized Jackson kernel* is given by

$$J_{\nu,k}(t) := \frac{1}{c_{\nu,k}}\left[\frac{\sin(\mu t/2)}{\sin(t/2)}\right]^{2l}, \quad t \in (0,\pi), \quad k \in \mathbb{Z}_+,$$

with the normalization constant $c_{\nu,k}$ computed through the formula

$$c_{\nu,k} = \int_0^\pi \left[\frac{\sin(\mu t/2)}{\sin(t/2)}\right]^{2l} V_m(t)(\sin t)^k\, dt.$$

Here, the constant V_m is that one introduced in A2. Clearly, the constants $c_{\nu,k}$ depend upon m too, but that will be not enforced in the notation adopted. On the other hand, it is easily seen that the normalization corresponds to

$$\int_0^\pi J_{\nu,k}(t)V_m(t)(\sin t)^k\, dt = 1.$$

The (integral) approximating operators themselves can now be defined through the family $\{T_t : t \in (0,\pi)\}$ of convolution operators.

PROPOSITION 1. *For each ν and k, the formula*

$$\Phi_{\nu,k}(f)(x) = \int_0^\pi J_{\nu,k}(t)T_t(f)(x)V_m(t)(\sin t)^k dt, \quad f \in L^2(S^m), \quad x \in S^m,$$

defines a bounded linear operator $\Phi_{\nu,k}$ from $L^2(S^m)$ into itself.

Proof. Minkowski's inequality for integrals ([9, p. 194]) implies that

$$\|\Phi_{\nu,k}(g)\|_2 \leqslant \int_0^\pi J_{\nu,k}(t)\|T_t(g)\|_2 V_m(t)(\sin t)^k\, dt, \quad g \in L^2(S^m).$$

Since T_t is a convolution operator, it follows that

$$\|\Phi_{\nu,k}(g)\|_2 \leqslant \|g\|_2 \int_0^\pi \|\mathscr{Z}_t^m\|_{1,m} J_{\nu,k}(t)V_m(t)(\sin t)^k\, dt \leqslant M\|g\|_2, \quad g \in L^2(S^m),$$

in which M is a uniform bound for the family $\{\mathscr{Z}_t^m : t \in (0,\pi)\}$. $\square$

In many cases, the formula

$$a_n(\mathscr{L}_K) = \min\{\|\mathscr{L}_K - U\| : \rho(U) \leqslant n-1\},$$

in which $\rho(U)$ is the rank of U, is a useful tool in either the exact computation or the estimation of the n-th approximation number $a_n(\mathscr{L}_K)$ of the operator of $\mathscr{L}_K$. As a matter of fact, $a_n(\mathscr{L}_K)$ coincides with the n-th eigenvalue of the operator in those situations. So, if $\rho(\Phi_{\nu,k}) < \infty$, it is clear that the composition $U = \Phi_{\nu,k} \circ \mathscr{L}_K$ is eligible to be used in the estimation of some of approximation numbers. Since $J_{\nu,k}$ is an even trigonometric polynomial of degree ν ([18]), it is reasonable to expect that $\rho(\Phi_{\nu,k}) < \infty$ for some special choices of T_t. The results that close this section will ratify that in the examples presented in the second half of Section 2.

We will write $\mathscr{H}_k^m$ to denote the space of all spherical harmonics of degree k in $m+1$ variables and will denote its dimension by $N(m,k)$. It is worthwhile to mention that ([5, p. 3])

$$N(m,k) \asymp k^{m-1}, \quad (n \to \infty).$$

The orthogonal decomposition $L^2(S^m) = \oplus_{k=0}^{\infty} \mathscr{H}_k^m$ is well-known while the orthogonal projection of $L^2(S^m)$ over $\mathscr{H}_k^m$ is given by the formula

$$\mathscr{Y}_k(g)(x) = \frac{N(m,k)}{\tau_m P_k^{(m-1)/2}(1)} \int_{S^m} P_k^{(m-1)/2}(x\cdot y) g(y)\, d\sigma_m(y), \quad g \in L^2(S^m), \quad x \in S^m,$$

in which $P_k^{(m-1)/2}$ is the usual Gegenbauer polynomial of degree k associated to the dimension m. The additional formula

$$\mathscr{Y}_k(M_t(g)) = \frac{\tau_{m-1}}{C_m(t) P_k^{(m-1)/2}(1)} \left(\int_0^t P_k^{(m-1)/2}(\cos h)(\sin h)^{m-1} dh \right) \mathscr{Y}_k(g),\ g \in L^2(S^m),$$

for the projections of the elements in the range of M_t was derived in [2]. A nice reference for the results on the analysis on the sphere mentioned above and others to be mentioned ahead is [5].

The propositions below provide estimates for the rank of the operator in Proposition 1, in the cases in which T_t is either the average operator M_t or the Stekelov-type mean operator E_t.

PROPOSITION 2. *The operator*

$$\Phi_{\nu,1}(f)(x) = \int_0^\pi J_{\nu,1}(t) M_t(f)(x) C_m(t) \sin t\, dt, \quad f \in L^2(S^m), \quad x \in S^m,$$

has rank at most $N(m+1, \nu+1)$.

Proof. For $f \in L^2(S^m)$ fixed, we will show that $\Phi_{\nu,1}(f)$ is a spherical polynomial of degree at most $\nu+1$. That will imply the estimate announced in the statement of the proposition due to two facts: the space of all spherical polynomials of degree at most $\nu+1$ is precisely $\oplus_{k=0}^{\nu+1}\mathscr{H}_k^m$ and its dimension is $\sum_{k=0}^{\nu+1}N(m,k)=N(m+1,\nu+1)$. The proof will be complete as long as we show that $\mathscr{Y}_k(\Phi_{\nu,1}(f))=0,\ k=\nu+2,\nu+3,\ldots$. Direct computation reveals that

$$\mathscr{Y}_k(\Phi_{\nu,1}(f))(x)=\int_0^\pi J_{\nu,1}(t)\mathscr{Y}_k(M_t(f))(x)C_m(t)\sin t\,dt,\quad x\in S^m,$$

while the formula prior to the statement of the theorem leads to

$$\mathscr{Y}_k(\Phi_{\nu,1}(f))=\frac{\tau_{m-1}}{P_k^{(m-1)/2}(1)}\left\{\int_0^\pi J_{\nu,1}(t)\left[\int_0^t P_k^{(m-1)/2}(\cos h)(\sin h)^{m-1}dh\right]\sin t\,dt\right\}\mathscr{Y}_k(f).$$

The inner integral can be put into the form

$$\int_0^t P_k^{(m-1)/2}(\cos h)(\sin h)^{m-1}dh=-\int_1^{\cos t}P_k^{(m-1)/2}(u)(1-u^2)^{(m-2)/2}du.$$

Invoking the classical equality ([26, p. 81–82])

$$\frac{d}{du}\left[-\frac{m-1}{k(k+m-1)}(1-u^2)^{m/2}P_{k-1}^{(m+1)/2}(u)\right]=(1-u^2)^{(m-2)/2}P_k^{(m-1)/2}(u),$$

we deduce that

$$\int_0^t P_k^{(m-1)/2}(\cos h)(\sin h)^{m-1}dh=\frac{m-1}{k(k+m-1)}(\sin t)^m P_{k-1}^{(m+1)/2}(\cos t).$$

Consequently,

$$\mathscr{Y}_k(\Phi_{\nu,1}(f))=\frac{\tau_{m-1}}{P_k^{(m-1)/2}(1)}\frac{m-1}{k(k+m-1)}\left[\int_0^\pi J_{\nu,1}(t)P_{k-1}^{(m+1)/2}(\cos t)(\sin t)^{m+1}dt\right]\mathscr{Y}_k(f).$$

Since $J_{\nu,1}(t)$ is a polynomial of degree ν with respect to $\cos t$, it is easily seen that we can write it in the form

$$J_{\nu,1}(\cos t)=\sum_{j=0}^{\nu}a_jP_j^{(m+1)/2}(\cos t),\quad a_1,a_2,\ldots,a_\nu\in\mathbb{R}.$$

In particular, the well-known orthogonality relation ([19, p. 98])

$$\int_0^\pi P_k^{(m+1)/2}(\cos t)P_l^{(m+1)/2}(\cos t)(\sin t)^{m+1}dt=0,\quad k\neq l,$$

implies that

$$\int_0^\pi J_{\nu,1}(t)P_{k-1}^{(m+1)/2}(\cos t)(\sin t)^{m+1}dt=0,\quad k-1\geqslant\nu+1.$$

It is now clear that

$$\mathscr{Y}_k(\Phi_{\nu,1}(f))=0,\quad k=\nu+2,\nu+3,\ldots,$$

and the result follows. □

PROPOSITION 3. *The operator*

$$\Phi_{\nu,m}(f)(x) = \int_0^\pi J_{\nu,m}(t)E_t(f)(x)D_m(t)(\sin t)^m dt, \quad f \in L^2(S^m), \quad x \in S^m,$$

has rank at most $N(m+1,\nu+1)$.

Proof. Since it is analogous to the proof of the previous proposition, the details will be not included. □

4. Decay rates via the generalized Jackson kernels

The attention in this section will be directed to integral operators $\mathscr{L}_K$ of the form

$$\mathscr{L}_K(f)(x) = \int_{S^m} K(x,y)f(y)\, d\sigma_m(y), \quad x \in S^m, \quad f \in L^2(S^m).$$

that possess the features below:

– it is generated by an element K of $L^2(S^m \times S^m)$ (so, it is a linear operator from $L^2(S^m)$ into itself);

– the kernel K is $L^2(S^m)$-positive definite in the sense that

$$\int_{S^m} \mathscr{L}_K(f)(x)\overline{f(x)}d\sigma_m(x)\rangle \geqslant 0, \quad f \in L^2(S^m);$$

– the square root $\mathscr{L}_K^{1/2}$ of $\mathscr{L}_K$ is an integral operator on $L^2(S^m)$ generated by a hermitian kernel $K_{1/2} : S^m \times S^m \to \mathbb{C}$;

– there is a *recovery formula* for K from $K_{1/2}$, that is,

$$\int_{S^m} K_{1/2}(x,y)K_{1/2}(w,x)d\sigma_m(x) = K(w,y), \quad y,w \in S^m.$$

A usual concrete setting in which all the conditions above hold is described in [24].

An operator as above will be called a *positive integral operator* from now on. The category of positive integral operators includes those integral operators generated by a continuous and positive definite kernel in the usual sense, as one can ratify in [8]. A positive integral operator has countably many nonnegative eigenvalues which can be ordered as

$$\lambda_1(\mathscr{L}_K) \geqslant \lambda_2(\mathscr{L}_K) \geqslant \cdots \geqslant 0,$$

repetitions being included in accordance with algebraic multiplicities. After we order the eigenvalues of $\mathscr{L}_K^{1/2}$ in the same way, it holds

$$\lambda_n(\mathscr{L}_K^{1/2}) = (\lambda_n(\mathscr{L}_K))^{1/2} = a_n(\mathscr{L}_K))^{1/2}, \quad n = 1,2,\ldots. \tag{1}$$

and

$$\|\mathscr{L}_K\| \geqslant a_1(\mathscr{L}_K) \geqslant a_2(\mathscr{L}_K) \geqslant \cdots \geqslant 0.$$

For a general treatment on approximation numbers of operators, we refer the reader to [20] while a treatment in a setting similar to the one used here can be found in [11].

4.1. The essential inequalities

This subsection contains preliminary estimates for the norm of the operators

$$\mathscr{L}_K^{1/2} - \Phi_{\nu,k}(\mathscr{L}_K^{1/2}) : L^2(S^m) \to L^\infty(S^m)$$

when $\mathscr{L}_K$ is a positive integral operator generated by a (T_t, B, β)-Hölder kernel K. In particular, we remind the reader that the setting described in Section 2 needs to hold here, including the assumptions A1 and A2.

Two reasons justify why we will estimate $a_n(\mathscr{L}_K^{1/2})$ instead of $a_n(\mathscr{L}_K)$: formula (1) is available in the most important cases and the applications on decay rates for eigenvalues we seek demand the approximation numbers of $\mathscr{L}_K^{1/2}$.

In the three lemmata below, we will use the generalized Jackson kernel $J_{\nu,k}$. We remind the reader that ν is defined via two positive integers l and $\mu \geqslant 2$, through the formula $\nu = l(\mu-1)$. The integer k is supposed to be a fixed nonnegative integer throughout the subsection. We begin with an estimation for certain integrals involving the generalized Jackson kernels.

LEMMA 1. *Let γ be a positive real number. If $2l > \gamma + \alpha(m) + k + 1$, then*

$$\int_0^\pi J_{\nu,k}(t) t^\gamma V_m(t)(\sin t)^k \, dt \leqslant \frac{d_{m,\gamma,l}}{\mu^\gamma},$$

where $d_{m,\gamma,l}$ is a constant depending of m, γ, l and the constant $\alpha(m)$ from A2.

Proof. The idea of the proof is to detach the normalizing constant $c_{\nu,k}$ from the integral, to find a lower bound for it and an upper bound for the resulting integral. Clearly,

$$c_{\nu,k} \geqslant \int_0^{\pi/2} \left[\frac{\sin(\mu t/2)}{\sin(t/2)}\right]^{2l} V_m(t)(\sin t)^k \, dt \geqslant \frac{2^{2l+k}}{\pi^k} \int_0^{\pi/2} t^{k-2l} [\sin(\mu t/2)]^{2l} V_m(t) dt.$$

From the inequality $V_m(t) \geqslant c_m t^{\alpha(m)}$, $t \in (0, \pi/2)$, we obtain

$$c_{\nu,k} \geqslant c_m \frac{2^{2l+k}}{\pi^k} \int_0^{\pi/2} t^{\alpha(m)+k-2l} [\sin(\mu t/2)]^{2l} \, dt.$$

The change of variables $s = \mu t$ and the inequality $\mu \geqslant 2$ provide the estimate

$$\begin{aligned} \int_0^{\pi/2} t^{\alpha(m)+k-2l} [\sin(\mu t/2)]^{2l} \, dt &= \frac{1}{\mu^{\alpha(m)+k+1-2l}} \int_0^{\mu\pi/2} s^{\alpha(m)+k-2l} [\sin(s/2)]^{2l} \, ds \\ &\geqslant \frac{1}{\mu^{\alpha(m)+k+1-2l}} \int_0^{\pi} s^{\alpha(m)+k-2l} [\sin(s/2)]^{2l} \, ds, \end{aligned}$$

while an additional adjustment leads to

$$\int_0^{\pi/2} t^{\alpha(m)+k-2l} [\sin(\mu t/2)]^{2l} \, dt \geqslant \frac{\pi^{-2l}}{\mu^{\alpha(m)+k+1-2l}} \int_0^\pi s^{\alpha(m)+k} ds.$$

The final lower estimate for $c_{\nu,k}$ is

$$c_{\nu,k} \geqslant c_m \frac{2^{2l+k}}{\pi^{2l+k}} \frac{1}{\mu^{\alpha(m)+k+1-2l}} \int_0^\pi s^{\alpha(m)+k} ds$$
$$= \frac{c_m}{\alpha(m)+k+1} \frac{2^{2l+k}}{\pi^{2l-\alpha(m)-k-1}} \mu^{2l-\alpha(m)-k-1}.$$

Next, we move to an upper bound for the integral

$$I := \int_0^\pi \left[\frac{\sin(\mu t/2)}{\sin(t/2)}\right]^{2l} t^\gamma V_m(t)(\sin t)^k \, dt.$$

Since $V_m(t) \leqslant C_m t^{\alpha(m)}$, $t \in (0,\pi)$, it is clear that

$$I \leqslant C_m \pi^{2l} \int_0^\pi [\sin(\mu t/2)]^{2l} t^{\gamma+\alpha(m)+k-2l} dt.$$

Using the change of variables $s = \mu t/2$, we can estimate the integral appearing above as follows

$$\int_0^\pi t^{\gamma+\alpha(m)+k-2l} [\sin(\mu t/2)]^{2l} \, dt = \left(\frac{2}{\mu}\right)^{\gamma+\alpha(m)+k+1-2l} \int_0^{\mu\pi/2} s^{\gamma+\alpha(m)+k} \left(\frac{\sin s}{s}\right)^{2l} ds$$
$$\leqslant \left(\frac{2}{\mu}\right)^{\gamma+\alpha(m)+k+1-2l} \int_0^\infty s^{\gamma+\alpha(m)+k} \left(\frac{\sin s}{s}\right)^{2l} ds.$$

The assumption $2l > \gamma + \alpha(m) + k + 1$ guarantees the convergence of the improper integral. Proceeding, we have that

$$I \leqslant C_m \pi^{2l} \left(\frac{2}{\mu}\right)^{\gamma+\alpha(m)+k+1-2l} \int_0^\infty s^{\gamma+\alpha(m)+k} \left(\frac{\sin s}{s}\right)^{2l} ds.$$

Combining our findings and making some minor adjustments, it is promptly seen that the inequality in the statement of the lemma follows. □

LEMMA 2. *Let $\mathscr{L}_K$ be a positive integral operator generated by a (T_t, B, β)-Hölder kernel K. If $f \in L^2(S^m)$ and $x \in S^m$, then*

$$\left|\mathscr{L}_K^{1/2}(f)(x) - \Phi_{\nu,k}(\mathscr{L}_K^{1/2}(f))(x)\right|$$
$$\leqslant \|f\|_2 \int_0^\pi J_{\nu,k}(t) t^{\beta/2} [B(x) + T_t(B)(x)]^{1/2} V_m(t)(\sin t)^k \, dt.$$

Proof. Fix f and x and let us write $I_x(f)$ to denote the difference $\mathscr{L}_K^{1/2}(f)(x) - \Phi_{\nu,k}(\mathscr{L}_K^{1/2}(f))(x)$. The normalization for the Jackson kernels implies that

$$I_x(f) = \int_0^\pi J_{\nu,k}(t) \left[\mathscr{L}_K^{1/2}(f)(x) - T_t(\mathscr{L}_K^{1/2}(f))(x)\right] V_m(t)(\sin t)^k \, dt.$$

Hence,

$$|I_x(f)| \leqslant \int_0^\pi J_{\nu,k}(t)|D_t(x)|V_m(t)(\sin t)^k\,dt,$$

where

$$D_t(x) = \mathscr{L}_K^{1/2}(f)(x) - T_t(\mathscr{L}_K^{1/2}(f))(x), \quad t \in (0,\pi).$$

The proof will be complete as long as we can reach the estimate

$$|D_t(x)| \leqslant \|f\|_2 t^{\beta/2}[B(x) + T_t(B)(x)]^{1/2}, \quad t \in (0,\pi).$$

Since

$$\mathscr{L}_K^{1/2}(f)(x) = \int_{S^m} K_{1/2}(x,y)f(y)d\sigma_m(y), \quad f \in L^2(S^m),$$

it is easily seen that

$$\begin{aligned} D_t(x) = &\int_{S^m} K_{1/2}(x,y)f(y)d\sigma_m(y) \\ &-\frac{1}{\tau_m}\int_{S^m}\int_{S^m} \mathscr{Z}_t^m(x\cdot w)K_{1/2}(w,y)f(y)d\sigma_m(w)d\sigma_m(y), \end{aligned}$$

while a change in the integration order leads to

$$D_t(x) = \frac{1}{\tau_m}\int_{S^m}\left(\tau_m K_{1/2}(x,y) - \int_{S^m} \mathscr{Z}_t^m(x\cdot w)K_{1/2}(w,y)d\sigma_m(w)\right)f(y)d\sigma_m(y).$$

To proceed, we apply Hölder's inequality to deduce that

$$|D_t(x)| \leqslant \frac{1}{\tau_m}\|I_x^t\|_2\|f\|_2, \quad t \in (0,\pi),$$

in which

$$I_x^t(y) = \tau_m K_{1/2}(x,y) - \int_{S^m} \mathscr{Z}_t^m(x\cdot w)K_{1/2}(w,y)d\sigma_m(w), \quad y \in S^m, \quad t \in (0,\pi).$$

The rest of the proof will consist of a tricky estimation of the quantity $\|I_x^t\|_2$. A simple calculation leads to

$$\begin{aligned} \|I_x^t\|_2^2 = &\int_{S^m} \tau_m^2 K_{1/2}(x,y)K_{1/2}(y,x)d\sigma_m(y) \\ &+\int_{S^m}\Big[-\tau_m K_{1/2}(x,y)\int_{S^m} \mathscr{Z}_t^m(x\cdot w)K_{1/2}(y,w)d\sigma_m(w) \\ &-\tau_m K_{1/2}(y,x)\int_{S^m} \mathscr{Z}_t^m(x\cdot w)K_{1/2}(w,y)d\sigma_m(w) \\ &+\int_{S^m}\int_{S^m} \mathscr{Z}_t^m(x\cdot w)\mathscr{Z}_t^m(x\cdot z)K_{1/2}(w,y)K_{1/2}(y,z)d\sigma_m(z)d\sigma_m(w)\Big]\,d\sigma_m(y). \end{aligned}$$

Interchanging the order of integration and applying the recovery formula, we deduce that

$$\begin{aligned}\|I_x^t\|_2^2 = {} & \tau_m^2 K(x,x) - \tau_m \int_{S^m} \mathscr{Z}_t^m(x\cdot w)K(x,w)d\sigma_m(w) \\ & -\tau_m \int_{S^m} \mathscr{Z}_t^m(x\cdot w)K(w,x)d\sigma_m(w) \\ & + \int_{S^m}\int_{S^m} \mathscr{Z}_t^m(x\cdot w)\mathscr{Z}_t^m(x\cdot z)K(w,z)d\sigma_m(z)\sigma_m(w).\end{aligned}$$

An additional adjustment produces the formula

$$\begin{aligned}\|I_x^t\|_2^2 \leqslant {} & \tau_m^2 \,|K(x,x) - T_t(K(x,\cdot))(x)| \\ & +\tau_m \int_{S^m} \mathscr{Z}_t^m(x\cdot w)\,|K(w,x) - T_t(K(w,\cdot))(x)|\,d\sigma_m(w).\end{aligned}$$

Now, introducing the inequality defining the Hölder condition, we obtain

$$\|I_x^t\|_2^2 \leqslant \tau_m^2 t^\beta B(x) + \tau_m t^\beta \int_{S^m} \mathscr{Z}_t^m(x\cdot w)B(w)d\sigma_m(w) = \tau_m^2 t^\beta [B(x) + T_t(B)(x)],$$

Combining all these findings lead to the inequality in the statement of the lemma. □

Next, we not only show that the operator $\mathscr{L}_K^{1/2} - \Phi_{\nu,k}(\mathscr{L}_K^{1/2}) : L^2(S^m) \to L^\infty(S^m)$ is well defined but we also bound the elements on its image. Two properties of the norm $\|\cdot\|_\infty$ are used in the arguments: the Minkowski's inequality for integrals ([9, p. 194]) and the inequality $\|\sqrt{f}\|_\infty \leqslant \|f\|_\infty^{1/2}$, $f \in L^\infty(S^m)$, which holds whenever f is a nonnegative function.

LEMMA 3. *Let $\mathscr{L}_K$ be a positive integral operator generated by a (T_t, B, β)-Hölder kernel K. If $f \in L^2(S^m)$, then*

$$\left\|\mathscr{L}_K^{1/2}(f) - \Phi_{\nu,k}(\mathscr{L}_K^{1/2}(f))\right\|_\infty \leqslant \|B\|_\infty^{1/2}(1+M)^{1/2}\left[\int_0^\pi J_{\nu,k}(t)t^{\beta/2}V_m(t)(\sin t)^k\,dt\right]\|f\|_2,$$

in which M is a uniform bound for the family $\{\mathscr{Z}_t^m : t \in (0,\pi)\}$ in $L^1([-1,1], d\omega_m)$.

Proof. Fix $f \in L^2(S^m)$ and write

$$G_\nu(f)(x) := \mathscr{L}_K^{1/2}(f)(x) - \Phi_{\nu,k}(\mathscr{L}_K^{1/2}(f))(x), \quad x \in S^m.$$

Lemma 2 and the remarks preceding the lemma imply that

$$|G_\nu(f)(x)| \leqslant \|f\|_2 \int_0^\pi J_{\nu,k}(t)t^{\beta/2}\,\|B + T_t(B)\|_\infty^{1/2}\, V_m(t)(\sin t)^k\,dt, \quad x \in S^m.$$

If M is a uniform bound for the family $\{\mathscr{Z}_t^m : t \in (0,\pi)\}$ in $L^1([-1,1], d\omega_m)$, we have that

$$\|T_t(B)\|_\infty \leqslant \|T_t\|\|B\|_\infty \leqslant M\|B\|_\infty, \quad t \in (0,\pi).$$

The inequality in the statement of the lemma follows. □

Since the embedding $L^\infty(S^m) \hookrightarrow L^2(S^m)$ is absolutely 2-summing, the operator $\mathscr{L}_K^{1/2} - \Phi_{\nu,k}(\mathscr{L}_K^{1/2}) : L^2(S^m) \to L^2(S^m)$ is Hilbert-Schmidt ([20, p. 56–56]). Denoting the Hilbert-Schmidt norm by $\|\cdot\|_{HS}$, the following result holds likewise.

LEMMA 4. *Let $\mathscr{L}_K$ be a positive integral operator generated by a (T_t, B, β)-Hölder kernel K. Then, there exists a constant C so that*

$$\left\|\mathscr{L}_K^{1/2} - \Phi_{\nu,k}(\mathscr{L}_K^{1/2})\right\|_{HS} \leqslant C\left[\int_0^\pi J_{\nu,k}(t) t^{\beta/2} V_m(t)(\sin t)^k\, dt\right].$$

4.2. Estimates for the approximation numbers of $\mathscr{L}_K$

This section contains the main result of the paper.

In the technical lemma below, we deduce a basic inequality for the approximation numbers of the operator $\mathscr{L}_K$ generated by a (T_t, B, β)-Hölder kernel K, under the assumption that the approximation operator $\Phi_{\nu,k}$ has finite rank.

LEMMA 5. *Let $\mathscr{L}_K$ be a positive integral operator generated by a (T_t, B, β)-Hölder kernel K. If for a fixed k, the approximation operator $\Phi_{\nu,k}$ has finite rank r, then*

$$r a_{2r}(\mathscr{L}_K) \leqslant \left\|\mathscr{L}_K^{1/2} - \Phi_{\nu,k}\mathscr{L}_K^{1/2}\right\|_{HS}^2.$$

Proof. Since the rule that assigns to every operator the sequence of its approximation numbers is an s-scale, it is easily seen that

$$a_j(\mathscr{L}_K^{1/2}) \leqslant a_{j-r}(\mathscr{L}_K^{1/2} - \Phi_{\nu,k}\mathscr{L}_K^{1/2}) + a_{r+1}(\Phi_{\nu,k}\mathscr{L}_K^{1/2}), \quad j > r \geqslant 0.$$

However, since $a_{r+1}(\Phi_{\nu,k}\mathscr{L}_K^{1/2})$ coincides with the $(r+1)$-th s-number of $\Phi_{\nu,k}\mathscr{L}_K^{1/2}$, if $\Phi_{\nu,k}$ has finite rank r, then we have that $a_{r+1}(\Phi_{\nu,k}\mathscr{L}_K^{1/2}) = 0$. This gives

$$r a_{2r}(\mathscr{L}_K) = r[a_{2r}(\mathscr{L}_K^{1/2})]^2 \leqslant \sum_{j=r+1}^{2r} [a_j(\mathscr{L}_K^{1/2})]^2 \leqslant \sum_{j=1}^{\infty} [a_j(\mathscr{L}_K^{1/2} - \Phi_{\nu,k}\mathscr{L}_K^{1/2})]^2.$$

which implies the estimate in the statement of the lemma. □

Under the same assumptions in the lemma, if $\rho(\Phi_{\nu,k}) \leqslant (q\nu)^{\alpha(m)}$ for some k, then a similar argument leads to

$$(q\nu)^{\alpha(m)} a_{2(q\nu)^{\alpha(m)}}(\mathscr{L}_K) \leqslant \left\|\mathscr{L}_K^{1/2} - \Phi_{\nu,k}\mathscr{L}_K^{1/2}\right\|_{HS}^2.$$

The main result in this section is this one.

THEOREM 1. *Let $\mathscr{L}_K$ be a positive integral operator generated by a (T_t, B, β)-Hölder kernel K. If for a fixed k, there exists a positive integer q so that $\rho(\Phi_{\nu,k}) \leqslant (q\nu)^{\alpha(m)}$, $\nu = 1, 2, \ldots$, then*

$$a_n(\mathscr{L}_K) = O(n^{-1-\beta/\alpha(m)}), \quad (n \to \infty).$$

Proof. Let us assume that $\rho(\Phi_{\nu,k}) \leqslant (q\nu)^{\alpha(m)}$, for some q. An application of Lemma 5, with the help of Lemma 4, leads to

$$(qn)^{\alpha(m)} a_{(qn)^{\alpha(m)}}(\mathscr{L}_K) \leqslant C \left[\int_0^{\pi} J_{n,k}(t) t^{\beta/2} V_m(t) (\sin t)^k \, dt \right]^2, \quad n \in \mathbb{Z}_+.$$

To proceed, choose an integer l in such a way that $2l$ is at least $\beta + \alpha(m) + k + 1$. If $n \in l\mathbb{Z}_+$, say, $n = l(\mu - 1)$, for some μ, then we can apply Lemma 1 to conclude that

$$(qn)^{\alpha(m)} a_{(qn)^{\alpha(m)}}(\mathscr{L}_K) \leqslant C \frac{d^2_{m,\beta/2,l}}{\mu^{\beta}} \leqslant C \frac{l^{\beta} d^2_{m,\beta/2,l}}{n^{\beta}}.$$

The same procedure can be repeated in order to obtain the same inequality in the cases in which $n \in j + l\mathbb{Z}_+$, $j \in \{1, 2, \ldots, l-1\}$. The final conclusion is

$$(qn)^{\alpha(m)} a_{(qn)^{\alpha(m)}}(\mathscr{L}_K) \leqslant \frac{C'}{n^{\beta}}, \quad n \in \mathbb{Z}_+.$$

in which C' is a positive constant not depending upon n. However, this inequality implies that

$$a_{(qn)^{\alpha(m)}}(\mathscr{L}_K) \leqslant \frac{C''}{n^{\beta+\alpha(m)}}, \quad n \in \mathbb{Z}_+.$$

An elementary calculation leads to the asymptotic behavior of $\{a_n(\mathscr{L}_K)\}$ described in the statement of the theorem. □

To end the section, we return to positive integral operators generated by a kernel satisfying either an averaged Hölder condition or a Stekelov-mean Hölder condition, as they provide relevant examples for Theorem 1. Indeed, combining Propositions 2 and 3 and the previous theorem leads to the following result.

COROLLARY 1. *If $\mathscr{L}_K$ is a positive integral operator generated by a kernel satisfying either the averaged Hölder condition (Example 1) or the Stekelov-mean Hölder condition (Example 2), then*

$$a_n(\mathscr{L}_K) = O(n^{-1-\beta/m}), \quad (n \to \infty).$$

5. Final remarks

Most of the concepts and constructions made in this paper can be recovered when we replace the unit sphere with a compact symmetric space of rank 1. Indeed, this space is a Riemannian manifold possessing a harmonic analysis structure very similar to that available on the spheres. A good source of information on compact symmetric spaces of rank 1, including concepts and results needed in a possible extension of the results proved here, is the survey paper [21]. We believe the new arguments needed in the detailing of such extension would not justify the writing of an additional paper.

The decay presented in Theorem 1 seems to be optimal within the setting considered. Restricting ourselves to the two motivational examples of Section 2, we tried for some time to construct a concrete example matching exactly the decay provided by the corresponding results proved in the paper. Unfortunately, we were unable to either construct such an example or substantiate optimality.

Recently, we have developed a new technique to deduce sharp decay rates for the sequence of eigenvalues of positive integral operators based on growth and integrability of Fourier coefficients ([13, 14]). This technique allows one to work in an even more general setting, replacing all the arguments involving the usual spherical convolutions with that of spherical convolutions with measures. In particular, this approach permits the inclusion of integral operators generated by kernels satisfying Hölder assumptions defined by families of general multiplier operators.

A final remark concerns the choice $B \in L^\infty(S^m)$ we have made in our definition for the Hölder assumption. On one hand, the choice is satisfactory because, in relevant concrete cases the function B is, in fact, constant. On the other, being a restriction, it may be not. However, a more general assumption on B, such as $B \in L^1(S^m)$, only appears in purely theoretical results.

Acknowledgement. We thank an anonymous referee for pointing out corrections and some relevant remarks. The actual format of the paper includes an upgrade raised by him.

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(Received November 24, 2014)

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